Vol 16 No 8, August 2007
1009-1963/2007/16(08)/2415-07

Chinese Physics

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Wigner function and tomogram of the Hermite polynomial state*

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(Received 8 October 2006; revised manuscript received 9 November 2006)

Using the coherent state representation of Wigner operator and the technique of integration within an ordered product (IWOP) of operators, this paper derives the Wigner function for the Hermite polynomial state (HPS). The tomogram of the HPS is calculated with the aid of newly introduced intermediate coordinate-momentum representation in quantum optics.

Keywords: Hermite polynomial state, IWOP technique, Wigner function, tomogram
PACC: 4250, 0365

1. Introduction

The Wigner function[1,2] in phase space is an important statistical method for studying fields in quantum optics. Recently, various photon states have been generated by the micromaser[3] and Wigner functions of some cavity fields can be measured in the whole phase space by a scheme based on interaction between cavity fields and atoms.[4,5] Therefore, it is very interesting to construct and measure the Wigner functions of quantum states by using some theoretical schemes. In this work, we shall calculate the Wigner function of the Hermite polynomial state (HPS) which is the minimum uncertainty state for the variables which describe amplitude-squared squeezing.

The HPS satisfies the following eigenvalue equation:[6]

$$(Y_1 + \mathrm{i}\lambda Y_2)|m,\lambda\rangle = \chi|m,\lambda\rangle, \tag{1}$$

where λ is real, χ is complex,

$$\chi = \mathrm{i}(1-\lambda^2)^{1/2}(m+1/2), \quad \text{for} \quad 0<\lambda<1, \tag{2}$$

$$\chi = (\lambda^2-1)^{1/2}(m+1/2), \quad \text{for} \quad \lambda \geq 1, \tag{3}$$

and

$$Y_1 = (a^2 + a^{+2})/2, \quad Y_2 = (a^2 - a^{+2})/(2\mathrm{i}), \tag{4}$$

a (a^+) is the usual annihilation (creation) operator. The state $|m,\lambda\rangle$ is taken as

$$|m,\lambda\rangle = C_m(\lambda)S(\xi)H_m(\mathrm{i}\gamma(\lambda)a^+)|0\rangle. \tag{5}$$

Where $H_m(x)$ is the m-order Hermite polynomials, its expansion formula is[7]

$$H_m(x) = \sum_{k=0}^{[m/2]} \frac{(-1)^k m!}{k!(m-2k)!}(2x)^{m-2k}, \tag{6}$$

and $S(\xi) = \exp[(\xi a^{+2} - \xi^* a^2)/2]$ is the single-mode squeezing operator,[8]

$$\begin{aligned} S(\xi) = {} & \mathrm{sech}^{1/2} r \exp\Big[\frac{1}{2}a^{+2}\mathrm{e}^{\mathrm{i}2\theta}\tanh r\Big] \\ & \times \exp[a^+ a \ln\cosh r] \\ & \times \exp\Big[-\frac{1}{2}a^2\mathrm{e}^{-\mathrm{i}2\theta}\tanh r\Big], \end{aligned} \tag{7}$$

where $\xi = r\exp(\mathrm{i}\theta)$ is the squeezing parameter. The parameters r and θ are defined as

(i) for $0<\lambda<1$,

$$\begin{aligned} & \theta = \pi/2, \quad \tanh r = \Big(\frac{1-\lambda}{1+\lambda}\Big)^{1/2}, \\ & \gamma(\lambda) = \exp(\mathrm{i}\pi/4)[(1-\lambda^2)^{1/2}/2]^{1/2}, \end{aligned} \tag{8}$$

(ii) for $\lambda \geq 1$,

$$\begin{aligned} & \theta = 0, \quad \tanh r = \Big(\frac{\lambda-1}{\lambda+1}\Big)^{1/2}, \\ & \gamma(\lambda) = [(\lambda^2-1)^{1/2}/(2\lambda)]^{1/2}. \end{aligned} \tag{9}$$

$C_m(\lambda)$ is the normalization constant of the state $|m,\lambda\rangle$ given by

$$[C_m(\lambda)]^{-2} = \langle 0|H_m(-\mathrm{i}\gamma^*(\lambda)a)H_m(\mathrm{i}\gamma(\lambda)a^+)|0\rangle$$

*Project supported by the National Natural Science Foundation of China (Grant No 10574060) and the Natural Science Foundation of Shandong Province of China (Grant No Y2004A09).

