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The separability of two-mode Gaussian state under amplification and symmetric damping*

Chen Xiao-Yu (陈小余)[a)†], Jiang Li-Zhen (蒋丽珍)[a)], and Wu Liang-Neng (邬良能)[b)]

[a)] *College of Information and Electronic Engineering, Zhejiang Gongshang University, Hangzhou* 310018, *China*
[b)] *College of Science, China Institute of Metrology, Hangzhou* 310034, *China*

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The performances of a two-mode Gaussian state under parametric amplification, symmetric amplitude damping and thermal noise are studied. The time-dependent complex correlation matrix of the state in evolution is given. The separability of the final two-mode Gaussian state is examined under symmetric amplification and asymmetric amplification separately.

Keywords: parametric amplifier, amplitude damping, characteristic function, separability
PACC: 0367, 4250, 0365

1. Introduction

Quantum continuous variable (CV) systems are playing an important role in quantum information theory and communication,[1,2] as evidenced by the implementations of CV teleportation schemes.[3,4] In practice, the decoherence is unavoidable. The decoherence may comprise amplitude damping,[5] phase damping[6] and thermal noise of the environment. Parametric amplification may be used to compensate the amplitude loss. We consider a two-mode Gaussian system under parametric amplification, symmetric amplitude damping and thermal noise in this paper.

For a continuous variable system under simultaneous amplitude damping, parametric amplification and thermal noise, the density matrix obeys the following master equation:[7−9]

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\frac{\mathrm{i}}{\hbar}[H,\rho] + \mathcal{L}\rho, \tag{1}$$

with the quadratic Hamiltonian

$$H = \hbar \sum_{j,k=1}^{s} \frac{\mathrm{i}}{2}(\eta_{jk}\hat{a}_j^\dagger \hat{a}_k^\dagger - \eta_{jk}^* \hat{a}_j \hat{a}_k),$$

where η is a complex symmetric matrix and represents the parametric amplification, $\hat{a}_j$ and $\hat{a}_j^\dagger$ are the annihilation and creation operators of jth mode respectively. $\mathcal{L}$ describes amplitude damping under thermal noise; $\mathcal{L}\rho = \sum_{j=1}^{s} \frac{\Gamma_j}{2}\{(\bar{n}_j+1)L[\hat{a}_j]\rho + \bar{n}_j L[\hat{a}_j^\dagger]\rho - w_j^* M[\hat{a}_j]\rho - w_j M[\hat{a}_j^\dagger]\rho\}$, where Γ_j is the damping coefficient of jth mode. The Lindblad super-operators are defined as $L[\hat{o}]\rho \equiv 2\hat{o}\rho\hat{o}^\dagger - \hat{o}^\dagger\hat{o}\rho - \rho\hat{o}^\dagger\hat{o}$, and $M[\hat{o}]\rho \equiv 2\hat{o}\rho\hat{o} - \hat{o}\hat{o}\rho - \rho\hat{o}\hat{o}$. The requirement for positivity of the density operator imposes the constraint $|w_j|^2 \leq \bar{n}_j(\bar{n}_j+1)$. In thermal equilibrium, i.e. $w_j = 0, \bar{n}_j$ is equal to the environmental average thermal photon number of jth mode.[10] If $w_j \neq 0$, then the thermal bath j is said to be 'squeezed'.

The evolution equation of the density matrix has been transformed into the diffusion equation of its characteristic function.[11] For any initial state, the time-dependent characteristic function can be worked out completely when all modes undergo the same amount of damping, that is, $\Gamma_1 = \Gamma_2 = \cdots = \Gamma_s = \Gamma$.[11] We should mention that the solution of the master equation of non-degenerate optical parametric amplification has been given.[12]

2. Symmetric damping of two-mode Gaussian system

In this section, we will consider the symmetric damping of the two modes (modes a and b): that is, $\Gamma_a = \Gamma_b = \Gamma$. The canonical operators of the two-mode system are $\hat{X}_a, \hat{P}_a$, and $\hat{X}_b, \hat{P}_b$. Letting

