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Nonchaotic random behaviour in the second order autonomous system*

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Evidence is presented for the nonchaotic random behaviour in a second-order autonomous deterministic system. This behaviour is different from chaos and strange nonchaotic attractor. The nonchaotic random behaviour is very sensitive to the initial conditions. Slight difference of the initial conditions will generate wholly different phase trajectories. This random behaviour has a transient random nature and is very similar to the coin-throwing case in the classical theory of probability. The existence of the nonchaotic random behaviour not only can be derived from the theoretical analysis, but also is proved by the results of the simulated experiments in this paper.

Keywords: chaos, nonchaotic, random, autonomous system
PACC: 0545, 0547, 0570

1. Introduction

Chaos provides a link between the deterministic system and the random system,[1] but it is not the only one that exists. There is another relationship which is called the nonchaotic random behaviour. The nonchaotic random behaviour comes from a deterministic system and is very sensitive to the initial conditions,[2] but it is different from chaos and strange nonchaotic attractor.[3–5] Under different initial conditions, the phase trajectory motions in the deterministic system are completely different. The parameter space that generates the nonchaotic random behaviour is not zero, and it will not generate strange attractor. The nonchaotic random behaviour is similar to the coin-throwing case in classical theory of probability. Since people know that random phenomenon can be generated in a deterministic system, the nonchaotic random behaviour also shows how a random phenomenon is produced in a deterministic system.

2. Initial conditions and motion in the deterministic system

The second-order autonomous differential equations are given by

$$\begin{aligned} C\frac{\mathrm{d}u}{\mathrm{d}t} &= i = Q(i,u), \\ 2L\frac{\mathrm{d}i}{\mathrm{d}t} &= -\left(u + f(i)\right) = P(i,u), \end{aligned} \tag{1}$$

where $P(i,u)$, $Q(i,u) \in C^0$, L and C are constants.

Equation (1) is the variance of Liénard equation, with $f(i)$ being a nonlinear function

$$f(i) = i^2. \tag{2}$$

Thus, Eq.(1) becomes

$$2L\frac{\mathrm{d}i}{\mathrm{d}t} = -(u + i^2), \tag{3a}$$

$$C\frac{\mathrm{d}u}{\mathrm{d}t} = i. \tag{3b}$$

Obviously, its equilibrium state is the focus-origin.

By comparing these two equations, the equation of tangent to the phase portrait of the system is obtained:

$$\frac{2\mathrm{d}i}{\mathrm{d}u} = \frac{-C(u + i^2)}{Li}. \tag{4}$$

This equation is easily integrated as compared with Eqs.(3a) and (3b).[6]

The solution of this equation is $i = u(i; A)$, where A is the integration constant. For different A, a group

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of curves on the $i-u$ plane are obtained, and expressed as follows:

$$\left(i^2+u-\frac{L}{C}\right)\mathrm{e}^{u\frac{C}{L}}=A; \tag{5}$$

for each point on the curves, the slope of its tangent is determined by Eq.(4).

Integration constant A is determined by the initial condition (i_0,u_0) of the system expressed as Eq.(3). For the convenience of studying the influence of A on the characteristics of the solutions, let $C=L=1$, thus Eq.(5) becomes

$$\left(i^2+u-1\right)\mathrm{e}^{u}=A. \tag{6}$$

When $A=-1$, we have

$$\left(i^2+u-1\right)\mathrm{e}^{u}=-1, \tag{7}$$

its phase trajectory corresponds to an isolated point ($u=0$, $i=0$), which is also the singular point (0,0) of the equation.

When

$$-1<\left(i^2+u-1\right)\mathrm{e}^{u}<0, \tag{8}$$

the phase trajectories of Eq.(8) are the closed curves encircling the singular point, and they correspond to the dotted closed phase trajectories in Fig.1. These closed phase trajectories show that the system undertakes periodic motions. When time $t\to\infty$, the waveforms of the periodic oscillation can be observed in time domain. They are named the quadratic-type oscillation.

