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Study on an extended Boussinesq equation*

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An extended Boussinesq equation that models weakly nonlinear and weakly dispersive waves on a uniform layer of water is studied in this paper. The results show that the equation is not Painlevé-integrable in general. Some particular exact travelling wave solutions are obtained by using a function expansion method. An approximate solitary wave solution with physical significance is obtained by using a perturbation method. We find that the extended Boussinesq equation with a depth parameter of $1/\sqrt{2}$ is able to match the Laitone's (1960) second order solitary wave solution of the Euler equations.

Keywords: Painlevé-integrability, exact soliton solutions, approximate solution
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1. Introduction

The models of nonlinear weakly-dispersive waves considering the information of wave profile and the depth-mean horizontal velocity have been presented by many authors.[1−4] Among them, an extended Boussinesq (EB) equation

$$\begin{aligned} &\eta_t + [(1+\eta)w]_x + aw_{xxx} - b\eta_{xxt} = 0, \\ &w_t + \eta_x + ww_x + c\eta_{xxx} - dw_{xxt} = 0, \end{aligned} \tag{1}$$

has aroused considerable interest in this area recently.[5−7] Here η is the wave elevation; w is the projection of velocity on the x-axis. The equation is first derived in Ref.[8] from the Euler equations with a perturbation scheme under the assumption that the wave amplitude is small and wave-length is large compared with the water depth, scaled to be 1 here. The leading order nonlinear and dispersive effects are balanced in the same way as for the Korteweg de Vries (KdV) equation

$$\eta_t + \eta_x + \frac{3}{2}\eta\eta_x + \frac{1}{6}\eta_{xxx} = 0, \tag{2}$$

which models unidirectional water waves. The EB equation can be used to model the interaction of bidirectional water waves. Here the speed w is assumed to be at a particular water depth that is denoted by a parameter $\gamma \in [0,1]$, where $\gamma = 0$ means being at the water bottom and $\gamma = 1$, on the water surface. The four parameters a, b, c and d are related to the depth parameter γ by the relationship

$$a + b = \frac{1}{2}\left(\gamma^2 - \frac{1}{3}\right), \tag{3}$$

$$c + d = \frac{1}{2}\left(1 - \gamma^2\right) \geq 0, \tag{4}$$

with the following restriction:

$$a + b + c + d = \frac{1}{3}, \tag{5}$$

which is a requirement for the leading order dispersion of linear waves.

Under the assumption of the perturbation scheme used to derive the EB model, we have the following relationship between η and w:

$$\begin{aligned} &\eta_t = -w_x + \text{h.o.t.}, \\ &\eta_x = -w_t + \text{h.o.t.}, \end{aligned} \tag{6}$$

$$\begin{aligned} &\eta_{xxt} = -w_{xxx} + \text{h.o.t.}, \\ &\eta_{xxx} = -w_{xxt} + \text{h.o.t.}, \end{aligned} \tag{7}$$

where "h.o.t." stands for the higher order terms. Substituting Eq.(7) into the last two terms of the EB equation (1) and neglecting higher order terms yield

$$\begin{aligned} &\eta_t + [(1+\eta)w]_x + (a+b)w_{xxx} = 0, \\ &w_t + \eta_x + ww_x - (c+d)w_{xxt} = 0, \end{aligned} \tag{8}$$

which is a Boussinesq model of constant water-depth. The mathematical properties, such as soliton solu-

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tion, Painlevé integrability, Lie symmetries and solitary wave solutions of Boussinesq model have been studied.[9,10] The difference between the two systems, EB (1) and Boussinesq (8), is small in the physical essence, since we have neglected only some higher order terms. But in the mathematical nature, such as their wellposedness, these two systems are quite different as illustrated in Ref.[6] with both linearized systems and nonlinear systems. Chen[11] has shown that the exact travelling wave solutions of the EB equation may have multiple peaks. The phenomenon of double-peak soliton has been reported in Ref.[12] and studied by Zhang and Li[13] with the integrable Kaup–Boussinesq[14,15] equation, i.e. the EB equation with $b = c = d = 0$.

