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Impulsive control of chaotic systems with exogenous perturbations*

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The impulsive control of chaotic systems, which are subjected to unbounded exogenous perturbations, is considered. By using the theory of impulsive differential equation together with the fuzzy control technique, the authors propose an impulsive robust chaos controlling criterion expressed as linear matrix inequalities (LMIs). Based on the proposed control criterion, the procedure for designing impulsive controllers of common (perturbed) chaotic systems is provided. Finally, a numerical example is given to demonstrate the obtained theoretical result.

Keywords: impulsive control law, chaos controlling, linear matrix inequality
PACC: 0545

1. Introduction

From the end of the last century on, chaos control, one of the most interesting and challenging tasks in nonlinear science, has attracted more and more attention of both engineers and scientists. Ott *et al*[1] have proposed a chaos controlling method, called OGY method, which has sparked a great number of schemes on controlling chaos.[2−15]

Impulsive control has arisen from a variety of applications[16,17] including the successful application to various chaotic systems.[18−21] In Ref.[21] is discussed the impulsively asymptotical stabilization using the Lyapunov stability theory, and Refs.[18,21] proposed impulsive control criteria by using the comparison theorem. However, all the above-mentioned control laws are difficult to design[18,19] or inapplicable to the common chaotic systems.[21] On the other hand, the Takagi–Sugeno (T–S) fuzzy model can represent a highly nonlinear dynamic system by combining its simple local linear dynamic systems with their linguistic descriptions,[22] thus enabling the well-established linear system theory to be applied to the study of the properties of those systems. Since chaotic systems are typically nonlinear ones, a vast literature dealing with them is based on the fuzzy T–S model.[23,24] Therefore, this paper is devoted to discussing the control of chaotic systems with exogenous perturbations based on the T–S fuzzy model via impulsive control.

The remainder of this paper is organized as follows. On the basis of the T–S fuzzy model of chaotic systems with exogenous perturbations, an impulsive control criterion is proposed which is expressed as linear matrix inequalities (LMIs), based on which a general procedure is developed to design impulsive control laws for such chaotic systems. It is noticeable that this paper concentrates on the exponential stability instead of asymptotical stability of the controlled systems as reported in a vast literature.

Notation $A > 0 (< 0)$ means A is a symmetrical positive (negative) definite matrix, and $A > B (A < B)$ refers to $A - B > 0 (A - B < 0)$. A^{T} represents the transpose of A, and $\|x(t)\|$ denotes the Euclidean norm of vector $x(t)$. All the matrices each have an appropriate dimension, unless otherwise specified.

2. Problem formulation

Consider an autonomous chaotic system described as follows:

$$\dot{x}(t) = f(x(t)), \tag{1}$$

where $x(t) \in R^n$ is the state variable, and $f \in C[R^n, R^n]$. Using the technique introduced in

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Ref.[22], we can construct a fuzzy model for system (1) as follows:

Rule i IF z_1 is μ_{i1}, ..., and z_p is μ_{ip}, THEN $\dot{x}(t) = A_i x(t)$, in which z_1, ..., z_p are the premise variables, $\mu_{i1}, \ldots, \mu_{ip}$ are fuzzy sets, $A_i \in R^{n\times n}$ is a constant matrix. Then the overall fuzzy system is represented as

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(t) A_i x(t), \tag{2}$$

where r is the number of fuzzy rules, $h_i(t) = w_i(t)/\sum_{i=1}^{r} w_i(t)$, $w_i(t) = \prod_{l=1}^{p} \mu_{il}(z_l(t))$, and $\mu_{il}(z_l(t))$ is the grade of membership of $z_l(t)$ in μ_{il}. Of course, we have $h_i(t) \geq 0$ and $\sum_{i=1}^{r} h_i(t) = 1$. System (2) can exactly represent system (1) locally.

If system (1) is subjected to an exogenous perturbation $g(x(t), t) \in R^n$, then the perturbed chaotic system can be represented as

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(t) A_i x(t) + g(x(t), t), \tag{3}$$

where $g(x(t), t)$ satisfies $\|g(x(t), t)\| \leq e \|x(t)\|$, with e being a positive scalar. The aim of this paper is to design an impulsive control law to stabilize the perturbed system.

