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Analysis and implementation of a new hyperchaotic system*

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Based on a modified Lorenz system, a relatively simple four-dimensional continuous autonomous hyperchaotic system is proposed by introducing a state feedback controller. The system consists of four coupled first-order ordinary differential equations with three nonlinear cross-product terms. Some dynamical properties of this hyperchaotic system, including equlibria, stability, Lyapunov exponent spectrum and bifurcation, are analysed in detail. Moreover, an electronic circuit diagram is designed for demonstrating the existence of the hyperchaos, and verifying computer simulation results.

Keywords: hyperchaos, state feedback, dynamical analysis, circuit implementation
PACC: 0545

1. Introduction

In 1963, Lorenz discovered chaos a three-dimensional autonomous system, which produces the well-known Lorenz chaotic attractor.[1] It is due to the first chaotic attractor that the Lorenz system has served as a prototype model for studying the chaotic behaviour for quite a long time. Recently, there has been increasingly aroused the interest in creating the chaos, particularly, since Chen found a new chaotic system called the Chen system from an engineering feedback control (or, chaotification).[2,3] Motivated by the consideration of chaotification, great effort has been made to construct the chaotic systems in the past few years. Typical examples are Lü system,[4] unified system,[5] modified Lorenz system,[6] and so on.[7−15]

In general, a hyperchaotic system is defined as a chaotic system with more than one positive exponents, implying that its dynamics extend in several different directions simultaneously. Therefore, hyperchaotic systems have more complex dynamical behaviours which can be used to improve the security of chaotic communication systems. Therefore, the topic of theoretical design and circuitry realization of various hyperchaotic signals has recently become hotspots in the research area.[16−21]

Very recently, hyperchaos has been found numerically and experimentally by adding a simple state feedback controller,[16−19] or a sinusoidal parameter perturbation controller[20] to the generalized Lorenz system,[22] Chen system[2] or Lü system,[4] or a unified chaotic system.[5]

This paper shows that the modified Lorenz system found in Ref.[6] can be controlled to become hyperchaotic via a very simple state feedback controller. The hyperchaotic nature is then verified not only by calculating its Lyapunov exponent spectrum and investigating its bifurcation routes but also by realizing an electronic circuit. Various attractors and several bifurcation routes evolving from periodic, chaotic and hyperchaotic to hyperchaos behaviour are observed.

2. The new hyperchaotic system and its basic properties

2.1. From chaos to hyperchaos

The well-known Lorenz system[1] is described by

$$\begin{aligned} \dot{x} &= a(y-x), \\ \dot{y} &= cx - y - xz, \\ \dot{z} &= xy - bz. \end{aligned} \tag{1}$$

Adding a cross-product term to the first equation

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of the Lorenz system yields a new modified Lorenz system expressed as[8]

$$\begin{aligned} \dot{x} &= a(y-x)+yz, \\ \dot{y} &= cx-y-xz, \\ \dot{z} &= xy-bz, \end{aligned} \tag{2}$$

where a, b and c are the real constants. When $a=35$, $b=8/3$, and $c=25$, the system has a chaotic attractor. Here, its Lyapunov exponents are $\mathrm{LE}_1=0.7250$, $\mathrm{LE}_2=0$ and $\mathrm{LE}_3=-39.3893$.

Based on system (2), one can construct a new fourth-order hyperchaotic system by introducing a state feedback controller, as follows:

$$\begin{aligned} \dot{x} &= a\,(y-x)+yz, \\ \dot{y} &= cx-y-xz+u, \\ \dot{z} &= xy-bz, \\ \dot{u} &= -kx, \end{aligned} \tag{3}$$

where k is a constant to be determined.

In order to create hyperchaos from system (3), the system should have two positive Lyapunov exponents along with a zero valued exponent and a negative Lyapunov exponent. Our numerical experiments show that system (3) has a hyperchaotic attractor for $a=25$, $b=8/3$, $c=25$ and $k=12$, as depicted in Figs.1(a)–(f).