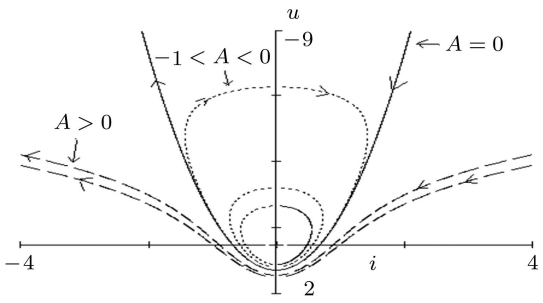


Fig.1. The different phase trajectories described by Eqs.(7)–(10).The boundary phase trajectory is expressed by the solid line, it divides the plane into two portions; closed phase trajectories are expressed by the dotted lines, while the divergent phase trajectories by dashed lines.

When $A\geq 0$, we have

$$\left(i^2+u-1\right)\mathrm{e}^{u}\geq 0; \tag{9}$$

the phase trajectories of Eq.(9) are divergent curves complying with i, and they correspond to the dashed curves in Fig.1. They are called divergent phase trajectories.

When $A=0$,

$$i^2=1-u; \tag{10}$$

the phase trajectory is a parabola with a truncated length equal to 1, it is the solid curve in Fig.1.

When $\lim|A|\to 0$, Eq.(10) is a homoclinic phase trajectory of the closed phase trajectories and divergent phase trajectories.[7] This homoclinic phase trajectory is also the boundary line. The closed phase trajectories lie inside the boundary, while the divergent phase trajectories lie outside, as shown in Fig.1.

If the integration constant A is considered as varying and passing through the zero point (e.g. A varies from -0.1 to 0.5), then Eq.(6) will lead to a three-dimensional curved surface as shown in Fig.2. This curved surface has the shape of a big truncated funnel with a large opening. According to the value of A, the projection on the $u-i$ plane can be closed curves or divergent curves.

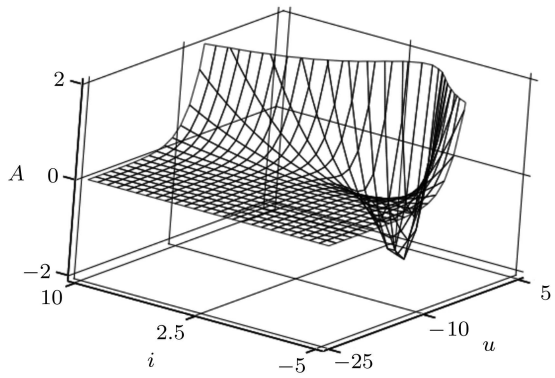


Fig.2. The three-dimensional curved surface of the solution of Eq.(3),obtained when the initial condition A is considered as varying and passing through the zero point.

The value of A is determined by the initial condition (i_0,u_0), and it determines the shapes of the phase trajectories of the solution related to the differential equation (3). It can be seen from Fig.1 that, if the initial condition (i_0, u_0) lies in the internal region of the homoclinic phase trajectory, the phase trajectories of the solution are limiting cycles (closed curves); if the initial condition (i_0, u_0) lies outside the homoclinic curve, the phase trajectories of the solution are divergent curves.

3. The deterministic character and stochastic property of a system

System (1) has deterministic properties, and this indicates that its parameters and variables are all deterministic. When $|A| \gg 0$, the initial points lie far away from the homoclinic line, the future states or shapes of the phase trajectories are also deterministic. When $A = 0$, the initial points lie on the homoclinic line, the future states or shapes of the phase trajectories will be undetermined, i.e. the stochastic property appears.