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$$= \sum_{n=0}^{[m/2]} \frac{(m!)^2[4|\gamma(\lambda)|^2]^{m-2n}}{(m-2n)!(n!)^2}. \quad (10)$$

The state is of interest because it can be generated by a degenerate parametric amplifier.[9] Especially, the state $|1,\lambda\rangle$ can be produced by using a one-photon number state as the input for a degenerate parametric amplifier because it is a squeezed one-photon state. We want to derive the Wigner function characterizing the HPS, and examine how this Wigner function varies with λ and m respectively. We also want to obtain the tomogram of the HPS. To simplify the calculations, we shall use the coherent state representation of Wigner operator and introduce the so-called intermediate coordinate-momentum representation.

2. The coherent state representation of squeezed Wigner operator

The Wigner function of the state $|m,\lambda\rangle$ is

$$\begin{aligned} W(\alpha,\alpha^*) &= \langle m,\lambda|\Delta(\alpha,\alpha^*)|m,\lambda\rangle \\ &= [C_m(\lambda)]^2\langle 0|H_m(-\mathrm{i}\gamma^*(\lambda)a)S^{-1}(\xi) \\ &\quad \times \Delta(\alpha,\alpha^*)S(\xi)H_m(\mathrm{i}\gamma(\lambda)a^+)|0\rangle, \end{aligned} \quad (11)$$

where $\Delta(\alpha,\alpha^*)$ is the coherent state representation of the Wigner operator, $\alpha = (x+\mathrm{i}p)/\sqrt{2}$.

Using the original definition of the Wigner function[10]

$$\Delta(\alpha,\alpha^*) = \frac{1}{2\pi^2}\int \mathrm{d}^2 z \exp[z(a^+-\alpha^*)-z^*(a-\alpha)], \quad (12)$$

and the following Glauber disentanglement theorem[11]

$$\exp(A+B) = \exp(A)\exp(B)\exp\Big(-\frac{[A,B]}{2}\Big), \quad (13)$$

as well as the technique of integration within an ordered product (IWOP) of operators, we can deduce the normal ordering form of the following Wigner operator

$$\begin{aligned} \Delta(\alpha,\alpha^*) &= \frac{1}{2\pi^2}\int \mathrm{d}^2 z : \exp\Big[-\frac{1}{2}|z|^2 \\ &\quad + z(a^+-\alpha^*) - z^*(a-\alpha)\Big] : \\ &= \frac{1}{\pi} : \exp[-2(a^+-\alpha^*)(a-\alpha)] :, \end{aligned} \quad (14)$$

where : : denotes normal ordering.

The Wigner operator $\Delta(\alpha,\alpha^*)$ is also expressed as[10]

$$\Delta(\alpha,\alpha^*) = \int \frac{\mathrm{d}^2 z}{\pi^2}|\alpha+z\rangle\langle\alpha-z|\mathrm{e}^{\alpha z^*-z\alpha^*}, \quad (15)$$

since the normal ordering form of Eq.(15) is the same as that of Eq.(14), i.e.,

$$\begin{aligned} \Delta(\alpha,\alpha^*) &= \int \frac{\mathrm{d}^2 z}{\pi^2}|\alpha+z\rangle\langle\alpha-z|\mathrm{e}^{\alpha z^*-z\alpha^*} \\ &= \int \frac{\mathrm{d}^2 z}{\pi^2} : \exp\{-|z|^2+(\alpha+z)a^+ \\ &\quad + (\alpha^*-z^*)a - a^+a + \alpha z^* - z\alpha^* - |\alpha|^2\} : \\ &= \frac{1}{\pi} : \exp[-2(a^+-\alpha^*)(a-\alpha)] :, \end{aligned} \quad (16)$$

where $|\alpha \pm z\rangle$ is the coherent state.[12]

On the other hand, the squeezed operator $S(\xi)$ yields

$$S^{-1}(\xi)aS(\xi) = a\cosh r + a^+\mathrm{e}^{\mathrm{i}\theta}\sinh r, \quad (17)$$

$$S^{-1}(\xi)a^+S(\xi) = a^+\cosh r + a\mathrm{e}^{-\mathrm{i}\theta}\sinh r, \quad (18)$$

where

$$[a\cosh r + a^+\mathrm{e}^{\mathrm{i}\theta}\sinh r,\ a^+\cosh r + a\mathrm{e}^{-\mathrm{i}\theta}\sinh r] = 1. \quad (19)$$