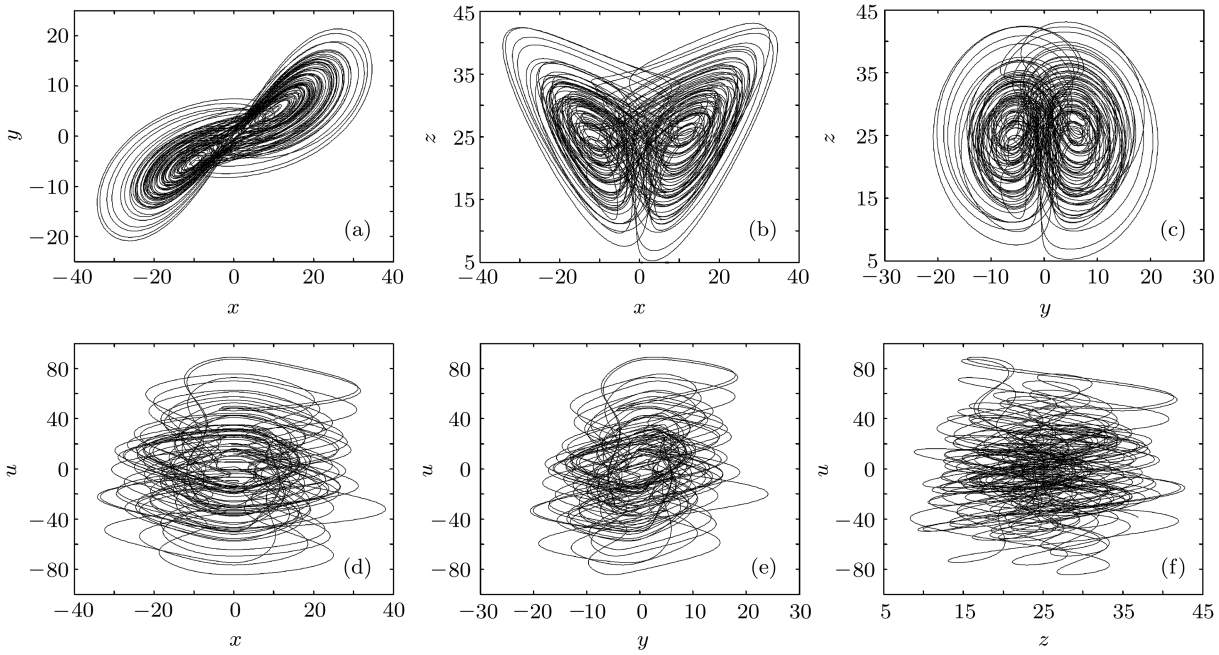


Fig.1. Phase portraits of hyperchaotic system (3) with $a=25$, $b=8/3$, $c=25$ and $k=12$. (a) $x-y$; (b) $x-z$; (c) $y-z$; (d) $x-u$; (e) $y-u$; (f) $z-u$.

2.2. Basic properties of the new hyperchaotic system

The hyperchaotic system (3) has the following basic properties:

(1) Symmetry: It is easy to see that system (3) has a natural symmetry under the following transformation:

$$(x,y,z,u)\to(-x,-y,z,-u)\,. \tag{4}$$

That is, system (3) is symmetrical about the z-axis.

(2) Dissipativity: Obviously, from system (3) one has

$$\nabla V=\frac{\partial\dot{x}}{\partial x}+\frac{\partial\dot{y}}{\partial y}+\frac{\partial\dot{z}}{\partial z}+\frac{\partial\dot{u}}{\partial u}=-(a+b+1). \tag{5}$$

Therefore, when $(a+b+1)>0$, system (3) is dissipative, with an exponential contraction rate being

$$\dot{V}=\mathrm{e}^{-(a+b+1)}. \tag{6}$$

This means that a volume element V_0 is con-

tracted by the flow into a volume element in time t:

$$V(t) = V_0 \mathrm{e}^{-(a+b+1)t}.$$

(3) Equilibrium and stability: The equilibrium of system (3) satisfies the following equations:

$$\begin{aligned} &a(y-x)+yz=0, \quad cx-y-xz+u=0, \\ &xy-bz=0, \quad \text{and} \quad -kx=0. \end{aligned} \tag{7}$$

Obviously, $S_0 = (0,\ 0,\ 0,\ 0)$ is a unique equilibrium.