By mapping the phase trajectories of Eqs.(8)–(10), we have

$$L^- = \{u, i : -1 < \left(i^2 + u - 1\right) \mathrm{e}^u > 0\}, \tag{11a}$$

and this shows a set of closed phase trajectories of quadratic-type oscillation;

$$L^+ = \{u, i : (i^2 + u - 1)\mathrm{e}^u > 0\}, \tag{11b}$$

this shows a set of divergent phase trajectories;

$$L^0 = \{u, i : i^2 = 1 - u\}, \tag{11c}$$

this is the homoclinic phase trajectory (boundary line between L^- and L^+).

All the points ($A = 0$) which satisfy the initial conditions for L^0 are lie on a line on the two-dimensional $u - i$ plane. In the actual system or numerical calculation, the motions, under the initial condition for L^0, are not stable, even belong to a stochastic system described by the classical probability theory. The motion may deviate from the phase trajectory of L^0, because in an actual system there may exist interferences (electromagnetic and heat noises and truncation errors in numerical calculations).

In numerical calculation, when $\sqrt{C/L}$ is irrational, the points satisfying the initial condition of L^0 cannot consistently move on L^0 from start to finish, in fact, there are much more irrational numbers than rational ones. For example, when $L/C = 1/3$, Eq.(5) becomes

$$\left(i^2 + u - 3\right) e^{u/3} = A. \tag{12}$$

When $A = 0$,

$$i^2 = 3 - u; \tag{13}$$

at $u = 0$, $i^2 = 3$, $i = \sqrt{3}$. In numerical calculation, the value of i is obtained as an initial condition, however due to the limitation of numerical value for length and the truncation error, $i_0 \neq \sqrt{3}$ is always true. When $i_0 > \sqrt{3}$, the phase trajectories tend to move from L^0 to L^-; if $i_0 < \sqrt{3}$, the phase trajectories tend to move from L^0 to L^+.

Let $\Delta\alpha$ be the interference in an actual electric circuit or the truncation error in the numerical calculation, then Eq.(6) can be expressed as

$$\left(i^2 + u - 1\right) \mathrm{e}^u = (A + \Delta a). \tag{14}$$

When $A = 0$, it becomes

$$\left(i^2 + u - 1\right) \mathrm{e}^u = \Delta a; \tag{15}$$

so long as $\Delta\alpha \neq 0$, Eq.(1) will not be expressed as the homoclinic phase trajectory.

Taking the case of throwing a coin in the classical probability theory as an example. Let the backside of the coin represent the phase trajectories of L^-, the front side represent the phase trajectories of L^+, and the edge of the corn represent the phase trajectory of L^0. Suppose the coin is rolling on a plane, then three possible cases will happen:

(i) standing steadily on its edge; (ii) lying down with the front side upward; (iii) lying down with the backside upward. Through mapping, the result of the above deterministic dynamic system is similar to that of throwing a coin in classical probability theory.

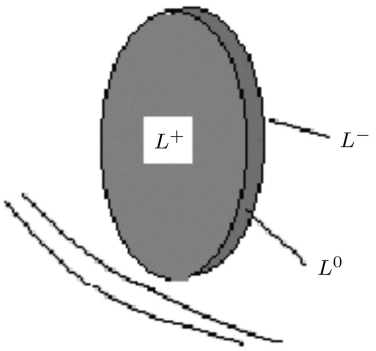


Fig.3. L^+ and L^- mapping into the front and back sides of a coin respectively. The periphery of the coin represents L_0.

Let the probabilities of the three cases be

$$\begin{aligned} P(L^-) &= n_1; \\ P(L^+) &= n_2; \\ P(L^0) &= n_3; \\ n_1 + n_2 + n_3 &= 1. \end{aligned} \tag{16}$$

The phase trajectory starting from any point of L^0 is taken to be a fundamental event ω, then all phase trajectories starting from the points of L^0 form a sample space Ω, and (Ω, ω, n) constructs a stochastic space, with $X(\omega)$ being a stochastic variable in this space. The study of the solutions (quantitative, qualitatively or numerical) of the deterministic differential equation (3) turns into a study of the distribution and digital characteristics of $X(\omega)$.