For system (3), an impulsive control law is defined as a series: $\{t_k, C_k x(t_k)\}$, $k \in N$, N is a set of natural numbers, $0 = t_0 < t_1 < t_2 < \cdots < t_k < t_{k+1} < \cdots$, $t_k \to \infty$ as $k \to \infty$, C_k is a constant matrix to be determined. Then we obtain the following impulsive differential system:

$$\begin{aligned} &\dot{x}(t) = \sum_{i=1}^{r} h_i(t) A_i x(t) + g(x(t), t), && t \neq t_k, \\ &\Delta x(t) = C_k x(t), && t = t_k, \end{aligned} \tag{4}$$

where $\Delta x(t_k) = x(t_k^+) - x(t_k)$, $x(t_k^+) = \lim_{h\to 0^+} x(t_k + h)$, and $k \in N$.

The following definition and lemma are needed.

Definition 1[25] Let $V : R_+ \to R_+$, where R_+ is a set of all the nonnegative real numbers. The upper right-hand derivative of $V(x(t))$ is

$$D^+V(x(t)) = \lim_{s\to 0^+} \sup \frac{1}{s} \left[V(x(t+s)) - V(x(t))\right].$$

Lemma 1[26] Assume $x, y \in R^n$. For any scalar $\varepsilon > 0$, the following inequality holds:

$$2x^{\mathrm{T}} y \leq \varepsilon x^{\mathrm{T}} x + \frac{1}{\varepsilon} y^{\mathrm{T}} y.$$

3. Main result

In this section, we propose a stability criterion for system (4), and then provide a procedure for designing an impulsive controller for system (3) according to the proposed condition.

Theorem 1 Given $\{t_k\}, k \in N$. If there exist an arbitrary matrix $C_k, k \in N$, and scalars $\alpha > 0$, $\beta > 0$, $\varepsilon > 0$, $\gamma > 0$, $b_k \geq 0$, such that $\prod_{t_k<t} b_k \leq \exp(-\beta t)$, $\forall t \geq 0$ and the following inequalities (5), (6), and (7) hold, then system (4) is exponentially stable:

$$\theta = \frac{\alpha}{\gamma} < \beta, \tag{5}$$

$$\begin{bmatrix} -b_k I & I + C_k^{\mathrm{T}} \\ I + C_k & -I \end{bmatrix} \leq 0, k \in N, \tag{6}$$

$$\begin{bmatrix} \gamma(A_i + A_i^{\mathrm{T}}) + (\varepsilon - \alpha) I & \gamma e I \\ \gamma e I & -\varepsilon I \end{bmatrix} < 0. \quad i = 1, 2, \ldots, r. \tag{7}$$

Proof Choose the following Lyapunov function candidate:

$$V(x(t)) = \gamma x^{\mathrm{T}}(t) x(t).$$

First, assume that $t \in (t_{k-1}, t_k]$, $k \in N$, and we will have

$$\begin{aligned} D^+V(x(t)) &= 2\gamma x^{\mathrm{T}}(t)\dot{x}(t) \\ &= 2\gamma x^{\mathrm{T}}(t) \left(\sum_{i=1}^{r} h_i(t) A_i x(t) + g(x(t), t)\right) \\ &= \sum_{i=1}^{r} h_i(t) x^{\mathrm{T}}(t) \gamma (A_i + A_i^{\mathrm{T}}) x(t) + 2\gamma x^{\mathrm{T}}(t) g(x(t), t). \end{aligned} \tag{8}$$

Since $\varepsilon > 0$ and $\gamma > 0$, $\varepsilon/\gamma > 0$. The following inequality is obtained from Lemma 1:

$$\begin{aligned} 2\gamma x^{\mathrm{T}}(t)g(x(t),t) &\leq \gamma\left(\frac{\varepsilon}{\gamma}x^{\mathrm{T}}(t)x(t)+\frac{\gamma}{\varepsilon}g^{\mathrm{T}}(x(t),t)g(x(t),t)\right) \\ &= \varepsilon x^{\mathrm{T}}(t)x(t)+\frac{\gamma^2}{\varepsilon}g^{\mathrm{T}}(x(t),t)g(x(t),t) \\ &= \varepsilon x^{\mathrm{T}}(t)x(t)+\frac{\gamma^2}{\varepsilon}\left\|g(x(t),t)\right\|^2 \\ &\leq \varepsilon x^{\mathrm{T}}(t)x(t)+\frac{\gamma^2 e^2}{\varepsilon}\left\|x(t)\right\|^2 \\ &= \varepsilon x^{\mathrm{T}}(t)x(t)+\frac{\gamma^2 e^2}{\varepsilon}x^{\mathrm{T}}(t)x(t). \end{aligned} \tag{9}$$

Combining Eq.(8), expression (9) and the fact $\sum_{i=1}^{r} h_i(t) = 1$, we have

$$D^+V(x(t)) \leq \sum_{i=1}^{r} h_i(t)x^{\mathrm{T}}(t)\left(\gamma(A_i+A_i^{\mathrm{T}})+\left(\varepsilon+\frac{\gamma^2e^2}{\varepsilon}\right)I\right)x(t). \tag{10}$$