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†Corresponding author. E-mail: chenxy@cjlu.edu.cn

operator vector $R = (\hat{X}_a, \hat{P}_a, \hat{X}_b, \hat{P}_b)$ and real vector $z = (x_a, y_a, x_b, y_b)$, the characteristic function of a state ρ is defined as $\chi(z) = \mathrm{Tr}[\rho \exp(\mathrm{i}Rz^{\mathrm{T}})]$. A state ρ is called Gaussian if its characteristic function has the form

$$\chi(z) = \exp\left[\mathrm{i}zm_{\mathrm{re}}^{\mathrm{T}} - \frac{1}{2}z\gamma_{\mathrm{re}}z^{\mathrm{T}}\right], \tag{2}$$

where m_{re} is a vector and γ_{re} is a symmetric matrix. It can be shown that m_{re} is indeed the mean and γ_{re} is the correlation matrix (CM). Since we deal with annihilation and creation operators in a master equation, a characteristic function with complex parameters is more convenient to calculate. The characteristic function with complex parameters is defined as $\chi(\mu, \mu^*) = \mathrm{Tr}[\rho D(\mu)]$, where $D(\mu) = \exp\left[\sum_{i=a,b}(\mu_i \hat{a}_i^\dagger - \mu_{\mathrm{i}}^* \hat{a}_i)\right]$ is the displacement operator, and $\mu = (\mu_a, \mu_b)$. By equalizing the displacement operator with Weyl operator $\exp(\mathrm{i}Rz^{\mathrm{T}})$ and using the operator relations $\hat{a}_{\mathrm{i}} = \frac{1}{\sqrt{2}}(\hat{X}_i + \mathrm{i}\hat{P}_i)$, and $\hat{a}_i^\dagger = \frac{1}{\sqrt{2}}(\hat{X}_i - \mathrm{i}\hat{P}_i)$, we obtain $\mu_i = \frac{1}{\sqrt{2}}(-y_i + \mathrm{i}x_i)$. Thus $(\mu_a, \mu_b, -\mu_a^*, -\mu_b^*) = zL$, with a unitary matrix

$$L = \frac{1}{\sqrt{2}}\begin{bmatrix} \mathrm{i} & 0 & \mathrm{i} & 0 \\ -1 & 0 & 1 & 0 \\ 0 & \mathrm{i} & 0 & \mathrm{i} \\ 0 & -1 & 0 & 1 \end{bmatrix}. \tag{3}$$

The characteristic function of the two-mode Gaussian state can be written as

$$\chi(\mu, \mu^*) = \exp\left[(\mu, -\mu^*)m^{\mathrm{T}} - \frac{1}{2}(\mu, -\mu^*)\gamma(\mu^*, -\mu)^{\mathrm{T}}\right], \tag{4}$$

where $m = \mathrm{i}m_{\mathrm{re}}L^*$ is a complex vector, and $\gamma = L^\dagger \gamma_{\mathrm{re}} L$ is a complex matrix. The complex vector m has a clear physical meaning: it is the mean of the annihilation and creation operators, that is, $m = (\langle \hat{a}_a^\dagger \rangle, \langle \hat{a}_b^\dagger \rangle, \langle \hat{a}_a \rangle, \langle \hat{a}_b \rangle) = \mathrm{Tr}[\rho \mathcal{C}]$, where $\mathcal{C} = (\hat{a}_a^\dagger, \hat{a}_b^\dagger, \hat{a}_a, \hat{a}_b)$. The physical meaning of the complex matrix γ is also clear: it is indeed the CM of the annihilation and creation operators, that is, $\gamma_{ij} = \frac{1}{2}\mathrm{Tr}[\rho\{\Delta\mathcal{C}_{\mathrm{i}}, \Delta\mathcal{C}_j^\dagger)\}]$, where $\{\Delta\mathcal{C}_{\mathrm{i}}, \Delta\mathcal{C}_j^\dagger\} = \Delta\mathcal{C}_{\mathrm{i}}\Delta\mathcal{C}_j^\dagger + \Delta\mathcal{C}_j^\dagger\Delta\mathcal{C}_{\mathrm{i}}$, and $\Delta\mathcal{C} = \mathcal{C} - m$. We may regard γ as a complex correlation matrix (CCM) of the state.