Therefore, from Eqs.(12), (13), (17) and (18) we have

$$\begin{aligned} &S^{-1}(\xi)\Delta(\alpha,\alpha^*)S(\xi) \\ &= \frac{1}{2\pi^2}\int \mathrm{d}^2 z : \exp\Big\{-|z|^2\Big(\sinh^2 r + \frac{1}{2}\Big) \\ &\quad + z(a\mathrm{e}^{-\mathrm{i}\theta}\sinh r + a^+\cosh r - \alpha^*) \\ &\quad - z^*(a^+\mathrm{e}^{\mathrm{i}\theta}\sinh r + a\cosh r - \alpha) \\ &\quad + \frac{1}{2}(z^2\mathrm{e}^{-\mathrm{i}\theta} + z^{*2}\mathrm{e}^{\mathrm{i}\theta})\sinh r\cosh r\Big\} : \\ &= \frac{1}{\pi} : \exp[-2(a^+ - \alpha^*\cosh r + \alpha\mathrm{e}^{-\mathrm{i}\theta}\sinh r) \\ &\quad \times (a - \alpha\cosh r + \alpha^*\mathrm{e}^{\mathrm{i}\theta}\sinh r)] : . \end{aligned} \quad (20)$$

Comparing Eq.(20) with Eq.(14), we obtain the coherent state representation of squeezed Wigner operator

$$\begin{aligned} \Delta(\beta,\beta^*) &= S^{-1}(\xi)\Delta(\alpha,\alpha^*)S(\xi) \\ &= \int \frac{\mathrm{d}^2 z}{\pi^2}|\beta+z\rangle\langle\beta-z|\mathrm{e}^{\beta z^*-z\beta^*}, \end{aligned} \quad (21)$$

where

$$\beta = \alpha\cosh r - \alpha^*\mathrm{e}^{\mathrm{i}\theta}\sinh r. \quad (22)$$

So the Wigner function of the state $|m,\lambda\rangle$ becomes

$$W(\alpha,\alpha^*) = [C_m(\lambda)]^2\langle 0|H_m(-\mathrm{i}\gamma^*(\lambda)a)$$

$$\times \int \frac{\mathrm{d}^2 z}{\pi^2} |\beta + z\rangle\langle\beta - z| \mathrm{e}^{\beta z^* - z\beta^*} \times H_m(\mathrm{i}\gamma(\lambda) a^+)|0\rangle. \quad (23)$$

3. The Wigner function of the state $|m, \lambda\rangle$

Now in terms of Eq.(23) we obtain

$$W(\alpha, \alpha^*) = [C_m(\lambda)]^2 \int \frac{\mathrm{d}^2 z}{\pi^2} H_m(-\mathrm{i}\gamma^*(\lambda)(\beta + z)) \times H_m(\mathrm{i}\gamma(\lambda)(\beta^* - z^*)) \times \exp[-|\beta|^2 - |z|^2 + \beta z^* - z\beta^*]. \quad (24)$$

Using the generating function formula of $H_n(x)$[13]

$$H_n(x) = \frac{\partial^n}{\partial t^n} \mathrm{e}^{2xt - t^2}\Big|_{t=0}, \quad (25)$$

and the following integral formula:

$$\int \frac{\mathrm{d}^2 z}{\pi} \exp(\zeta|z|^2 + \xi z + \eta z^* + f z^2 + g z^{*2}) = \frac{1}{\sqrt{\zeta^2 - 4fg}} \exp\Big[\frac{-\zeta\xi\eta + \xi^2 g + \eta^2 f}{\zeta^2 - 4fg}\Big], \quad (26)$$

with

$$\mathrm{Re}(\zeta \pm f \pm g) < 0, \quad \mathrm{Re}\Big(\frac{\zeta^2 - 4fg}{\zeta \pm f \pm g}\Big) < 0. \quad (27)$$

