Vol 16 No 8, August 2007
1009-1963/2007/16(08)/2188-06

Chinese Physics

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Entanglement-enhanced classical communication through an amplitude-damping channel*

Hou Li-Zhen(侯丽珍) and Fang Mao-Fa(方卯发)†

Department of Physics, Hunan Normal University, Changsha 410081*, China*

(Received 21 February 2006; revised manuscript received 15 January 2007)

We consider the problem of trying to send a single classical bit through an amplitude-damping channel when two transmissions through the channel are available as a resource. It is demonstrated that two entangled transmissions can enhance the receiver's capability of making a correct inference under certain conditions compared with two product-state transmissions.

Keywords: two transmissions, entanglement, amplitude-damping channel, average probability of error

PACC: 0365

1. Introduction

A cental problem in quantum information theory is how to reliably transmit information.[1−7] The sender (Alice) encodes the information in appropriate quantum states which are sent to the receiver (Bob) via a noisy quantum channel. At the receiving end, Bob performs a general quantum mechanical measurement and uses the acquired data to make a venturous guess at the information which Alice has sent him. Due to the effects of noise, Bob perhaps cannot make a completely correct inference. So, many researchers are interested in how to improve the receiver's ability of making a correct inference, i.e. how to improve the classical communication.[8−11] They have developed several methods to solve this problem: for example, quantum superdense coding, which uses the previously shared entanglement and can enhance the classical communication;[8] the two-mode squeezed states, which can enhance the classical communication over a quantum Gaussian channel;[9] and the use of entangled input signals, which may enhance the information transmission compared with the use of product input signals through a time-correlated quantum amplitude damping channel.[10] In Ref.[11], Bennett and co-workers considered the problem of trying to send a single classical bit through a two-pauli channel when two transmissions through the channel are available as a resource, and calculated the average probability of error for the receiver to identify the bit which the sender has sent him. They have found that entanglement can enhance the classical communication for the two-pauli channel, and predicted that this effect may be a property of most noisy channels. However, to the best of our knowledge, the further demonstration of this forecast of using other noisy quantum channels has not yet been reported so far.

Among noisy quantum channels, the amplitude-damping channel is a very important kind of noisy quantum channel.[12] It can describe the energy dissipation processes of some physical systems: for example, the dynamics of an atom which can spontaneously emit a photon, the process of a spin system at high temperature which approaches an equilibrium with its environment, and the evolution of the state of a photon in an interferometer or cavity when it is subjected to scattering and attenuation. Although each of these processes has its own unique features, all their general behaviours are well characterized by amplitude damping. However, when a classical bit is transmitted through an amplitude-damping channel from one place to another, whether or not two entangled transmissions can enhance the classical communication has not been studied. In the present paper, our task is to use an amplitude-damping channel as effectively as possible for communicating a single bit, 0 or 1, provided that only two transmissions through the channel are allowed. Alice encodes the bit in the two transmis-

*Project supported by the National Natural Science Foundation of China (Grant No 10374025), and the Natural Science Foundation of Hunan Normal University (Grant No 22040640).

†Corresponding author. E-mail: mffang@hunnu.edu.cn

sions, and Bob tries to infer the real bit by measuring the output of the channel. Classically, any attempt to protect the bit against noise due to redundancy is always useless when only two transmissions are available. This is because Bob cannot perform any sort of 'majority vote' error correction on two transmissions. However, in a quantum field, things become very interesting. Our aim is to demonstrate that, for an amplitude-damping channel, two entangled transmissions can indeed enhance the classical communication under certain situations compared with two product-state transmissions.

The remainder of this paper is organized as follows. In Section 2, a brief introduction to noisy channel and the general formula about the average probability of error are presented. In particular, we introduce the amplitude-damping channel and analyse what can be done with it upon one transmission. In Section 3, we study two transmissions with product-state inputs and entangled transmissions. A brief summary and some remarks are given in Section 4.