Linearizing system (3) at equilibrium S_0 provides an eigenvalue $\lambda_1 = -b$ along with the following characteristic equation:

$$\lambda^3 + (a+1)\lambda^2 + (a-ac)\lambda + ak = 0. \tag{8}$$

According to the Routh–Hurwitz condition, the real parts of the roots λ of Eq.(8) are negative if and only if

$$(a+1) > 0, \quad ak > 0, \text{and} \quad (a+1)(a-ac) - ak > 0,$$

namely, if and only if

$$c < 1 - k/(a+1). \tag{9}$$

To make the equilibrium S_0 unstable, thereby yielding a possibility for chaos or hyperchaos to occur, the coefficients of system (3) may be chosen to satisfy the following conditions, which are obtained according to the Routh–Hurwitz criterion:

$$a, b, k > 0, \quad \text{and} \quad c > 1 - k/(a+1). \tag{10}$$

When $a = 25$, $b = 8/3$, $c = 25$ and $k = 12$, the eigenvalues of equilibrium S_0 are -40.8629, 14.3513, 0.5116 and -2.6667. Then S_0 is a two-dimensional unstable saddle point.

3. Bifurcation analysis

The basic dynamics of the new hyperchaotic system (3) can be summarized in the following Lyapunov exponent spectra (to clearly see positive Lyapunov exponents, the vertical axis is limited in a certain range, so we cannot see the fourth curve in Figs.2–5) and bifurcation diagrams.

3.1. $a = 25$, $b = 8/3$ and $c = 25$ being given and k varied

The Lyapunov exponent spectra and the corresponding bifurcation diagram of state variable x versus parameter k are shown in Fig.2, where $a = 25$, $b = 8/3$ and $c = 25$. When $k \in [0, 43.0)$, the system is hyperchaotic with two positive Lyapunov exponents $(\mathrm{LE}_1, \mathrm{LE}_2)$; while for $k \in [43.0, 66.5)$, the system has a single positive Lyapunov exponent LE_1, implying that the system becomes chaotic. As k further increases in the region $k > 66.5$, the system follows a series of quasi-period and period bifurcations and eventually evolves into a line shown in Fig.2(b), implying a period-1 orbit. The periodic nature is characterized by a zero valued exponent along with three negative Lyapunov exponents.

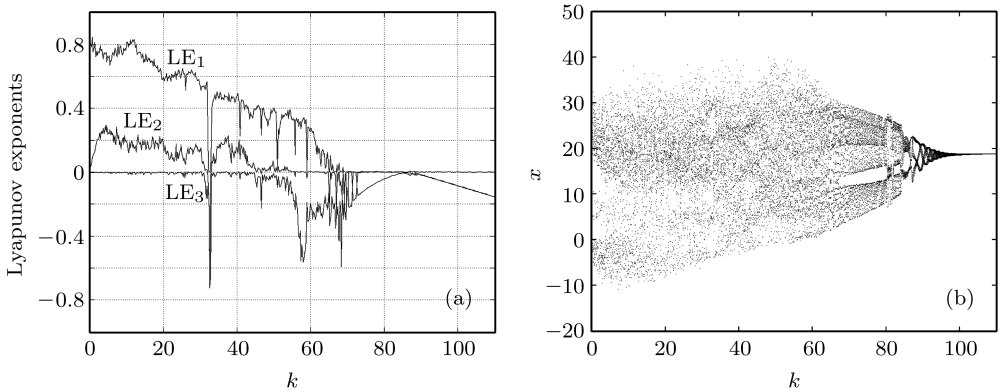


Fig.2. Spectra of Lyapunov exponents and bifurcation diagram of system (3) with $a = 25$, $b = 8/3$, $c = 25$, $k \in [0, 110]$. (a) Spectra of Lyapunov exponents; (b) bifurcation diagram of x with respect to k.