we find that Eq.(24) becomes

$$\begin{aligned} W(\alpha, \alpha^*) &= [C_m(\lambda)]^2 \frac{\partial^{2m}}{\partial t^m \partial t'^m} \int \frac{\mathrm{d}^2 z}{\pi^2} \exp[-|z|^2 - \mathrm{i}2\gamma^*(\lambda)(\beta + z)t - t^2 \\ &\quad + \mathrm{i}2\gamma(\lambda)(\beta^* - z^*)t' - t'^2 - |\beta|^2 + \beta z^* - z\beta^*]|_{t=t'=0} \\ &= [C_m(\lambda)]^2 \frac{\partial^{2m}}{\partial t^m \partial t'^m} \exp[-4|\gamma(\lambda)|^2 t t' + \mathrm{i}4(\gamma(\lambda)\beta^* t' - \gamma^*(\lambda)\beta t) - t^2 - t'^2 - 2|\beta|^2]|_{t=t'=0} \\ &= [C_m(\lambda)]^2 \sum_{n=0}^{m} \binom{m}{n} \Big[\frac{\partial^n}{\partial t^n} H_m(\mathrm{i}2\gamma(\lambda)\beta^* - 2|\gamma(\lambda)|^2 t)\Big] \\ &\quad \times \frac{\partial^{m-n}}{\partial t'^{m-n}} \exp[-\mathrm{i}4\gamma^*(\lambda)\beta t - t^2 - 2|\beta|^2]|_{t=0}. \end{aligned} \quad (28)$$

Then using the well-known recurrence relations of $H_n(x)$[13]

$$\frac{\mathrm{d}^l}{\mathrm{d}x^l} H_n(x) = \frac{2^l n!}{(n-l)!} H_{n-l}(x), \quad (29)$$

we finally obtain the Wigner function of the state $|m, \lambda\rangle$ in the following form:

$$W(\alpha, \alpha^*) = [C_m(\lambda)]^2 \sum_{n=0}^{m} \frac{2^n (m!)^2}{n![(m-n)!]^2} \times |H_{m-n}(-2\gamma^*(\lambda)\beta)|^2 \exp[-2|\beta|^2], \quad (30)$$

where β is given by Eq.(22).

Now we would like to discuss changes of the Wigner function with varying parameters λ and m from Figs.1 and 2. In Fig.1 there are plots of the Wigner function of the HPS with $m = 10$ and for different values of λ ($\lambda = 1, 1.02$ and 1.2). When $\lambda = 1$ (Fig.1(a)), we clearly see that the Wigner function $W(\alpha, \alpha^*)$ displays a single peak and has Gaussian shape, which is similar to that of the Wigner function of vacuum state, centred at the origin of phase space. This is reasonable and accurate because the HPS becomes the vacuum state for taking $\lambda = 1$. By increasing the parameter λ, the single peak begins to split into two peaks and they are compressed along the Imα (or p) direction as can be seen in Figs.1(b) and 1(c). The features result from the squeezed parameter $r \neq 0$. By reason that λ plays the role of a squeezing parameter, the two peaks are further compressed with increasing λ. Here from the phase space point of view the squeezed properties of the HPS can be well interpreted and described. The greater the value of λ, the smaller the value of two peaks, the fur-

ther the distance between the two peaks and finally two independent peaks occur. Similarly, when the parameter λ decreases from 1 to 0, the Wigner function $W(\alpha, \alpha^*)$ obeys the same rule. Comparing Fig.1(c) with Fig.2, we also find that the greater the value of m, the greater the value of two peaks and the further the distance between the two peaks.