The real symmetric property of γ_{re} indicates that the CCM has the form

$$\gamma = \begin{bmatrix} \alpha & \beta^* \\ \beta & \alpha^* \end{bmatrix}, \tag{5}$$

with $\alpha = \alpha^\dagger$, and $\beta = \beta^{\mathrm{T}}$. Take the following denotations:

$$\alpha = \begin{bmatrix} \alpha_a & \alpha_c \\ \alpha_c^* & \alpha_b \end{bmatrix}, \quad \beta = \begin{bmatrix} \beta_a & \beta_c \\ \beta_c & \beta_b \end{bmatrix}, \tag{6}$$

$$\gamma_i = \begin{bmatrix} \alpha_i & \beta_i^* \\ \beta_i & \alpha_i^* \end{bmatrix}, \quad i = a, b, c. \tag{7}$$

In the form of CCM, the Peres–Horodecki criterion[13,14] for separability of the state can be written as

$$\det\gamma_a \det\gamma_b + \left(\frac{1}{4} - |\det\gamma_c|\right)^2 - \mathrm{Tr}(\gamma_a\sigma_3\gamma_c\sigma_3\gamma_b\sigma_3\gamma_c^\dagger\sigma_3) \geq \frac{1}{4}(\det\gamma_a + \det\gamma_b). \tag{8}$$

For the state evolution described by the master equation (1), if the state initially is a Gaussian state with its characteristic function $\chi(\mu, \mu^*, 0) = \exp\left[(\mu, -\mu^*)m^{\mathrm{T}}(0) - \frac{1}{2}(\mu, -\mu^*)\gamma(0)(\mu^*, -\mu)^{\mathrm{T}}\right]$, it will remain a Gaussian state in later evolution. The time evolution of the complex first moment $m^{\mathrm{T}}(t)$ is neglected in the following for it is irrelative to the separability and the entanglement of the state.

The time evolution of the CCM is[11]

$$\gamma(t) = \mathrm{e}^{-\Gamma t}\begin{bmatrix} \cosh^*(|\eta|\,t) & \eta^*\dfrac{\sinh(|\eta|\,t)}{|\eta|} \\ \dfrac{\sinh(|\eta|\,t)}{|\eta|}\eta & \cosh(|\eta|\,t) \end{bmatrix}\left(\gamma(0) - \begin{bmatrix} \alpha_0 & \beta_0^* \\ \beta_0 & \alpha_0^* \end{bmatrix}\right)\begin{bmatrix} \cosh^*(|\eta|\,t) & \eta^*\dfrac{\sinh(|\eta|\,t)}{|\eta|} \\ \dfrac{\sinh(|\eta|\,t)}{|\eta|}\eta & \cosh(|\eta|\,t) \end{bmatrix} + \begin{bmatrix} \alpha_0 & \beta_0^* \\ \beta_0 & \alpha_0^* \end{bmatrix}, \tag{9}$$

where α_0 and β_0 are the solutions of the following matrix equations:

$$\eta\alpha_0 + \alpha_0^*\eta - \Gamma\beta_0 + \Gamma w = 0, \quad (10)$$

$$\Gamma\alpha_0 - \eta^*\beta_0 - \beta_0^*\eta - \Gamma\left(\bar{n} + \frac{1}{2}\boldsymbol{I}_2\right) = 0, \quad (11)$$

with $\bar{n} = \mathrm{diag}\{\bar{n}_a, \bar{n}_b\}$, and $w = \mathrm{diag}\{w_a, w_b\}$. The density operator of the state can be retrieved by operator integral of

$$\rho(t) = \int \left[\frac{\mathrm{d}^2\mu}{\pi}\right] \exp\left[(\mu - \mu^*)m^{\mathrm{T}}(t) - \frac{1}{2}(\mu, -\mu^*)\gamma(t)(\mu^*, -\mu)^{\mathrm{T}}\right] D(-\mu).$$

Since η is a symmetric matrix, we can express it as $\eta = \eta_0\sigma_0 + \eta_1\sigma_1 + \eta_3\sigma_3$, where $\sigma_0 = \boldsymbol{I}_2$ and $\sigma_i (i = 1, 2, 3)$ are Pauli matrices. The σ_2 term is nullified due to the symmetry of η. The single mode amplification parameters for modes a and b are $\frac{1}{2}(\eta_0 + \eta_3)$ and $\frac{1}{2}(\eta_0 - \eta_3)$, respectively, while $\eta_1$represents the inter-mode amplification. By using the algebra of Pauli matrices, the cosh and sinh functions can be further simplified as follows:[11]

$$\cosh(|\eta|\, t) = \cosh(\kappa_+ t)P_+ + \cosh(\kappa_- t)P_-, \quad (12)$$

$$\frac{\sinh(|\eta|\, t)}{|\eta|}\eta = \frac{\sinh(\kappa_+ t)}{\kappa_+}P_+\eta + \frac{\sinh(\kappa_- t)}{\kappa_-}P_-\eta, \quad (13)$$