3.2. $b = 8/3, c = 25$ and $k = 12$ being given and a varied

Now, $b = 8/3$, $c = 25$ and $k = 12$ are given and a is varied. The corresponding spectra of Lyapunov exponents and the bifurcation diagrams versus parameter a are shown in Figs.3(a) and 3(b), respectively.

In the region $a \in [5.0, 9.4)$, system (3) exhibits a periodic orbit with no positive Lyapunov exponent. When $a \in [9.3, 18.0)$, the system becomes hyperchaotic with two positive Lyapunov exponents (LE_1, LE_2). When a increases further in the region $a \in [18.0, 19.6)$, the system undergoes a periodic window and then becomes hyperchaotic again at $a = 19.6$. The hyperchaotic nature is also characterized by two positive Lyapunov exponents LE_1 and LE_2. With a increasing in the range $a \in (35.0, 54.0]$, the system enters into a chaotic region with a single positive Lyapunov exponent as shown in Fig.3(a).

As a increases in the range of $a \in (55.7, 70.0)$, the system exhibits a sequence of transitions between period-3 and period-1 and finally becomes a period-1 orbit at $a = 63.1$. In contrast with standard period-doubling bifurcations, the distinctive feature of the bifurcation is a jump between period-3 and period-1.

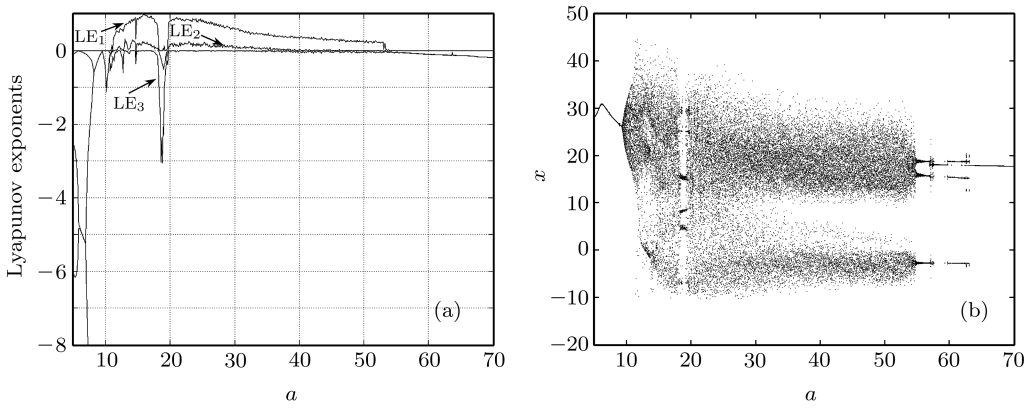


Fig.3. Spectra of Lyapunov exponents and bifurcation diagrams of system (3) with $b = 8/3$, $c = 25$, $k = 12$, $a \in [5, 70]$.(a) Spectra of Lyapunov exponents; (b) bifurcation diagram of x.