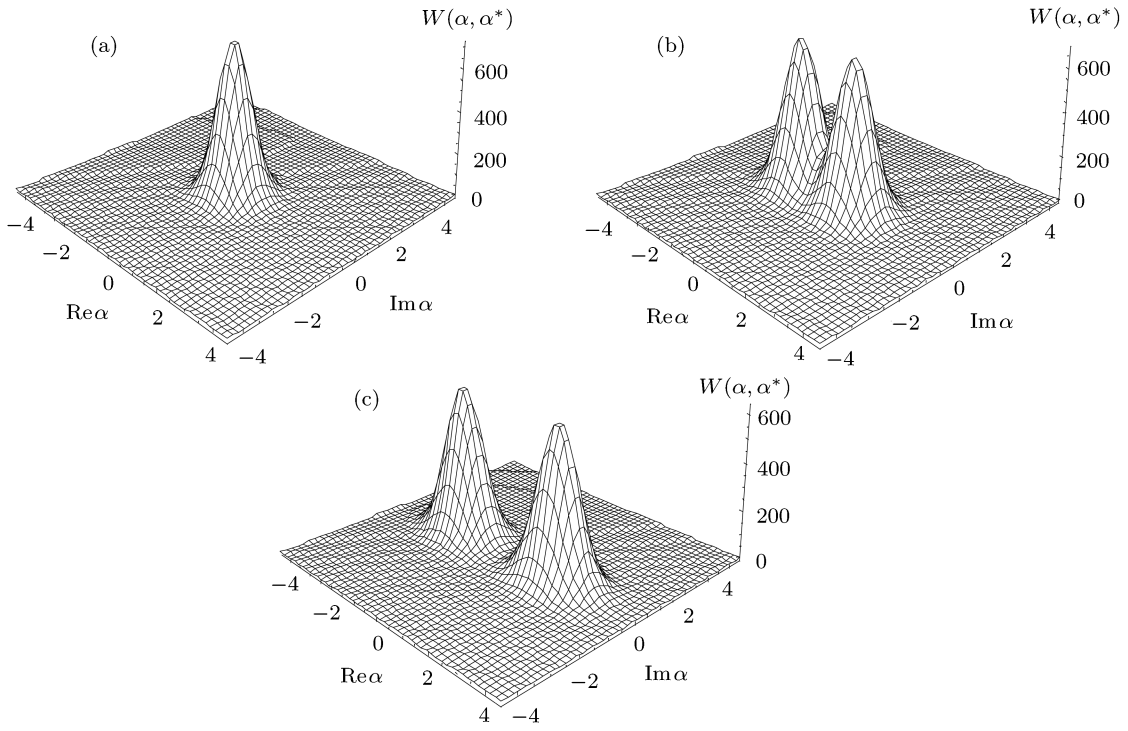


Fig.1. The Wigner function of the HPS for $m = 10$ and (a) $\lambda = 1$, (b) $\lambda = 1.02$, (c) $\lambda = 1.2$.

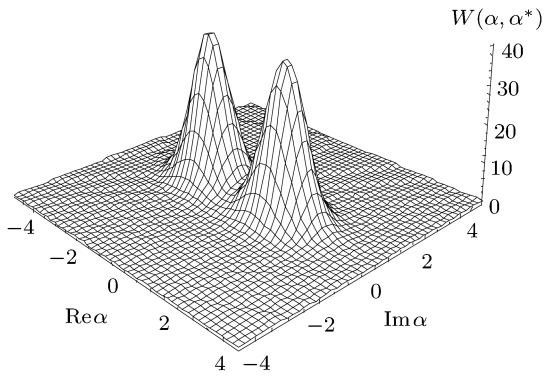


Fig.2. The Wigner function of the HPS for $m = 6$ and $\lambda = 1.2$.

4. Tomogram of the state $|m, \lambda\rangle$

In recent years, tomogram approach, whose property is a standard positive probability distribution function describing the quantum state in quantum statistics and quantum optics, has brought much interests of physicists. As is well-known, x-ray or optical tomographic imaging techniques (classical tomography) derive two-dimensional data from a three-dimensional object to obtain a slice image of the internal structure and thus have the ability to peer inside the object noninvasively. In the context of phase space theory of quantum statistics Vogel and Risken[14] pointed out that the probability distribution for the rotated quadrature phase $X_\theta = [a^+ \exp(\mathrm{i}\theta) + a\exp(-\mathrm{i}\theta)]/\sqrt{2}$ can be expressed in terms of Wigner function, and the reverse is also true (named as Vogel–Risken relation), i.e., one can obtain the Wigner distribution by tomographic inversion of a set of measured probability distributions, $P_\theta(x_\theta)$, of the quadrature amplitude. Smithey *et al*[15] also pointed out that once the distribution $P_\theta(x_\theta)$ is obtained, one can use the inverse Radon transformation familiar in tomographic imaging to obtain the Wigner distribution and density matrix. Thus tomographic approach of quantum theories offers a description in terms of tomographic probabilities. In this section we continue

to derive the tomogram of $|m,\lambda\rangle$, which is defined as

$$T(x,f,g)=\iint_{\infty}\mathrm{d}p'\mathrm{d}x'\delta(x-fx'-gp')W(x',p'). \quad (31)$$

However, it will be very difficult to evaluate the integration if we directly substitute Eq.(30) into Eq.(31), fortunately we use the following relation between the Wigner operator and the projector of intermediate coordinate-momentum[13]

$$|x\rangle_{f,g\ f,g}\langle x|=\iint_{\infty}\mathrm{d}p'\mathrm{d}x'\delta(x-fx'-gp')\Delta(x',p'). \quad (32)$$