where $\kappa_\pm = \sqrt{(C \pm B)}, C = |\eta_0|^2 + |\eta_1|^2 + |\eta_3|^2$, $B = \sqrt{C^2 - A^2}$, and $A = \left|\eta_0^2 - \eta_1^2 - \eta_3^2\right|$. The operators $P_\pm = \frac{1}{2}(\sigma_0 \pm \boldsymbol{\sigma}\cdot\boldsymbol{b}), \boldsymbol{b}$ is a real unit vector and equals $(\eta_0\eta_1^* + \eta_1\eta_0^*, \mathrm{i}\eta_3\eta_1^* - \mathrm{i}\eta_1\eta_3^*, \eta_0\eta_3^* + \eta_3\eta_0^*)/B$. The operators $P_\pm$ are projective operators and they are orthogonal to each other, $P_\pm^2 = P_\pm$, $P_+P_- = P_-P_+ = 0$, and $P_\pm^\dagger = P_\pm$.

In the following, we will investigate the separability of the final state. We consider the simple environment of $\bar{n}_a = \bar{n}_b = \bar{n}_0$ and $w = 0$. The environment noise is a simple (without squeezing, for $w = 0$) thermal noise with mean photon number $\bar{n}_0$ for both modes.

3. The symmetric amplification

The symmetric amplification is the case in which the two modes have the same amount of amplification, thus $\eta_3 = 0$. The solutions of β_0 and α_0 matrices are (see Appendix)

$$\beta_0 = 2\Gamma\left(\bar{n}_0 + \frac{1}{2}\right)\Delta_1^{-1}\{[(\Gamma^2 - 4c_0)\eta_0 + 4c_1\eta_1]\sigma_0 + [(\Gamma^2 - 4c_0)\eta_1 + 4c_1\eta_0]\sigma_1\}, \quad (14)$$

$$\alpha_0 = \Gamma^2\left(\bar{n}_0 + \frac{1}{2}\right)\Delta_1^{-1}\{(\Gamma^2 - 4c_0)\sigma_0 + 4c_1\sigma_1\}. \quad (15)$$

Let $\bar{n}_0' = \bar{n}_0 + \frac{1}{2}$, and introduce the dimensionless quantities $\eta_0' = 2\eta_0/\Gamma, \eta_1' = 2\eta_1/\Gamma$ (thus $\eta' = 2\eta/\Gamma$), and $h_\pm = |\eta_0' \pm \eta_1'|$, then we will have $\Gamma^2\Delta_1^{-1}[(\Gamma^2 - 4c_0) \pm 4c_1] = 1/(1 - h_\pm^2)$. Since $\eta_3 = 0$, we have $b_2 = b_3 = 0$, and $b_1 = \pm 1$. Without loss of generality, we set $b_1 = 1$, that is, the real quantity $\eta_0\eta_1^* + \eta_1\eta_0^*$ is positive, thus $h_+ > h_-$, and $\kappa_\pm = \frac{\Gamma}{2}h_\pm$. Hence $P_\pm = \frac{1}{2}(\sigma_0 \pm \sigma_1)$,

$$\alpha_0 = K_+P_+ + K_-P_-, \beta_0 = \alpha_0\eta', \quad (16)$$

$$\cosh(|\eta|\, t) = \cosh(h_+ t')P_+ + \cosh(h_- t')P_-, \quad (17)$$

$$\frac{\sinh(|\eta|\, t)}{|\eta|}\eta = \left[\frac{\sinh(h_+ t')}{h_+}P_+ + \frac{\sinh(h_- t')}{h_-}P_-\right]\eta'. \quad (18)$$

where $K_\pm = \bar{n}_0'/(1 - h_\pm^2)$, and $t' = \frac{\Gamma}{2}t$. If the initial state is a vacuum state with CCM, $\gamma(0) = \frac{1}{2}\sigma_0 \otimes \sigma_0$ (This may comprise a coherent state which differs from a vacuum state only by the first moment), by the orthogonality of $P_\pm$, we arrive at (we will denote $\alpha(t), \beta(t)$ as α, β)

$$\alpha = u_+P_+ + u_-P_-, \quad \text{and}$$
$$\beta = (v_+P_+ + v_-P_-)\eta', \quad (19)$$