2. The amplitude-damping channel

What is a noisy quantum channel, and how can it be described mathematically? In previous studies on the noisy channel problem[13−15] apparently different formulas have been used to describe the noisy channel. In fact, all the formulas can be shown to be equivalent to each other. In general, the action of a noisy channel on the physical system is described by mapping Φ from input density operators into the output density operators. In all cases, the mapping is assumed to come about via an interaction between the system of interest and an independently prepared environment, to which neither Alice nor Bob has access. Thus a noisy channel is described formally by a mapping of the form

$$\rho \longrightarrow \Phi(\rho) = \mathrm{tr}_E(U(\rho \otimes \sigma)U^\dagger), \tag{1}$$

where σ is the initial state of the environment, U is the unitary interaction between the system and environment, and tr_E denotes a partial trace over the environment's Hilbert space.

The most general form for all possible noise models is given by the Kraus representation theorem.[16] This states that Eq.(1) can be written in the form

$$\Phi(\rho) = \sum_i A_i \rho A_i^\dagger, \tag{2}$$

where the operators A_i satisfy the completeness relation

$$\sum_i A_i^\dagger A_i = I. \tag{3}$$

Conversely, any set of operators A_i satisfying Eq.(3) can be used in Eq.(2) to give rise to a valid noisy channel in the sense of Eq.(1).

For the amplitude-damping channel,[12] the corresponding Kraus A_i operators are

$$A_1 = \begin{pmatrix} \sqrt{x} & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad A_2 = \begin{pmatrix} 0 & 0 \\ \sqrt{1-x} & 0 \end{pmatrix}. \tag{4}$$

This channel can be interpreted as follows: with a probability of x, it leaves the qubit alone; with a probability of $1-x$, it flips the qubit.

In the language of the Kraus theorem,[16] the situation of two transmissions through the channel is described by the mapping

$$R \longrightarrow \Phi(R) = \sum_{i,j} (A_i \otimes A_j) R (A_i \otimes A_j)^\dagger, \tag{5}$$

where R denotes a density operator in the d^2-dimensional Hilbert space $\mathcal{H}_d \otimes \mathcal{H}_d$.

In our problem, Alice encodes the classical bit that she wishes to transmit to Bob by preparing a quantum system in one of two states, R_0 or R_1, in $\mathcal{H}_d \otimes \mathcal{H}_d$. The action of the channel via Eq.(5) leads to one of two possible output density operators ($\tilde{R}_0$ and $\tilde{R}_1$) for Bob at the receiving end.

For the problem of distinguishing between the two output density operators $\tilde{R}_0$ and $\tilde{R}_1$, we imagine Bob performing a general quantum mechanical measurement, or positive operator-valued measure (POVM) $\{E_b\}$, and then using the acquired data to make a venturous guess at the identity of the density operator. Depending on which bit s Alice has sent, the probability of Bob's measurements will be given by $\mathrm{tr}(\tilde{R}_s E_b)$. Clearly, the best strategy for Bob to identify the bit — upon finding an outcome b — is to guess at the value of s such that $\mathrm{tr}(\tilde{R}_s E_b)$ is the largest. We assume that inputs are equally likely, this gives rise to an average probability of error given by

$$P_e(\{E_b\}) = \frac{1}{2} \sum_b \min \left\{ \mathrm{tr}(\tilde{R}_0 E_b), \mathrm{tr}(\tilde{R}_1 E_b) \right\}. \tag{6}$$

This makes it clear that the best measurement on Bob's part is to choose $\{E_b\}$ that can minimize this expression. It turns out that this measurement can be described by a standard von Neumann measurement of the Hermitian operator $\Gamma = \tilde{R}_1 - \tilde{R}_0$.[17] Moreover, with this measurement, Eq.(6) reduces to[18]

$$\begin{aligned} P_e &= \frac{1}{2} - \frac{1}{4}\mathrm{tr}\left|\tilde{R}_0 - \tilde{R}_1\right| \\ &= \frac{1}{2} - \frac{1}{4}\mathrm{tr}\left|\Phi(R_0) - \Phi(R_1)\right|. \end{aligned} \tag{7}$$