Thus the tomogram of $|m,\lambda\rangle$ is

$$T(x,f,g)=\iint_{\infty}\mathrm{d}p'\mathrm{d}x'\delta(x-fx'-gp')\times\langle m,\lambda|\Delta(x',p')|m,\lambda\rangle=|_{f,g}\langle x|m,\lambda\rangle|^2, \quad (33)$$

which shows that for tomographic approach there exists the intermediate coordinate-momentum state $|x\rangle_{f,g}$, and the Radon transforms of the Wigner operator are just the pure-state density matrices $|x\rangle_{f,g\,f,g}\langle x|$. As a result, the tomogram of quantum states can be considered as the module-square of the states' wave function in this representation. This is a new way to derive tomogram of quantum states. Using the coherent state's completeness relation we can obtain

$$_{f,g}\langle x|m,\lambda\rangle=C_m(\lambda)_{f,g}\langle x|\int\frac{\mathrm{d}^2\alpha}{\pi}|\alpha\rangle\langle\alpha|S(\xi)H_m(\mathrm{i}\gamma(\lambda)a^+)S^{-1}(\xi)\int\frac{\mathrm{d}^2\beta}{\pi}|\beta\rangle\langle\beta|S(\xi)|0\rangle. \quad (34)$$

By virtue of Eqs.(6), (7) and (18) we have

$$\begin{aligned}
_{f,g}\langle x|m,\lambda\rangle&=C_m(\lambda)\mathrm{sech}^{1/2}r_{f,g}\langle x|\int\frac{\mathrm{d}^2\alpha}{\pi}|\alpha\rangle\langle\alpha|H_m(\mathrm{i}\gamma(\lambda)(a^+\cosh r-a\mathrm{e}^{-\mathrm{i}\theta}\sinh r))\\
&\times\int\frac{\mathrm{d}^2\beta}{\pi}|\beta\rangle\langle\beta|\exp\Big[-\frac{1}{2}a^{+2}\mathrm{e}^{2\mathrm{i}\theta}\tanh r\Big]|0\rangle\\
&=C_m(\lambda)\mathrm{sech}^{1/2}r_{f,g}\langle x|\int\frac{\mathrm{d}^2\alpha}{\pi}|\alpha\rangle H_m(\mathrm{i}\gamma(\lambda)(\alpha^*\cosh r-\beta\mathrm{e}^{-\mathrm{i}\theta}\sinh r))\\
&\times\int\frac{\mathrm{d}^2\beta}{\pi}\exp\Big[-\frac{1}{2}\beta^{*2}\mathrm{e}^{2\mathrm{i}\theta}\tanh r-|\beta|^2-\frac{1}{2}|\alpha|^2+\alpha^*\beta\Big]\\
&=C_m(\lambda)\mathrm{sech}^{1/2}r\sum_{n=0}^{[m/2]}\sum_{k=0}^{[m-2n]}\binom{m-2n}{k}\frac{(-)^nm![\mathrm{i}2\gamma(\lambda)]^{m-2n}}{n!(m-2n)!}\\
&\times(\cosh r)^k(-\mathrm{e}^{-\mathrm{i}\theta}\sinh r)^{m-2n-k}\int\frac{\mathrm{d}^2\alpha}{\pi}{}_{f,g}\langle x|\alpha\rangle\alpha^{*k}\exp(-|\alpha|^2)\\
&\times\int\frac{\mathrm{d}^2\beta}{\pi}\beta^{m-2n-k}\exp\Big[-\frac{1}{2}\beta^{*2}\mathrm{e}^{2\mathrm{i}\theta}\tanh r-|\beta|^2+\alpha^*\beta\Big]. \quad (35)
\end{aligned}$$

Then using the following integral formula:[16]

$$\int\frac{\mathrm{d}^2z}{\pi}z^n\exp(\zeta|z|^2+\xi z+\eta z^*+gz^{*2})=-\Big(\frac{1}{\zeta}\Big)^{2n+1}\exp[\frac{1}{\zeta^2}(g\xi^2-\zeta\xi\eta)]\sum_{k=0}^{[n/2]}\frac{n!}{k!(n-2k)!}(2\xi g-\zeta\eta)^{n-2k}(g\zeta^2)^k, \quad (36)$$

and the intermediate coordinate-momentum state

$$|x\rangle_{f,g}=[\pi(f^2+g^2)]^{1/4}\exp\Big[-\frac{x^2}{2(f^2+g^2)}+\frac{\sqrt{2}x}{f-ig}a^+-\frac{f+\mathrm{i}g}{2(f-\mathrm{i}g)}a^{+2}\Big]|0\rangle, \quad (37)$$