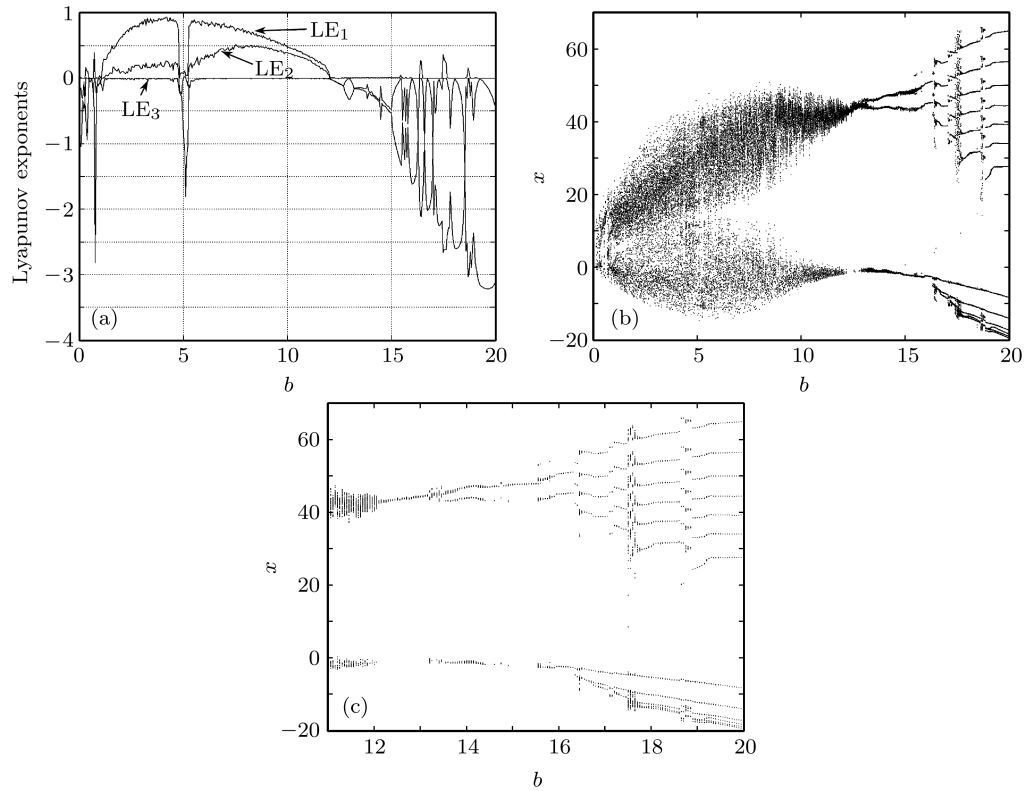


Fig.4. Spectra of Lyapunov exponents and bifurcation diagrams of system (3) with $a = 25$, $c = 8/3$, $k = 12$, $b \in [0, 20]$: (a) Spectra of Lyapunov exponents; (b) and (c) bifurcation diagrams of x.

3.3. $a = 25, c = 8/3$ and $k = 12$ being given and b varied

Figure 4 shows the spectra of Lyapunov exponents and the corresponding bifurcation diagrams versus the parameter b for the parameter region [0, 20], from which very rich dynamical behaviours can be clearly observed.

From Fig.4(a) it can be seen that the range $0 < b < 0.97$ is periodic, in which there appear two small chaotic windows. As b increases in the range, $0.97 \leq b < 12.26$, the system enters into a hyperchaotic region first, then a chaotic region at $b = 4.82$, and later a hyperchaotic region again at $b = 5.43$.

When b increases further in the region, $12.6 \leq b \leq 20$, the complex dynamical behaviours of this system are observed. First, in the region $12.6 \leq b \leq 15.3$, both period-1 and period-3 obits co-exist and appear alternately as b increases gradually as shown in Fig.4(c). Here, compared with standard period-doubling bifurcation and period-adding bifurcation, the dynamical behaviours are also featured by a jump, and period-1 and period-3 appearing alternatively.

Next, the period-3 orbit which appears at $b = 15.5$ continues until $b = 16.3$, where this system suddenly switches over to a period-5 orbit without traversing any bifurcation region, and a very small period-3 window $16.32 \leq b \leq 16.34$ is born. After this window, the system enters into a small chaotic region, which continues until $b = 16.42$, where a period-7 orbit is born via a saddle-node bifurcation. Arising from this bifurcation, a series of odd-periodic windows of increasing periodicity (7, 9, 11, 13) occurs and finally leads to a period-13 at $b = 18.9$. In fact, the transition observed is $7 \to 9 \to$ chaos $\to 11 \to$ chaos $\to 13$.