According to the Schur complement, inequality (7) is equivalent to

$$\gamma(A_i+A_i^{\mathrm{T}})+\left(\varepsilon-\alpha+\frac{\gamma^2e^2}{\varepsilon}\right)I<0, \quad i=1,2,\ldots,r,$$

i.e.

$$\gamma(A_i+A_i^{\mathrm{T}})+\left(\varepsilon+\frac{\gamma^2e^2}{\varepsilon}\right)I<\alpha I, \quad i=1,2,\ldots,r\,. \tag{11}$$

Based on inequalities (10) and (11), one can easily obtain

$$D^+V(x(t)) < \alpha x^{\mathrm{T}}(t)x(t) = \frac{\alpha}{\gamma}V(x(t)) = \theta V(x(t)), \quad t\in(t_{k-1}, \quad t_k]\,, \quad k\in N. \tag{12}$$

Again, by the Schur complement, it follows from (6) that $\left(I+C_k^{\mathrm{T}}\right)(I+C_k)-b_kI\leq 0$ which implies

$$\left(I+C_k^{\mathrm{T}}\right)(I+C_k)\leq b_kI. \tag{13}$$

So we obtain

$$V(x(t_k^+)) = \gamma x^{\mathrm{T}}(t)\left(I+C_k^{\mathrm{T}}\right)(I+C_k)\,x(t) \leq \gamma x^{\mathrm{T}}(t)b_kIx(t) = b_kV(x(t)). \tag{14}$$

When $t\in(t_0,t_1]$, given $x(0)=x_0$, from inequality (12), one has $V(x(t)) < V(x_0)\exp(\theta t)$. Then, it follows from inequality (14) that $V(x(t_1^+)) \leq b_1V(x(t_1)) < b_1V(x_0)\exp(\theta t_1)$. In a similar way, we can acquire

$$V(x(t)) < b_1V(x_0)\exp(\theta t), t\in(t_1,t_2]\,, \quad \text{and} \quad V(x(t_2^+)) \leq b_2V(x(t_2)) < b_1b_2V(x_0)\exp(\theta t_2).$$

Repeating this process, we obtain the following result finally:

$$V(x(t)) \leq \prod_{t_k<t} b_kV(x_0)\exp(\theta t), \quad t\geq 0.$$

Since $\prod_{t_k<t} b_k \leq \exp(-\beta t), \forall t\geq 0$, the following expression can be directly obtained:

$$\gamma\left\|x(t)\right\|^2 = \gamma x^{\mathrm{T}}(t)x(t) = V(x(t)) < V(x_0)\exp(-(\beta-\theta)t), \quad t\geq 0. \tag{15}$$

Arbitrarily choose $\delta>0$ and let $\|x_0\|<\delta$, then we will have

$$\gamma\left\|x(t)\right\|^2 < \gamma x_0^{\mathrm{T}}x_0\exp(-(\beta-\theta)t) \leq \gamma\delta^2\exp(-(\beta-\theta)t).$$

So

$$\|x(t)\| < \delta \exp\left(\frac{-(\beta-\theta)t}{2}\right). \tag{16}$$

$\beta - \theta > 0$ holds because of inequality (5), which together with inequality (16) implies that system (4) is exponentially stable. The proof ends here.

Now, we provide the following procedures to design an impulsive controller for a perturbed chaotic system (3):

Step 1 According to the given system (1), determine the system parameters $A_i (i = 1, 2, \ldots, r)$ and $b_j (j = 1, 2, \ldots, n)$. Determine the series of times $t_k, k \in N$ and define $d_k = t_{k+1} - t_k$, $k \in N$.

Step 2 Use inequality (7) to compute α and γ, then choose an arbitrary scalar β such that $\dfrac{\alpha}{\gamma} < \beta$.

Step 3 Determine $b_k, k \in N$ such that $b_1 \leq \exp(-\beta(t_1 + d_1))$, $b_k \leq \exp(-\beta d_k)$, and $k \in N \backslash \{1\}$. These scalars should satisfy the condition $\prod_{t_k<t} b_k \leq \exp(-\beta t), \forall t \geq 0$.

Step 4 Utilize inequality (6) to compute $C_k, k \in N$. Thus, the final impulsive control law is given by $\{(t_k, C_k), k \in N\}$.