The stochastic domain not only exists on phase trajectory of L^0 but also on the phase trajectories of L^+ and L^- under the conditions of disturbance when they lie in the very vicinity of L^0.

Let the maximum value of disturbance be $\delta(\Delta i, \Delta u)$, and substituted into Eq.(10). We obtain

$$\begin{aligned}(i+\Delta i)^2 &= 1-(u+\Delta u) = i^2+2i\Delta i+(\Delta i)^2 \\ &= 1-u-\Delta u, \qquad (17)\\ i^2+u-1 &= -2i\Delta i-\Delta u-(\Delta i)^2. \qquad (18)\end{aligned}$$

So long as $-2i\Delta i-\Delta u-(\Delta i)^2 \neq 0$, the phase trajectories will leave L^0 towards L^- or L^+.

From the above analysis, we can observe how the parametric conditions and the initial conditions for a second-order autonomous system can cause a stochastic domain.

4. Results of the simulation experiment

In order to verify the above views, a second-order nonlinear autonomous circuit is used to carry out the simulation experiments. The circuit is shown in Fig.4, where the linear inductor $L_1 = 1\,\mathrm{mH}$, the linear capacity $C_1 = 100\,\mu\mathrm{F}$, and $f(i) = i^2$ is the $u-i$ character of the nonlinear resistor.

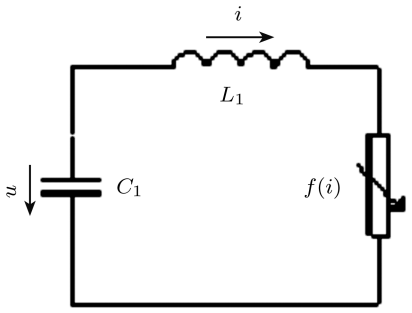


Fig.4. The second-order autonomous nonlinear circuit.

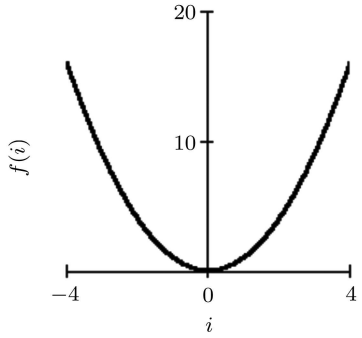


Fig.5. The u-i character of nonlinear component $f(i)$.

Let the current i and the voltage u be the state variables, then the circuit in Fig.4 can be represented as

$$\frac{\mathrm{d}i}{\mathrm{d}t} = -(u+i^2)\times 10^3, \qquad (19a)$$

$$\frac{\mathrm{d}u}{\mathrm{d}t} = i\times 10^6. \qquad (19b)$$

Substituting L_1 and C_1 into Eq.(5), we have an equation which meets the initial condition (i_0, u_0) of homoclinic phase trajectory

$$i^2+u-\frac{L_1}{2C_1} = i^2+u-5 = 0. \qquad (20)$$

Now replace (i, u) of Eq.(20) by one of the initial conditions $(i_0 = 0, u_0 = 5)$. The initial condition determines the value of A, and it also determines the shape of the phase trajectory. When $i_0 = 0, u_0 > 5$, divergent phase trajectories appear, and when $i_0 = 0, u_0 < 5$, quadratic-type oscillations with closed phase trajectories are revealed. We use the common software Pspice to simulate the circuitry in Fig.4 (other software also can be used to carry out the numerical calculation, such as Matlab, and Mathematica etc.). Three experiments have been carried out.