with

$$u_\pm = K_\pm + \left(\frac{1}{2} - K_\pm\right)C_\pm - K_\pm S_\pm h_\pm, \quad (20)$$

$$v_\pm = K_\pm - K_\pm C_\pm + \left(\frac{1}{2} - K_\pm\right)S_\pm/h_\pm, \quad (21)$$

where $C_\pm = \cosh(2h_\pm t')\mathrm{e}^{-2t'}$, and $S_\pm = \sinh(2h_\pm t')\mathrm{e}^{-2t'}$. A direct calculation with algebra programming will simplify the separability criterion (8) into

$$(u_+^2 - v_+^2 h_+^2)(u_-^2 - v_-^2 h_-^2) + \frac{1}{2}(v_+ v_- g - u_+ u_-) + \frac{1}{16} \geq 0, \qquad (22)$$

where $g = |\eta_0'|^2 - |\eta_1'|^2$.

If the damping is stronger than the amplification, that is, $\dfrac{\Gamma}{2} > \kappa_+$ or $h_+ < 1$, the state will tend to the final Gaussian state which is characterized by the residue CCM $\gamma(\infty) = \begin{bmatrix} \alpha_0 & \beta_0^* \\ \beta_0 & \alpha_0^* \end{bmatrix}$ for any initial Gaussian state. In this situation, $C_\pm = S_\pm = 0$, and $u_\pm = v_\pm = K_\pm$, hence the separability condition will be

$$\left(\bar{n}_0 + \frac{1}{2}\right)^2 \geq \frac{1}{4}\left[1 - g + \sqrt{(1-g)^2 - (1-h_+^2)(1-h_-^2)}\right], \qquad (23)$$

which can be written as

$$\left(\bar{n}_0 + \frac{1}{2}\right)^2 \geq \frac{1}{4}\left[1 - |\eta_0'|^2 + |\eta_1'|^2 + \sqrt{4|\eta_1'|^2 + (\eta_0'\eta_1'^* - \eta_1'\eta_0'^*)^2}\right].$$

When the damping is weaker than the amplification, that is $\dfrac{\Gamma}{2} < \kappa_+$ or $h_+ > 1$, suppose that $h_- < 1$, at $t' \to \infty$, then we will have $C_-, S_- \to 0$; $u_-, v_- \to K_-$; $C_+ - S_+ \to 0$; $u_+ - v_+ h_+ \to K_+(1-h_+)$. Substitute all these into inequality (22), divide the inequality by C_+ and take the limit of $t' \to \infty$, then the separability criterion will turn into

$$[1 - 2(1+h_+)K_+]\,K_+(1-h_+)K_-^2(1-h_-^2) + \frac{1}{2h_+}(h_+ - g)\left[(h_+ + 1)K_+ - \frac{1}{2}\right]K_- \geq 0, \qquad (24)$$

which has the solution

$$\left(\bar{n}_0 + \frac{1}{2}\right)^2 \geq \frac{(1+h_+)(h_+ - g)}{4h_+}. \qquad (25)$$

If η_0 and η_1 have the same phase angle ($\Delta\phi = \phi_0 - \phi_1 = 0$; ϕ_0 and ϕ_1 are phase angles of the complex parameters η_0 and η_1, respectively), inequalities (23) and (25) can be uniformly written as $\left(\bar{n}_0 + \dfrac{1}{2}\right)^2 \geq \dfrac{1}{4}\left[(1 + |\eta_1'|)^2 - |\eta_0'|^2\right]$. In Figs.1 and 2, the separability conditions are displayed for $\Delta\phi = 0$ and $\Delta\phi = \dfrac{\pi}{2}$, respectively. The strong and weak amplifications are combined and plotted in the same figure.

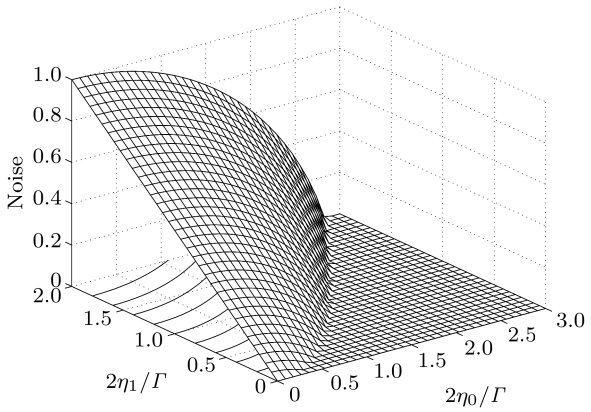


Fig.1. The separable boundary for $\Delta\phi = \phi_0 - \phi_1 = 0$. $\bar{n}_0$, η_0, η_1, and Γ are the noise, single-mode amplification parameter, inter-mode amplification parameter, and damping coefficient respectively. All entangled states are under the surface.