3.4. $a = 25, b = 8/3$ and $k = 12$ being given and c varied

The Lyapunov exponent spectra and bifurcation diagrams shown in Fig.5 reveal that periodic orbit, hyperchaotic orbit, chaotic and quasi-periodic orbit appear alternately with c increasing gradually from 0 to 180. It is noticed that with c increasing in the range $180 < c \leq 200$ the system eventually evolves into a periodic obit.

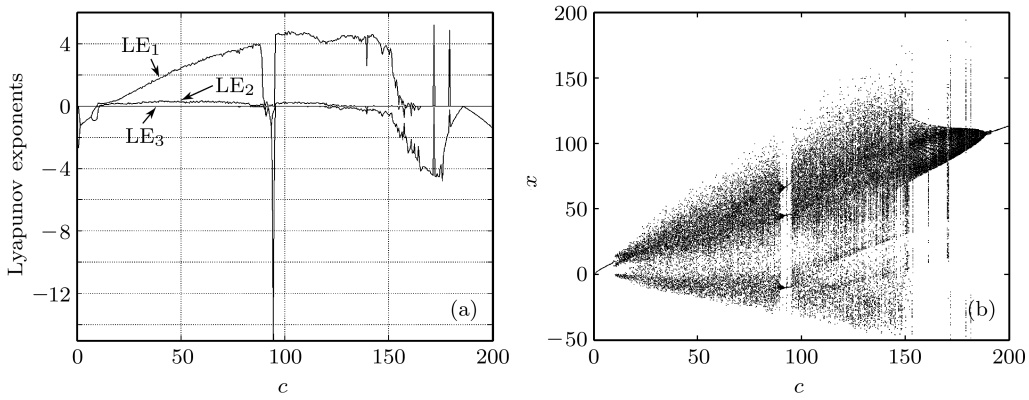


Fig.5. Spectra of Lyapunov exponents and bifurcation diagram of system (3) with $a = 25$, $b = 8/3$, $k = 12$, $c \in [0, 200]$. (a) Spectra of Lyapunov exponents; (b) bifurcation diagram of x.

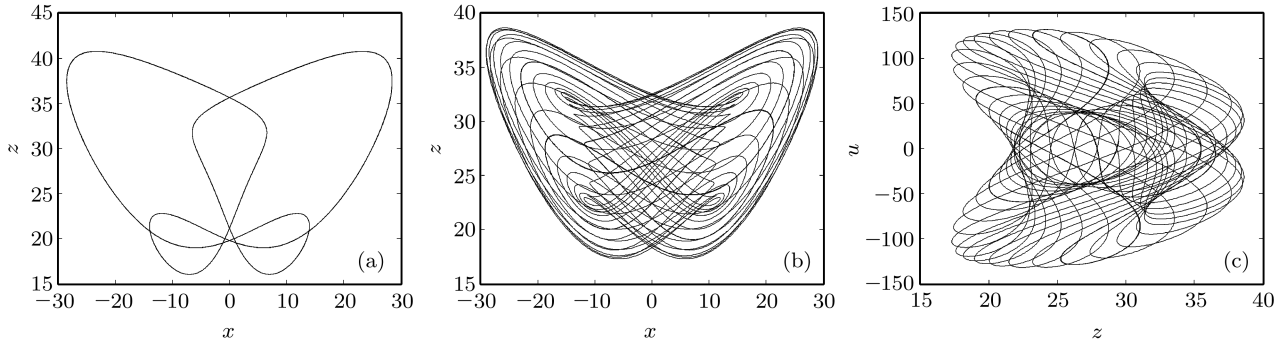


Fig.6. Periodic and quasi-periodic orbits: (a) periodic orbit with four Lyapunov exponents $LE_1 = 0$, $LE_2 = -0.7554$, $LE_3 = -0.7554$ and $LE_4 = -35.1561$, where $a = 33$, $b = 8/3$, $k = 67.5$ and $c = 25$; (b) and (c) quasi-periodic orbits with four Lyapunov exponents $LE_1 = LE_2 = 0$, $LE_3 = -0.3744$ and $LE_4 = -25.6932$, where $a = 22.4$,$b = 8/3$, $k = 67.5$ and $c = 25$.