Here $\mathrm{tr}|A|$, for any Hermitian operator A, should be regarded as the sum of the absolute values of A's eigenvalues. In a precise manner, for any two orthogonal pure state inputs $|0\rangle$ and $|1\rangle$ in $\mathcal{H}_d \otimes \mathcal{H}_d$, Eq.(7) reduces to

$$P_e = \frac{1}{2} - \frac{1}{4}\mathrm{tr}\left|\Phi(|0\rangle\langle 0|) - \Phi(|1\rangle\langle 1|)\right|. \tag{8}$$

Now, we consider only one transmission through the amplitude-damping channel. Suppose two possible inputs, ρ_+ and ρ_-, given in Bloch sphere representation by

$$\rho_\pm = \frac{1}{2}\left(I \pm \boldsymbol{a}\cdot\boldsymbol{\sigma}\right). \tag{9}$$

Here, $\boldsymbol{a} = (a_1, a_2, a_3)$ is any real vector of length 1 or less, and $\boldsymbol{\sigma}$ is a vector of Pauli matrices.

Through the channel, the corresponding two output density operators are expressed as

$$\Phi(\rho_+) = \frac{1}{2}\begin{pmatrix} x(1+a_3) & \sqrt{x}(a_1 - \mathrm{i}a_2) \\ \sqrt{x}(a_1 + \mathrm{i}a_2) & 2 - x(1+a_3) \end{pmatrix}, \tag{10}$$

$$\Phi(\rho_-) = \frac{1}{2}\begin{pmatrix} x(1-a_3) & -\sqrt{x}(a_1 - \mathrm{i}a_2) \\ -\sqrt{x}(a_1 + \mathrm{i}a_2) & 2 - x(1-a_3) \end{pmatrix}. \tag{11}$$

For communication, we should make the final states as distinguishable as possible. This means that we should pick the vector $\boldsymbol{a}$ such that $\Phi(\rho_+)$ and $\Phi(\rho_-)$ become as pure as possible. The eigenvalues of $\Phi(\rho_+)$ are

$$\frac{1}{2} \pm \frac{1}{2}\sqrt{[(x-1) + xa_3]^2 + x(a_1^2 + a_2^2)}, \tag{12}$$

and the eigenvalues of $\Phi(\rho_-)$ are

$$\frac{1}{2} \pm \frac{1}{2}\sqrt{[(1-x) + xa_3]^2 + x(a_1^2 + a_2^2)}. \tag{13}$$

And the appendix in Ref.(11) shows that the optimal input states, in which the distinguishability of the associated outputs can reach a maximum, are orthogonal pure states. Hence, clearly, when $a_1 = a_2 = 0$ and $a_3^2 = 1$, the inputs will be optimal and the outputs will turn as pure as possible.

Thus the probability of error in guessing at the identity of states will be worked out easily. It is just

$$P_e = \frac{1}{2}(1-x). \tag{14}$$

3. Two transmissions

Consider two product-state transmissions through the channel. We assume two inputs to be of the following forms:

$$R_0 = \rho_+ \otimes \rho_+ \quad \text{and} \quad R_1 = \rho_- \otimes \rho_-. \tag{15}$$

To make the final states as distinguishable as possible, we must choose the appropriate vector $\boldsymbol{a}$ such that R_0 and R_1 are orthogonal and pure states, at the same time, $\Phi(R_0)$ and $\Phi(R_1)$ turn as pure as possible.[11] Through tedious calculation and simple analysis, we know that if and only if $a_1 = a_2 = 0$ and $a_3^2 = 1$, the input states are optimal. Under this condition we have

$$\Phi(R_0) - \Phi(R_1) = \begin{pmatrix} \pm x^2 & 0 & 0 & 0 \\ 0 & \pm x(1-x) & 0 & 0 \\ 0 & 0 & \pm x(1-x) & 0 \\ 0 & 0 & 0 & \pm x(x-2) \end{pmatrix}. \tag{16}$$