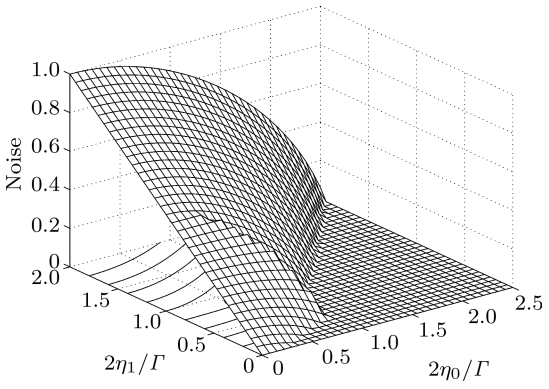


Fig.2. The separability boundary for $\Delta\phi = \phi_0 - \phi_1 = \dfrac{\pi}{2}$. $\bar{n}_0$, η_0, η_1, and Γ are the noise, single-mode amplification parameter, inter-mode amplification parameter, and damping coefficient respectively. All entangled states are under the surface.

The physical meaning of the separability criteria (23) and (25) shown by Figs.1 and 2 is quite clear. There are four factors which determine the separability of the state. They are the noise $\bar{n}_0$, the damping Γ, the single mode amplification η_0 and the inter-mode amplification η_1. We start with an entangled state (a point under the separable boundary surface) in Figs.1 and 2. If the variation of one of the four factors drives the point away from the separable boundary surface, the entanglement of the state is said to increase, otherwise it is said to decrease. The decrease or increase of the entanglement is influenced by the following factors: (1) the noise, whose increase can reduce the entanglement of the state as anticipated; (2) the single mode amplification, whose increase can also reduce the entanglement; (3) the inter-mode amplification, whose increase can increase the entanglement; (4) the damping Γ, whose increase can reduce the amplification/damping ratio $2\eta_0/\Gamma$ and $2\eta_1/\Gamma$ simultaneously. From Figs.1 and 2 we can see that the entanglement can be reduced in the cases of weak amplification and strong amplification separately. Thus noise and damping can reduce the entanglement of the state, which is universally received. The single mode amplification can also reduce the entanglement. The only factor that can increase the entanglement is the inter-mode amplification.

4. The asymmetric amplification

The asymmetric amplification is the case in which the two modes have the same amount of amplification but with different signs, thus $\eta_0 = 0$. The solutions of β_0 and α_0 matrices are

$$\beta_0 = 2\Gamma\left(\bar{n}_0 + \frac{1}{2}\right)\Delta_2^{-1}\{[(\Gamma^2 - 4c_0)\eta_1 + 4\mathrm{i}c_2\eta_3]\sigma_1 + [(\Gamma^2 - 4c_0)\eta_3 - 4\mathrm{i}c_2\eta_1]\sigma_3\}, \tag{26}$$

$$\alpha_0 = \Gamma^2\left(\bar{n}_0 + \frac{1}{2}\right)\Delta_2^{-1}\{(\Gamma^2 - 4c_0)\sigma_0 - 4c_2\sigma_2\}. \tag{27}$$

With the dimensionless quantities $\eta_1' = 2\eta_1/\Gamma$, and $\eta_3' = 2\eta_3/\Gamma$ (thus $\eta' = 2\eta/\Gamma$, the forms of the expressions (17), (18), (20), (21) and the separability criterion (22) are preserved. The differences from what are in the preceding section are $P_\pm = \frac{1}{2}(\sigma_0 \pm \sigma_2)$ (with $\mathrm{i}(\eta_3\eta_1^* - \eta_1\eta_3^*) > 0$ assumed), $h_\pm = |\eta_0' \pm \mathrm{i}\eta_3'|$, $g = |\eta_3'|^2 - |\eta_1'|^2$, and

$$\alpha_0 = K_+P_- + K_-P_+, \beta_0 = (K_+P_+ + K_-P_-)\eta', \tag{28}$$

$$\alpha = u_+P_- + u_-P_+, \beta = (v_+P_+ + v_-P_-)\eta'. \tag{29}$$

In the derivation, $P_\pm\eta' = \eta' P_\mp$ has been used.