In fact, if two parameters are fixed and the other two parameters are varied, complex dynamical behaviours can also be observed. For example, as shown in Fig.6, some periodic and quasi-periodic orbits are obtained from simulations.

4. Circuit realization of the new hyperchaotic system

In this section, an analogue circuit is designed to physically realize the new hyperchaotic system (3). The designed circuitry realizing Eq.(3) is shown in Fig.7. It consists of four channels which realize the four state variables x, y, z and u, respectively. Operational amplifiers and the associated circuitry jointly perform the operations of addition, subtraction and integration. Analogue multipliers are employed to perform the required multiplication operations.

The observations measured in the experimental setup are shown in Figs.8(a)–(i). The adjustable parameters a, b, c and k in system (3) can be implemented by adjusting the linear resistors R_{x1}–R_{x2}, R_{y1}, R_{z1} and R_u respectively as shown in Fig.7.

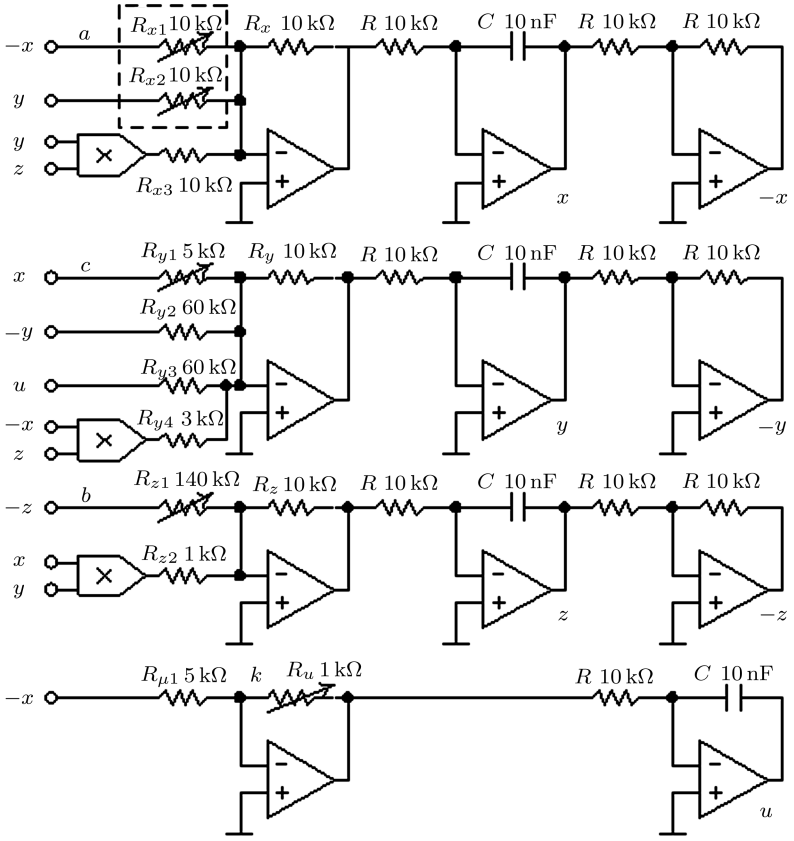


Fig.7. Circuit diagram for realizing the new hyperchaotic system (3). Operational amplifiers are LF347. Multiplier is AD633.

5. Conclusions and discussion

This paper has introduced a novel hyperchaotic system. Some basic dynamical behaviours are further explored by calculating its Lyapunov exponent spectra and bifurcation diagrams. Furthermore, this system has also been physically realized by using an analogue electronic circuit, showing a good agreement between computer simulations and experimental results. The attractive property of the proposed system is the existence of hyperchaotic attractor over a range of parameter values, which might be useful for applications in chaos communications and various chaos-based information systems.