So, according to Eq.(7), we obtain the best possible average error probability

$$P_e = \frac{1}{2}(1-x)^2. \tag{17}$$

Comparing Eq.(17) with Eq.(14), we find that two product-state transmissions can enhance the capability of the receiver judging information more correctly than one transmission through the amplitude-damping channel if only the channel parameter $x \neq 0, 1$. But it does not hold true for all noisy quantum channels, for example, the two-Pauli channel.[11]

Now, we turn to the problem of two entangled transmissions through the amplitude-damping channel. A convenient basis by which the input states for two transmissions are given is the Bell operator basis, where the four orthogonal basis vectors are given as

$$|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle|\uparrow\rangle \pm |\downarrow\rangle|\downarrow\rangle), \quad \text{and}$$

$$|\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle \pm |\downarrow\rangle|\uparrow\rangle). \tag{18}$$

We use the common notations that are given as

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{19}$$

for the eigenvectors of σ_3. With respect to this basis, an arbitrary set of two input states can be written as

$$|0\rangle = a_0|\Phi^+\rangle + b_0|\Phi^-\rangle + c_0|\Psi^+\rangle + d_0|\Psi^-\rangle, \tag{20}$$

and

$$|1\rangle = a_1|\Phi^+\rangle + b_1|\Phi^-\rangle + c_1|\Psi^+\rangle + d_1|\Psi^-\rangle. \tag{21}$$

According to Eq.(5), the output density operator can be expressed as

$$\tilde{R}_0 = \Phi(|0\rangle\langle 0|)$$

$$= \frac{1}{2}\begin{pmatrix} x^2A_0^2 & x\sqrt{x}A_0C_0 & x\sqrt{x}A_0D_0 & xA_0B_0 \\ x\sqrt{x}A_0C_0 & x[(1-x)A_0^2+C_0^2] & xC_0D_0 & \sqrt{x}[B_0C_0+(1-x)A_0D_0] \\ x\sqrt{x}A_0D_0 & xC_0D_0 & x[(1-x)A_0^2+D_0^2] & \sqrt{x}[B_0D_0+(1-x)A_0C_0] \\ xA_0B_0 & \sqrt{x}[B_0C_0+(1-x)A_0D_0] & \sqrt{x}[B_0D_0+(1-x)A_0C_0] & (1-x)^2A_0^2+(1-x)(C_0^2+D_0^2)+A_0B_0 \end{pmatrix}, \tag{22}$$

where

$$A_0 = a_0 + b_0, \quad B_0 = a_0 - b_0, \quad C_0 = c_0 + d_0, \quad \text{and} \quad D_0 = c_0 - d_0. \tag{23}$$

A similar expression of $\Phi(|1\rangle\langle 1|)$ can be obtained as long as 0 is exchanged for 1 in the expression (22) of $\Phi(|0\rangle\langle 0|)$.

In order to turn the input states into the orthogonal pure states, the corresponding coefficients must satisfy the following condition:

$$a_0^2 + b_0^2 + c_0^2 + d_0^2 = 1, \tag{24}$$

$$a_1^2 + b_1^2 + c_1^2 + d_1^2 = 1, \tag{25}$$

$$a_0a_1 + b_0b_1 + c_0c_1 + d_0d_1 = 0. \tag{26}$$

That is to say, the input states that satisfy the above condition are optimal input states for the two entangled transmissions. Though there exist many equivalent optimal entangled signals, only two inputs can always be taken to be of the form

$$|0\rangle = \cos\alpha|\Phi^+\rangle + \sin\alpha|\Psi^+\rangle, \tag{27}$$

and

$$|1\rangle = -\sin\alpha|\Phi^+\rangle + \cos\alpha|\Psi^+\rangle \tag{28}$$

without losing any performance.