we find that Eq.(35) becomes

$$_{f,g}\langle x|m,\lambda\rangle=\frac{C_m(\lambda)\mathrm{sech}^{1/2}r}{[\pi(f^2+g^2)]^{1/4}}\sum_{n=0}^{[m/2]}\sum_{k=0}^{[m-2n]}\binom{m-2n}{k}\frac{(-)^{m+n-k}m![\mathrm{i}2\gamma(\lambda)]^{m-2n}\cosh^k r}{n!(m-2n)!}$$

$$\times (\mathrm{e}^{-\mathrm{i}\theta}\sinh r)^{m-2n-k}\int\frac{\mathrm{d}^2\alpha}{\pi}\alpha^*(1-\alpha^*\mathrm{e}^{2\mathrm{i}\theta}\tanh r)$$
$$\times\exp\Big[-|\alpha|^2+\frac{\sqrt{2}x}{f+\mathrm{i}g}\alpha-\frac{f-\mathrm{i}g}{2(f+\mathrm{i}g)}\alpha^2-\frac{1}{2}\alpha^{*2}\mathrm{e}^{2\mathrm{i}\theta}\tanh r-\frac{x^2}{2(f^2+g^2)}\Big]. \tag{38}$$

Now using Eqs.(25) and (26) we obtain the following integral in Eq.(38):

$$\begin{aligned}
&\int\frac{\mathrm{d}^2\alpha}{\pi}\alpha^{*k}\exp\Big[-|\alpha|^2+\frac{\sqrt{2}x}{f+ig}\alpha-\frac{f-\mathrm{i}g}{2(f+\mathrm{i}g)}\alpha^2-\frac{1}{2}\alpha^{*2}\mathrm{e}^{2\mathrm{i}\theta}\tanh r-\frac{x^2}{2(f^2+g^2)}\Big]\\
&=\Big(\frac{\partial}{\partial\rho}\Big)^k\int\frac{\mathrm{d}^2\alpha}{\pi}\exp\Big[-|\alpha|^2+\frac{\sqrt{2}x}{f+\mathrm{i}g}\alpha-\frac{f-\mathrm{i}g}{2(f+\mathrm{i}g)}\alpha^2-\frac{1}{2}\alpha^{*2}\mathrm{e}^{2\mathrm{i}\theta}\tanh r-\frac{x^2}{2(f^2+g^2)}+\rho\alpha^*\Big]\Big|_{\rho=0}\\
&=\Big(\frac{f+\mathrm{i}g}{f+\mathrm{i}g-(f-\mathrm{i}g)\mathrm{e}^{2\mathrm{i}\theta}\tanh r}\Big)^{1/2}\exp\Big[\frac{x^2\mathrm{e}^{2\mathrm{i}\theta}\tanh r}{(f+\mathrm{i}g)^2-(f^2+g^2)\mathrm{e}^{2\mathrm{i}\theta}\tanh r}\Big]\\
&\quad\times\Big(\frac{\partial}{\partial\rho}\Big)^k\exp\Big[\frac{-\frac{1}{2}(f-\mathrm{i}g)\rho^2+\sqrt{2}x\rho}{f+\mathrm{i}g-(f-\mathrm{i}g)\mathrm{e}^{2\mathrm{i}\theta}\tanh r}\Big]\Big|_{\rho=0}\\
&=\Big[\frac{2(f+\mathrm{i}g)}{f-\mathrm{i}g}\Big]\Big\{\frac{f-\mathrm{i}g}{2[f+\mathrm{i}g-(f-\mathrm{i}g)\mathrm{e}^{2\mathrm{i}\theta}\tanh r]}\Big\}^{(k+1)/2}\exp\Big[-\frac{x^2\mathrm{e}^{2\mathrm{i}\theta}\tanh r}{(f+\mathrm{i}g)^2-(f^2+g^2)\mathrm{e}^{2\mathrm{i}\theta}\tanh r}\Big]\\
&\quad\times H_k\Big(\frac{2x}{f^2+g^2-(f-\mathrm{i}g)^2\mathrm{e}^{2\mathrm{i}\theta}\tanh r}\Big),
\end{aligned}\tag{39}$$