The solutions to the separability criterion have the same forms as those of inequalities (23) and (25). For $h_+ < 1$, the separability condition is

$$\left(\bar{n}_0 + \frac{1}{2}\right)^2 \geq \frac{1}{4}[1 - |\eta_3'|^2 + |\eta_1'|^2 + \sqrt{4|\eta_1'|^2 - (\eta_3'\eta_1'^* + \eta_1'\eta_3'^*)^2}]. \tag{30}$$

If the difference between the phase angles of η_1 and η_3 is $\frac{\pi}{2}$, the separability conditions of the two situations of $h_+ < 1$ and $h_+ > 1$ can be combined into $\left(\bar{n}_0 + \frac{1}{2}\right)^2 \geq \frac{1}{4}[(1 + |\eta_1'|)^2 - |\eta_3'|^2]$.

The physical meaning of separability condition is also evident. That is, the noise, the damping and the single mode amplification will reduce the entanglement of the state. The only factor that can increase the entanglement is the inter-mode amplification.

5. Conclusions

For any two-mode initial Gaussian state input, the output state at any time of evolution is obtained in the form of complex correlation matrix when

the simultaneous amplification, symmetric amplitude damping and thermal noise are considered as a channel. When the amplification is weaker than the damping, the ultimate ($t \to \infty$) complex correlation matrix of the state will tend to the residue complex correlation matrix (represented by α_0 and β_0 in the text) regardless of the input Gaussian state. The separability criteria are obtained both for symmetric amplification and asymmetric amplification. It can be written as a ratio between the noise and some expression of ratios of amplifications to damping. When the amplification is stronger than the damping, vacuum input (this may comprise a coherent state, for it differs from a vacuum state only by the first moment which is not relevant to the entanglement) is considered to simplify the calculation. The separability criterion can also be written as a ratio between the noise and some expression of ratios of amplifications to damping. The inter-mode amplification parameter η_1 plays a crucial role in entangling the output state, while noise, damping and single mode amplification are harmful to the entanglement of the state.

Appendix The solution of residue complex correlation matrix for symmetrical damping system

The solutions α_0 and β_0 to the algebraic equations (10) and (11) are obtained as follows. As Γ is proportional to identity matrix, we obtain α_0 from Eq.(11) and substitute it into Eq.(10). The resulting equation is an equation about β_0 and β_0^*, with $\beta_0 = \beta_{00}\sigma_0 + \beta_{01}\sigma_1 + \beta_{03}\sigma_3$. The conjugate equation is also an equation about β_0 and β_0^*. Cancelling out β_0^* yields an equation about β_0 as

$$\begin{aligned} & \Gamma^4\beta_0 - 2\Gamma^2(\beta_0\eta^*\eta + \eta\eta^*\beta_0) + (\eta\eta^*)^2\beta_0 - 2\eta\eta^*\beta_0\eta^*\eta + \beta_0(\eta^*\eta)^2 \\ = \; & \Gamma(\eta\eta^*\bar{n}'\eta + \eta\bar{n}'\eta^*\eta - \eta\eta^*\eta\bar{n}' - \bar{n}'\eta\eta^*\eta) + \Gamma^3(\eta\bar{n}' + \bar{n}'\eta) \\ & + \Gamma^4 w - \Gamma^2(\eta\eta^* w + w\eta^*\eta) + 2\Gamma^2\eta w^*\eta, \end{aligned}$$

where $\bar{n}' = \bar{n} + \frac{1}{2}\boldsymbol{I}_2$. Let $c_0 = C$ and $c_{\rm i} = b_i B$, then $\eta\eta^* = c_0\sigma_0 + \boldsymbol{c}\cdot\boldsymbol{\sigma}$, with $\boldsymbol{c} = (c_1, c_2, c_3)$ and $|\boldsymbol{c}| = B$. Let the terms on the right-hand side of the above equation equal $d_0\sigma_0 + d_1\sigma_1 + d_3\sigma_3$, compare the coefficients of $\sigma_{\rm i}$, the linear equation will be $Q\boldsymbol{\beta_0'} = \boldsymbol{d}$,where $\boldsymbol{\beta_0'} = (\beta_{00}, \beta_{01}, \beta_{03})^{\rm T}, \boldsymbol{d} = (d_0, d_1, d_3)^{\rm T}$,