In this paper our attention is directed to the study of the Painlevé property and travelling wave solutions of the EB equation. We have found new exact solutions and some of them are the solutions found in Ref.[11]. However, most of the exact solutions are not physically meaningful because they violate the assumptions under which the EB system is derived. In order to find a solitary wave solution with physical significance, we turn to a perturbation method and search for an asymptotic formula with one parameter of wave amplitude. We then compare our approximate formula with that of full water wave problem modelled by the Euler equations. The comparison shows that the EB system as a leading order model, is able to match the Laitone's[16] second order asymptotic formula of the Euler equations in terms of wave profile, wave number and wave speed.

2. Painlevé analysis

We begin our study with the Painlevé property of the EB system (1).

According to the definition of the Painlevé property, the general solutions of the EB system have the following expansions around an arbitrary singular manifold $\phi \equiv \phi(x,t) = 0$:

$$\eta = \sum_{j=0}^{\infty} \eta_j \phi^{j+\alpha}, \quad \text{and} \quad w = \sum_{j=0}^{\infty} w_j \phi^{j+\beta}, \tag{9}$$

where α and β are negative finite integers. Furthermore, if the model possesses the Painlevé property, there should be at least one primary branch, i.e. one possible choice of $\{\alpha, \beta, \eta_0, w_0\}$, such that enough arbitrary functions can be included in the Painlevé expansions.

By using a standard leading order analysis, i.e. substituting $\eta = \eta_0 \phi^{\alpha}$ and $w = w_0 \phi^{\beta}$ into the EB system and balancing the leading order terms, we find the following cases of choosing the values of α and β:

$$\text{(A)} \quad b = c = d = 0 \;\Rightarrow\; \alpha = -2, \beta = -1, \tag{10}$$

$$\left.\begin{array}{l}\text{(B1)} \quad c = d = 0, \quad b \neq 0 \quad \text{or} \\ \text{(B2)} \quad a = c = 0, \quad b, d \neq 0\end{array}\right\} \;\Rightarrow\; \alpha = -4, \beta = -2, \tag{11}$$

$$\text{(C)} \quad \text{otherwise} \;\Rightarrow\; \alpha = -2, \quad \beta = -2, \tag{12}$$

if $(a, b) \neq (0, 0)$. If $(a, b) = (0, 0)$, it is impossible to find suitable negative integers for α and β.

Case A $b = c = d = 0 \;\Rightarrow\; \alpha = -2$ and $\beta = -1$. The EB system turns into the Kaup–Boussinesq equation. It is known that the system has the Painlevé property.

Case B1 $c = d = 0,\ b \neq 0 \;\Rightarrow\; \alpha = -4$ and $\beta = -2$.

It is easy to know that

$$w_0 = 20 b \phi_x \phi_t, \quad \text{and} \quad \eta_0 = -200 b^2 \phi_t^2 \phi_x^2,$$

and the resonant points are $j = 4,\ 6$ and 10. But the resonant conditions are not satisfied.

Case B2 $a = c = 0,\ b, d \neq 0 \;\Rightarrow\; \alpha = -4$ and $\beta = -2$.

We have

$$w_0 = 20 b \phi_t \phi_x, \quad \text{and} \quad \eta_0 = 40 b \phi_t^2 \phi_x^2 (3d - 5b).$$

Substituting the above expansions together with the conditions $a = c = 0$ into the EB system and letting the coefficient of the power ϕ^k vanish for each k, we can find the following recursive relationships to determine the expansion functions η_j and w_j:

$$(j-4)[\eta_j - \phi_t\phi_x(j^2d - 5jd - 20b + 6d)w_j] = F_{1j}, \tag{13}$$

$$(j-6)[j(j-9)\eta_j + 40\phi_t\phi_x(5b - 3d)w_j] = F_{2j}, \tag{14}$$

where F_{1j} and F_{2j} are only functions of η_0, η_1, ..., η_{j-1}, w_0, w_1, ..., w_{j-1} and the derivatives of ϕ. From the recursive relations (13) and (14), we know that all the functions η_j and w_j are fixed except those that are related to the resonant points determined by the coefficient determinant of Eqs.(13) and (14):

$$\begin{aligned} &\begin{vmatrix} (j-4) & -\phi_t\phi_x(j^2d - 5jd - 20b + 6d)(j-4) \\ (j-6)j(j-9) & 40\phi_t\phi_x(5b - 3d)(j-6) \end{vmatrix} \\ =\ & \phi_t\phi_x(j+1)(j-4)(j-6)(j^3d - 15j^2d + 66jd - 20jb + 200b - 120d) = 0. \end{aligned} \tag{15}$$

It can be proved that there are not many resonant points for any $b \neq 0$ and $d \neq 0$, except $b = \frac{3}{5}d$. However, when $b = \frac{3}{5}d$, some of the resonant conditions at $j = 0, 4, 6, 6$ and 9 are not satisfied. Therefore, the system does not have the Painlevé property for this case.