The first experiment is to observe the dependence of sensitivity of circuit state on the initial conditions and the result shows that they are closely related. All the given circuit parameters and parameters used in numerical calculation are the same, and the maximum iterative step is $10\,\mu\mathrm{s}$. Let the initial conditions (0, 5.00000198) and (0, 5.00000199) as the disturbance parameters, both are very close to the homoclinic phase trajectory. The difference of them is only 1×10^{-8}, and they belong to the divergent phase trajectories of Eq.(19). Figure 6 is the divergent phase trajectories with initial value of (0, 5.00000199) after 2.2 ms and Fig.7 are the quadratic-type oscillation closed phase trajectories with initial value (0,

5.000000198) after 4.6 ms. A tiny difference between the initial conditions produces tremendous difference in the state of the circuit shown in Fig.4. The results of this experiment show that when the initial conditions are very close to the homoclinic phase trajectory, we cannot tell the exact characteristics of the solution of Eq.(19) with the initial conditions before simulation. In this circumstance, it becomes a random system described by Eq.(19).

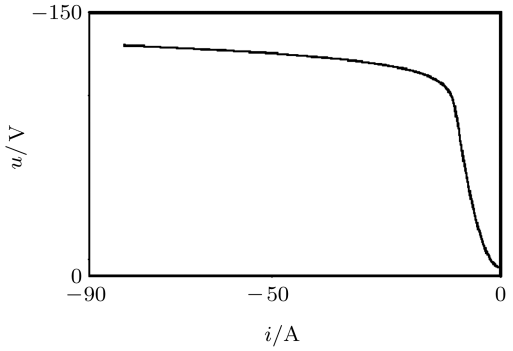


Fig.6. The divergent phase trajectories of the first experiment. The maximum iterative step is 10μs, and the initial condition of disturbance is (0, 5.00000199), in 2.2 ms.

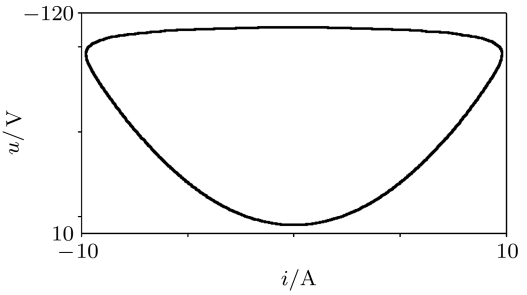


Fig.7. The quadratic-type oscillation phase trajectory of the first experiment. The maximum iterative step is 10 μs, and the initial condition of disturbance is (0, 5.00000198), in 4.6 ms.

Our second experiment shows that the sensitivity of the circuit state is related to the disturbance of the numerical calculation parameters. All the circuit parameters used are the same as above with the initial condition being (0, 5.000001). Choosing the maximum iterative step as 7.5 μs and 7.4 μs, the difference between them is only 0.1 μs. As the initial conditions are very close to the homoclinic phase trajectory, after 2.2 ms, the states of the circuit in Fig.4 changes greatly. Figure 8 shows the quadratic-type oscillation in time domain and the phase trajectory at a step = 7.5 μs. Figure 9 is the figure with divergent state variables in time domain and the phase trajectory at a step = 7.4 μs. The experiment shows that when the initial condition is very close to the homoclinic phase trajectory, a tiny difference in calculating condition would produce great difference in the state of the circuit in Fig.4.

The third experiment shows that when the initial conditions are far away from the homoclinic phase trajectory, the circuit state is not sensitive to all the disturbance parameters. Before simulating calculation we can tell the exact characteristics of the solution of Eq.(19) under the initial conditions. In this circumstance, the solution becomes a deterministic system described by Eq.(19) (omitted, for the result is obvious).

The results of the three experiments confirm three conclusions: (i) When the initial conditions of states are very close to the homoclinic phase trajectory, tiny changes of initial conditions will produce great changes in phase trajectories. (ii) When initial conditions of states are very close to homoclinic phase trajectory, tiny changes in integral step (parameters used in numerical calculation) will produce great changes in phase trajectory. (iii) When initial conditions are far away from the homoclinic phase trajectory, changes of initial conditions or integral step will not produce great changes in phase trajectories.