4. Numerical simulation

The well-known Lorenz chaotic system, which has been extensively discussed,[16,18,19,21,27–30] is described as follows:

$$\begin{aligned} \dot{x}_1 &= a(x_2 - x_1), \\ \dot{x}_2 &= cx_1 - x_2 - x_1 x_3, \\ \dot{x}_3 &= x_1 x_2 - bx_3, \end{aligned} \tag{17}$$

where a, b, c are all positive parameters and when $a = 10$, $b = 8/3$, and $c = 28$, system (17) has a chaotic attractor. According to Ref.[21], we can describe system (17) as the following fuzzy T–S model restricting $x_1 \in [-d, d]$, where d is a positive scalar and taken as 100 in this paper.

Rule $i (i = 1, 2)$: IF $x_1(t)$ is M_i, THEN $\dot{x}(t) = A_i x(t)$, where $x(t) = [x_1(t), \ x_2(t), \ x_3(t)]^{\mathrm{T}}$, $M_1(x_1(t)) = \dfrac{1}{2}\left(1 + \dfrac{x_1(t)}{b}\right)$, $M_2(x_1(t)) = \dfrac{1}{2}\left(1 - \dfrac{x_1(t)}{b}\right)$ and

$$A_1 = \begin{bmatrix} -a & a & 0 \\ c & -1 & -d \\ 0 & d & -b \end{bmatrix}, \quad A_2 = \begin{bmatrix} -a & a & 0 \\ c & -1 & d \\ 0 & -d & -b \end{bmatrix}.$$

Using fuzzy 'blending', the overall fuzzy system is represented as

$$\dot{x}(t) = \sum_{i=1}^{2} M_i(t) A_i x(t). \tag{18}$$

System (18) can exactly represent system (17) if $x_1 \in [-d, d]$. It is assumed that system (18) is subjected to an exogenous perturbation $g(x(t), t) = [x_1(t)\sin(t), \ x_2(t)\cos(t), \ x_3(t)\sin(t)\cos(t)]^{\mathrm{T}}$, so the perturbed Lorenz system is represented as

$$\dot{x}(t) = \sum_{i=1}^{2} M_i(t) A_i x(t) + g(x(t), t), \tag{19}$$

whose phase plot is shown in Fig1.

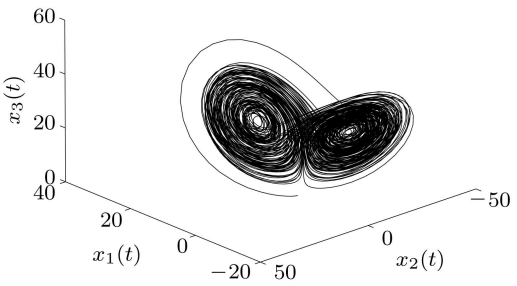


Fig.1. Phase plot of perturbed Lorenz system.

Now, we will design an impulsive control law $\{t_k, C_k x(t_k)\}$ for system (19). Let $t_k = 0.01k$. According to Theorem 1, it is easy to obtain $b_1 = 0.0600$, and $b_k = 0.2449$, $k = 2, 3, \ldots$. According to inequality (6), the following cases each satisfy the conditions in Theorem 1:

1) $C_1 = -0.7551I$, $C_k = -1.4949I$, $k = 2, 3, \ldots$;

2) $C_1 = -1.2449I$, $C_k = -0.5051I$, $k = 2, 3, \ldots$;

3) $C_1 = -1.2449I$, $C_k = -1.4949I$, $k = 2, 3, \ldots$;

4) $C_1 = -0.7551I$, $C_k = -0.5051I$, $k = 2, 3, \ldots$. The simulation results of cases 1) and 2), which run under initial condition $x_0(t) = [-3, 4, -2]^{\mathrm{T}}$, are shown in Figs.2 and 3 respectively.