Now we consider the average probability of error (P_e) in Bob's inference of the signal when choosing Eqs.(27) and (28) as input states. From Eq.(8) we must know the eigenvalues of $\Phi(|0\rangle\langle 0|) - \Phi(|1\rangle\langle 1|)$. According to Eq.(2), we obtain

$$\Phi(|0\rangle\langle 0|) - \Phi(|1\rangle\langle 1|) = \frac{1}{2}\begin{pmatrix} x^2\cos 2\alpha & x\sqrt{x}\sin 2\alpha & x\sqrt{x}\sin 2\alpha & x\cos 2\alpha \\ x\sqrt{x}\sin 2\alpha & -x^2\cos 2\alpha & -x\cos 2\alpha & \sqrt{x}(2-x)\sin 2\alpha \\ x\sqrt{x}\sin 2\alpha & -x\cos 2\alpha & -x^2\cos 2\alpha & \sqrt{x}(2-x)\sin 2\alpha \\ x\cos 2\alpha & \sqrt{x}(2-x)\sin 2\alpha & \sqrt{x}(2-x)\sin 2\alpha & x^2\cos 2\alpha \end{pmatrix}. \quad (29)$$

Let $y = \cos 2\alpha$, then $-1 \leq y \leq 1$. Thus P_e should be determined by the input state parameter y and channel parameter x, i.e. $P_e = P_e(x, y)$. Because $P_e(x, y) = P_e(x, -y)$, we consider only $0 \leq y \leq 1$ in this paper.

Numerical result demonstrates that the two entangled transmissions through the amplitude-damping channel are more effective at disabling noise than two product-state transmissions under certain conditions. To convey a feeling for the effectiveness of entangled transmissions, our results are tabulated below.

From Table 1, we know that for input state parameter values $0 \leq y \leq 0.5$ and channel parameter values $x \leq 0.5$, the optimal inputs are entangled states inputs, while for channel parameter values $x \geq 0.6$, the optimal inputs are product-state inputs. Table 1 only gives some representative points and roughly shows what the optimal input state is. In fact, for each input state parameter y, we can numerically find a turning point x_0, i.e. for $x \leq x_0$, the entangled input states are optimal inputs; for $x > x_0$, the optimal inputs are product-state inputs. For example, when $y = 0$, the turning point $x_0 = 0.585786$. In other cases, the turning points can be also obtained by making a similar numerical analysis.

Table 1. A comparison between the values of error probability P_e obtained in product states and entangled states.

x	P_e	P_e (entangled states)					
	(product states)	$y = 0$	$y = 0.1$	$y = 0.2$	$y = 0.3$	$y = 0.4$	$y = 0.5$
0.1	0.4050000	0.2872800	0.2871940	0.2892290	0.2934820	0.3001370	0.3094920
0.2	0.3200000	0.2136440	0.2129590	0.2150380	0.2200000	0.2280660	0.2396010
0.3	0.2450000	0.1657100	0.1644450	0.1662520	0.1712480	0.1796520	0.1918260
0.4	0.1800000	0.1312180	0.1294280	0.1307890	0.1353980	0.1434480	0.1552490
0.5	0.1250000	0.1047150	0.1024780	0.1032800	0.1071890	0.1143500	0.1250000
0.6	0.0800000	0.0828669	0.0803122	0.0805094	0.0834969	0.0893676	0.0982757
0.7	0.0450000	0.0632506	0.0606008	0.0602525	0.0622186	0.0665465	0.0733199
0.8	0.0200000	0.0439298	0.0415498	0.0408616	0.0418642	0.0445741	0.0490254
0.9	0.0050000	0.0232925	0.0217295	0.0210905	0.0213728	0.0225788	0.0247158

4. Summary

We have proved the entanglement-enhanced classical communication through an amplitude-damping channel. By increasing our resources from one to two transmissions and allowing those transmissions to be entangled, we can make a classical bit sent through the amplitude-damping channel more resilient against the effects of noise. That is to say, under some conditions, two entangled transmissions can enhance the receiver's capacity of making a correct inference compared with two product-state transmissions over the amplitude damping channel. We expect our results to be useful for the development of information communication. More work is underway to further investigate the effect of two entangled transmissions on other noisy quantum channels.

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