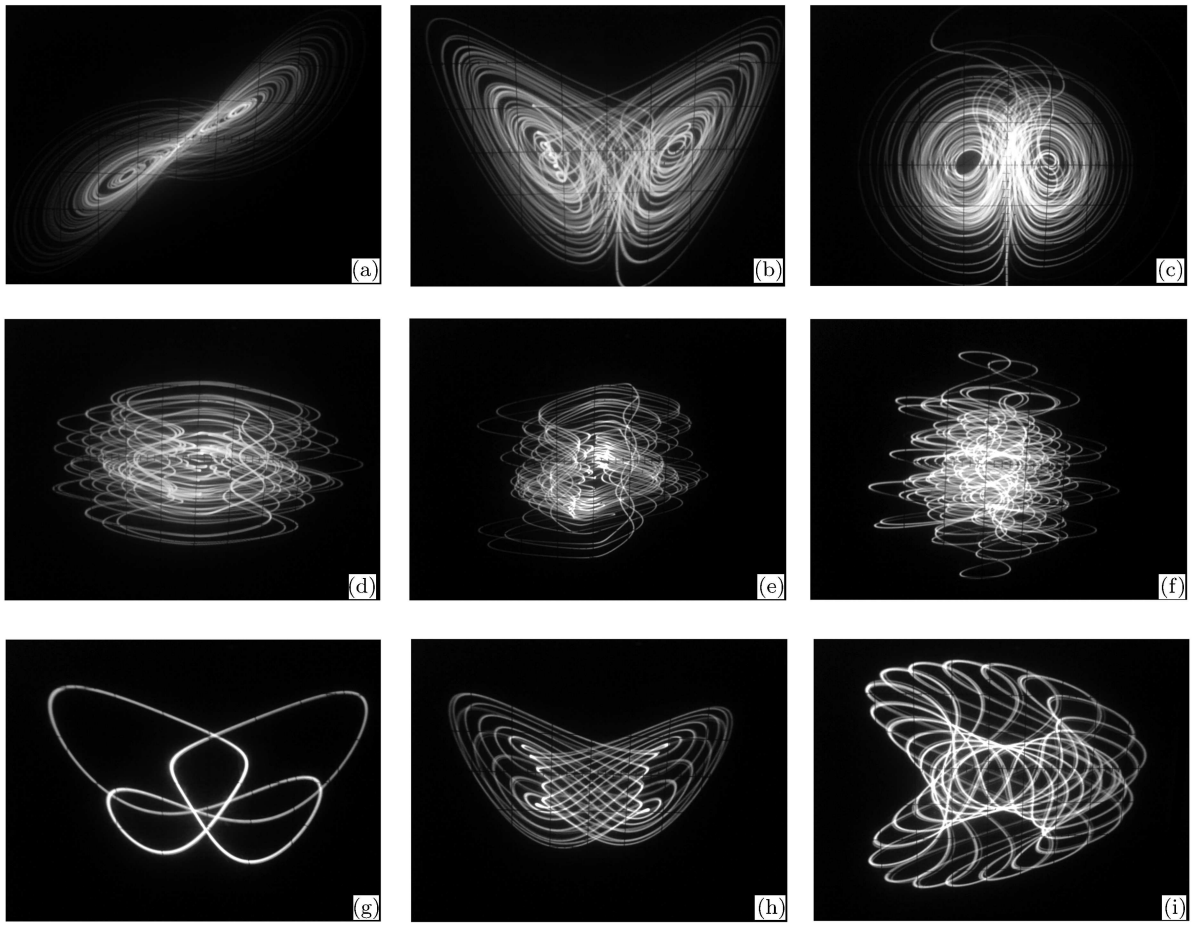


Fig.8. Experimental observations of the new hyperchaotic system (3): (a) x-y plane, (b) x-z plane, (c) y-z plane; (d) x-u plane; (e) y-u plane; (f)z-u plane; (g) periodic orbit on y-z plane; (h) and (i) quasi periodic orbits in x-z plane and z-u respectively. The power supplies of all active devices are at ± 14 volts of output voltage. The scales of both vertical and horizontal axes are 3V / div.

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