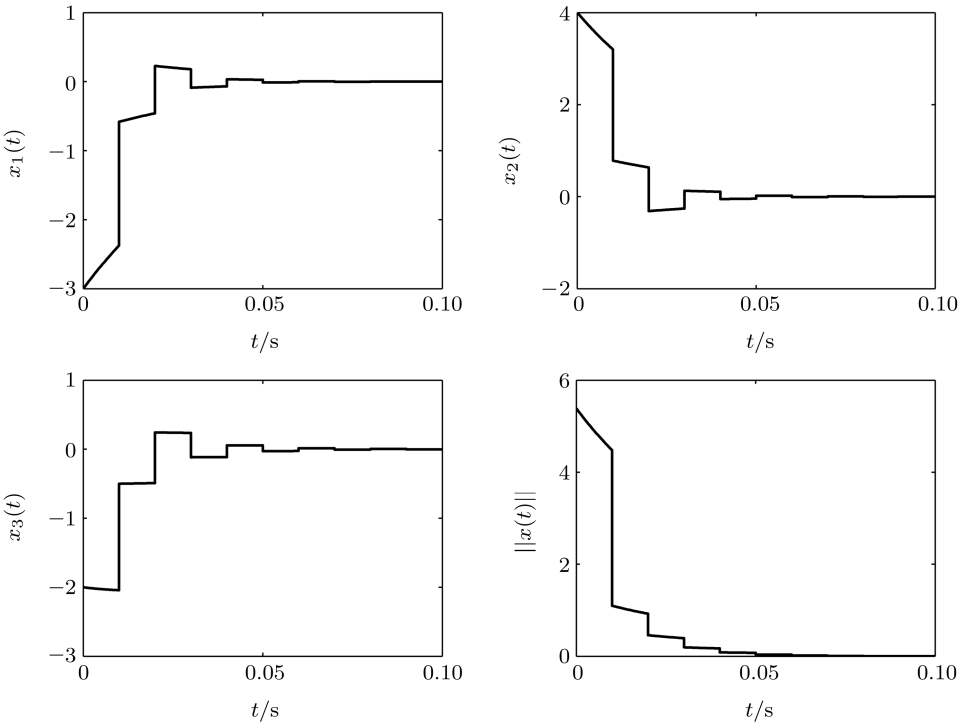


Fig.2. $C_1 = -0.7551I,\ \ C_k = -1.4949I,\ \ k = 2, 3, \ldots$.

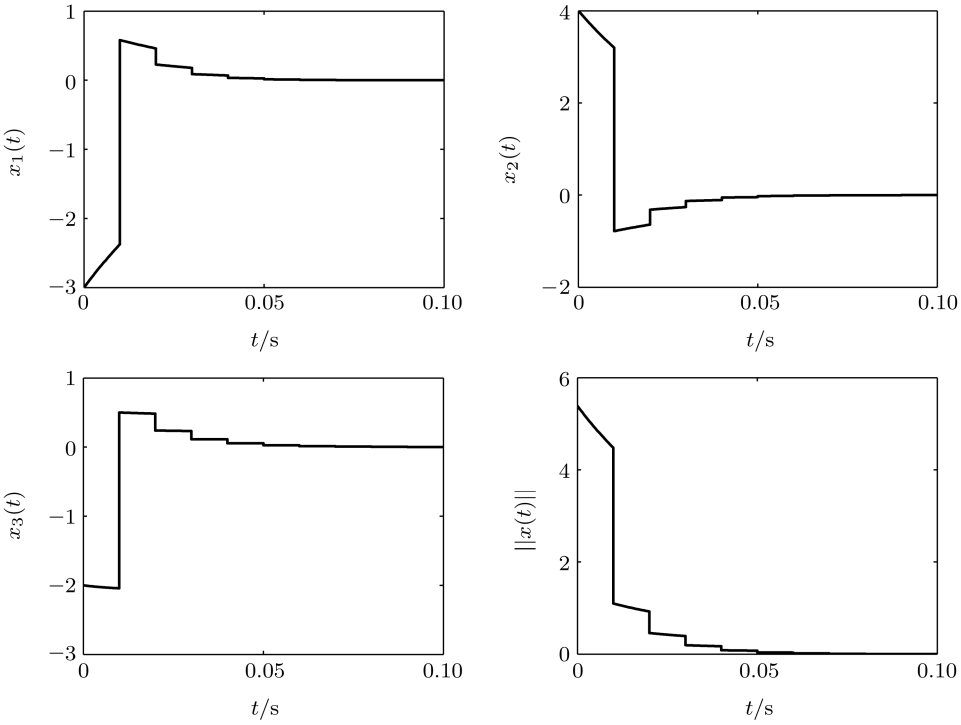


Fig.3. $C_1 = -1.2449I,\ \ C_k = -0.5051I,\ \ k = 2, 3, \ldots$.

From these figures one can see that the designed impulsive control law $\{t_k, C_k x(t_k)\}$, $k \in N$ is effective.

5. Conclusions

We have studied the impulsive control of a chaotic system with an exogenous perturbation based on its fuzzy T–S model. A criterion for designing the impulsive control law is proposed, and the procedure for designing the impulsive controller is formulated. Finally, a numerical example is provided to demonstrate the obtained theoretical result. It is noticeable that this criterion is applicable to common (perturbed) chaotic systems and easy to numerically simulate. We will extend the obtained result to the synchronization of chaotic systems in the future.

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