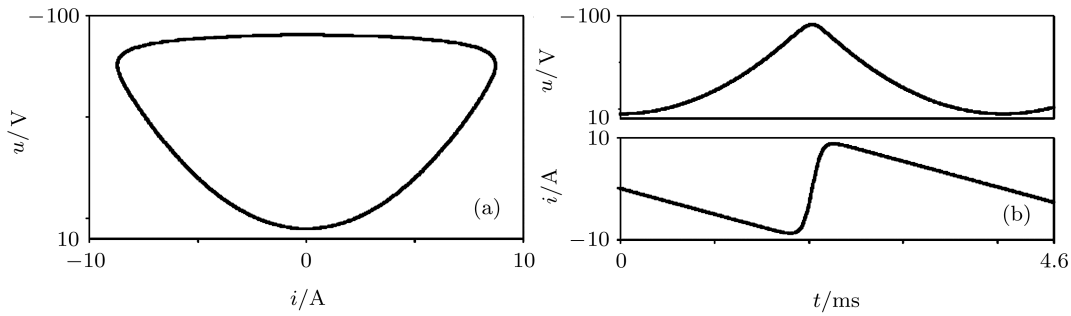


Fig.8. The quadratic-type oscillation phase trajectory and time domain waves graph of the second experiment. The initial condition is (0, 5.000001) and the maximum iterative step is 7.5 μs as disturbance parameter in 4.6 ms, phase trajectory (a)and time domain (b).

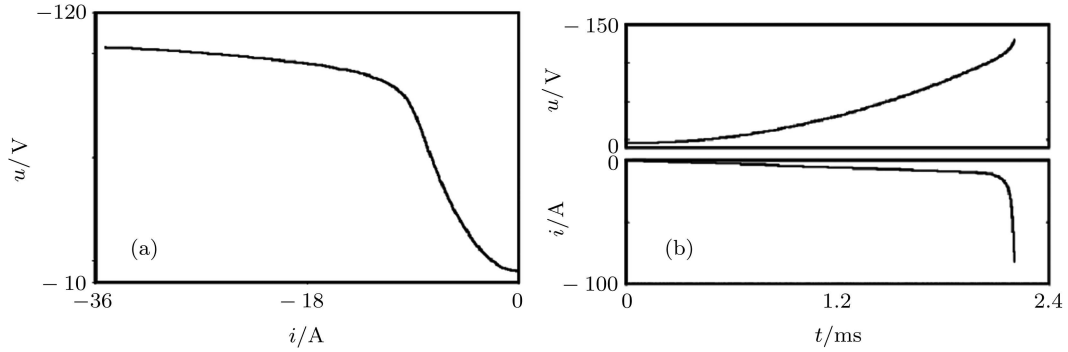


Fig.9. The divergent phase trajectory and time domain waves graph of the second experiment. The initial condition is (0,5.000001) and maximum iterative step is 7.4 μs as disturbance parameter in 2.2 ms, divergent phase trajectory (a) and time domain (b).

5. Conclusion

From the theoretical analysis and the results of the experiments, and by comparing chaos with the nonchaotic random behaviour in a deterministic system, we have found the following characteristics:

1. The nonchaotic random behaviour in a deterministic system is sensitive to the initial conditions as the chaos, even if there is only a tiny difference, the motion of the system will change greatly. A set of points with the initial conditions near the homoclinic phase trajectory will form a random region in the deterministic system. When the initial conditions are far away from the homoclinic phase trajectory, the system will have all the characteristics of a deterministic system.

2. Since the divergent phase trajectories of L^0 and L^+ cannot return to the starting point, it is impossible for them to form the strange attractors. This is the transient random behaviour[8] and very similar to the coin-throwing case in the classical theory of probability.

3. Although the nonchaotic random behaviour in a deterministic system is not rich as compared with the chaotic phenomena, [9,10] it can tell us clearly how the random behaviour can appear in a deterministic system and why sometimes a deterministic system may have random behaviour and sometimes not.

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