$$Q = \begin{bmatrix} \Gamma^4 - 4\Gamma^2 c_0 + 4c_2^2 & -4\Gamma^2 c_1 + 4{\rm i}c_2c_3 & -4\Gamma^2 c_3 - 4{\rm i}c_1c_2 \\ -4\Gamma^2 c_1 - 4{\rm i}c_2c_3 & \Gamma^4 - 4\Gamma^2 c_0 + 4c_3^2 & -4{\rm i}\Gamma^2 c_2 - 4c_1c_3 \\ -4\Gamma^2 c_3 + 4{\rm i}c_1c_2 & 4{\rm i}\Gamma^2 c_2 - 4c_1c_3 & \Gamma^4 - 4\Gamma^2 c_0 + 4c_1^2 \end{bmatrix}, \tag{A1}$$

and $\boldsymbol{\beta_0'} = Q^{-1}\boldsymbol{d}$, with $\alpha_0 = (\eta^*\beta_0 + \beta_0^*\eta)/\Gamma + \left(\bar{n} + \frac{1}{2}\boldsymbol{I}_2\right)$. The determinant of Q reads $\det Q = \Gamma^4[\Gamma^4 - 4c_0\Gamma^2 + 4\,|\boldsymbol{c}|^2][\Gamma^4 - 8c_0\Gamma^2 + 16(c_0^2 - |\boldsymbol{c}|^2)]$, and it can be further expressed as

$$\det Q = \Gamma^4[\Gamma^2 - 2(C+A)][\Gamma^2 - 2(C-A)][\Gamma^2 - 4(C+B)][\Gamma^2 - 4(C-B)]. \tag{A2}$$

The expression of Q^{-1} is relatively complicated, but it can be quite simple in some special cases. In the case of symmetric amplification of the two modes, that is $\eta_3 = 0$, we have $c_2 = c_3 = 0$. Let $\Delta_1 = (\Gamma^2 - 4c_0 - 4c_1)(\Gamma^2 - 4c_0 + 4c_1)$ and $\Omega_1 = \Gamma^4 - 4c_0\Gamma^2 + 4c_1^2$, then we will obtain

$$Q^{-1} = \begin{bmatrix} (\Gamma^2 - 4c_0)/(\Gamma^2\Delta_1) & 4c_1/(\Gamma^2\Delta_1) & 0 \\ 4c_1/(\Gamma^2\Delta_1) & (\Gamma^2 - 4c_0)/(\Gamma^2\Delta_1) & 0 \\ 0 & 0 & 1/\Omega_1 \end{bmatrix}. \tag{A3}$$

In the case of asymmetric amplification of the two modes, that is, $\eta_0 = 0$, we have $c_1 = c_3 = 0$. Let $\Delta_2 =$

$(\Gamma^2 - 4c_0 - 4c_2)(\Gamma^2 - 4c_0 + 4c_2)$ and $\Omega_2 = \Gamma^4 - 4c_0\Gamma^2 + 4c_2^2$, then we will have

$$Q^{-1} = \begin{bmatrix} 1/\Omega_2 & 0 & 0 \\ 0 & (\Gamma^2 - 4c_0)/(\Gamma^2\Delta_2) & 4\mathrm{i}c_2/(\Gamma^2\Delta_2) \\ 0 & -4\mathrm{i}c_2/(\Gamma^2\Delta_2) & (\Gamma^2 - 4c_0)/(\Gamma^2\Delta_2) \end{bmatrix}. \tag{A4}$$

In the case of no bilateral amplification, that is, $\eta_1 = 0$, we have $c_1 = c_2 = 0$. Let $\Delta_3 = (\Gamma^2 - 4c_0 - 4c_3)(\Gamma^2 - 4c_0 + 4c_3)$ and $\Omega_3 = \Gamma^4 - 4c_0\Gamma^2 + 4c_3^2$, then we will achieve

$$Q^{-1} = \begin{bmatrix} (\Gamma^2 - 4c_0)/(\Gamma^2\Delta_3) & 0 & 4c_3/(\Gamma^2\Delta_3) \\ 0 & 1/\Omega_3 & 0 \\ 4c_3/(\Gamma^2\Delta_3) & 0 & (\Gamma^2 - 4c_0)/(\Gamma^2\Delta_3) \end{bmatrix}. \tag{A5}$$

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