so the tomogram amplitude of the state $|m,\lambda\rangle$ can be obtained as

$$\begin{aligned}
{}_{f,g}\langle x|m,\lambda\rangle&=\frac{\sqrt{2}C_m(\lambda)(f^2+g^2)^{1/4}\mathrm{sech}^{1/2}r}{\pi^{1/4}(f-\mathrm{i}g)}\exp\Big[-\frac{x^2\mathrm{e}^{\mathrm{i}2\theta}\tanh r}{(f+\mathrm{i}g)^2-(f^2+g^2)\mathrm{e}^{\mathrm{i}2\theta}\tanh r}-\frac{x^2}{2(f^2+g^2)}\Big]\\
&\quad\times\sum_{n=0}^{[m/2]}\sum_{k=0}^{[m-2n]}\binom{m-2n}{k}\frac{(-)^n m![\mathrm{i}2\gamma(\lambda)]^{m-2n}}{n!(m-2n)!}(\cosh r)^k(-\mathrm{e}^{-\mathrm{i}\theta}\sinh r)^{m-2n-k}\\
&\quad\times\Big\{\frac{f-\mathrm{i}g}{2[f+\mathrm{i}g-(f-\mathrm{i}g)\mathrm{e}^{\mathrm{i}2\theta}\tanh r]}\Big\}^{(k+1)/2}\Big\{H_k\Big(\frac{2x}{[(f^2+g^2)-(f-\mathrm{i}g)^2\mathrm{e}^{\mathrm{i}2\theta}\tanh r]}\Big)\\
&\quad+\mathrm{e}^{\mathrm{i}2\theta}\tanh r\Big(\frac{f-\mathrm{i}g}{2[f+\mathrm{i}g-(f-\mathrm{i}g)\mathrm{e}^{\mathrm{i}2\theta}\tanh r]}\Big)^{1/2}H_{k+1}\Big(\frac{2x}{[(f^2+g^2)-(f-\mathrm{i}g)^2\mathrm{e}^{\mathrm{i}2\theta}\tanh r]}\Big)\Big\}.
\end{aligned}\tag{40}$$

Thus the tomogram of the state $|m,\lambda\rangle$ is

$$\begin{aligned}
T(x,f,g)=\Big|&\frac{\sqrt{2}C_m(\lambda)(f^2+g^2)^{1/4}\mathrm{sech}^{1/2}r}{\pi^{1/4}(f-\mathrm{i}g)}\exp\Big[-\frac{x^2\mathrm{e}^{\mathrm{i}2\theta}\tanh r}{(f+\mathrm{i}g)^2-(f^2+g^2)\mathrm{e}^{\mathrm{i}2\theta}\tanh r}-\frac{x^2}{2(f^2+g^2)}\Big]\\
&\times\sum_{n=0}^{[m/2]}\sum_{k=0}^{[m-2n]}\binom{m-2n}{k}\frac{(-)^n m![\mathrm{i}2\gamma(\lambda)]^{m-2n}}{n!(m-2n)!}(\cosh r)^k(-\mathrm{e}^{-\mathrm{i}\theta}\sinh r)^{m-2n-k}\\
&\times\Big\{\frac{f-\mathrm{i}g}{2[f+\mathrm{i}g-(f-\mathrm{i}g)\mathrm{e}^{\mathrm{i}2\theta}\tanh r]}\Big\}^{(k+1)/2}\Big\{H_k\Big(\frac{2x}{[(f^2+g^2)-(f-\mathrm{i}g)^2\mathrm{e}^{\mathrm{i}2\theta}\tanh r]}\Big)\\
&+\mathrm{e}^{\mathrm{i}2\theta}\tanh r\Big(\frac{f-\mathrm{i}g}{2[f+\mathrm{i}g-(f-\mathrm{i}g)\mathrm{e}^{\mathrm{i}2\theta}\tanh r]}\Big)^{1/2}H_{k+1}\Big(\frac{2x}{[(f^2+g^2)-(f-\mathrm{i}g)^2\mathrm{e}^{\mathrm{i}2\theta}\tanh r]}\Big)\Big\}\Big|^2.
\end{aligned}\tag{41}$$

Therefore, in experiment one can measure the module-square of the wavefunction $|m,\lambda\rangle$ in the intermediate coordinate-momentum representation, then the tomogram of the state $|m,\lambda\rangle$ is obtained and the state $|m,\lambda\rangle$ shall be measured.

In summary, we have used the coherent state representation of Wigner operator and the intermediate coordinate-momentum representation to derive the

Wigner function and tomogram of the HPS. It is found that, in the phase space, the variations of the Wigner function with the parameters λ and m well reveal nonclassical properties of the HPS. The tomogram of the HPS is calculated by virtue of the Radon transform between the Wigner operator and the projection operator of the intermediate coordinate-momentum state $|x\rangle_{f,g}$. The results may be useful for reference in doing experiments.

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