Case C $\alpha = -2$ and $\beta = -2$

i) $c = 0$ and $b \neq 2d$.

For this case, we have

$$w_0 = 12d\phi_t\phi_x, \quad \text{and} \quad \eta_0 = \frac{12ad\phi_x^2}{b - 2d}.$$

And the resonant points are determined by

$$(j-4)^2(bj^2 - 5bj - 12d + 6b)d(j+1)(j-6) = 0.$$

Then we know that there are not many resonant points under the conditions $c = 0$ and $b \neq 2d$.

ii) $c = 0,\ b = 2d$.

It can be proved that this case is reduced to either Case A or Case B2.

iii) $c \neq 0$.

For this case, we have

$$w_0 = \phi_x[\phi_t(6d + 3b) \pm 3\sqrt{\phi_t^2(2d - b)^2 + 8ca\phi_x^2}],$$

$$\eta_0 = -\frac{12ac\phi_x^3 + bw_0\phi_t - 12bd\phi_x\phi_t^2}{2c\phi_x}.$$

And the resonant points are determined by

$$(j-4)^2(j+1)(j-6)(j^2 - 5j + 12 - \frac{\phi_t(2d + b)w_0}{\phi_x(bd\phi_t^2 - ca\phi_x^2)}) = 0.$$

This gives us only two cases for which we have enough resonant points:

a) When $a = 0$ and $b = 2d$, the resonant points are $j = -1,\ 4,\ 4,\ 5,\ 6$;

b) When $a = d = 0$, the resonant points are $j = -1,\ 4,\ 4,\ 6$.

But some of resonant conditions are not satisfied.

From the above investigation, we can conclude that except for Case (A), system (1) has no Painlevé property.

3. Function expansion method and explicit exact travelling waves

Though the model holds for the non-Painlevé integrable case, we can still construct some exact solutions by using some suitable Painlevé truncated expansions. According to the Weiss–Tabor–Carnevale (WTC) method,[17] we know that the standard truncated Painlevé expansions of expressions (9) have the form

$$\eta = \sum_{j=0}^{4} \eta_j \phi^{j-4}, \quad \text{and} \quad w = \sum_{j=0}^{2} w_j \phi^{j-2},$$

where $\{\eta_4, w_2\}$ is a seed solution of the EB system and usually taken as a constant solution in order to obtain a single travelling wave solution of the model. In particular, $\eta_0 = \eta_1 = w_0 = 0$ correspond to Case A ($\alpha = -2$ and $\beta = -1$ with $b = c = d = 0$); $\eta_0 = \eta_1 = 0$ correspond to Case C ($\alpha = -2$ and $\beta = -2$); and all the coefficients being non-zero correspond to Case B ($\alpha = -4$ and $\beta = -2$ with $a = c = 0$ and $b, d \neq 0$ or $c = d = 0$ and $b \neq 0$).

In Refs.[18–20], it is pointed out that the Painlevé expansion form expressions (9) can be modified as follows:

$$\eta = \sum_{j=0}^{\infty} \eta_j f^{j-4}, \quad \text{and} \quad w = \sum_{j=0}^{\infty} w_j f^{j-2}, \tag{16}$$

with the expansion function f determined by

$$f_x = \sum_{j=0}^{N} S_j f^j, \quad \text{and} \quad f_t = \sum_{j=0}^{N} C_j f^j. \tag{17}$$

If we take $N \to \infty$ and select the functions S_j and C_j such that the summations become some closed forms, then they can be used to obtain some types of exact solutions of the EB system through the standard and nonstandard truncations of the extended Painlevé expansion.[21] For instance, a particular type of summation form of expression (17) is

$$f_x^2 = s^2(a_0 + a_1 f + a_2 f^2 + a_3 f^3 + a_4 f^4), \quad \text{and} \quad f_t = -\lambda f_x. \tag{18}$$

Its solutions are very rich. They include $\sin, \tan, \exp, \tanh$ and Jacobian elliptic function sn , etc.[22−25] Some of them are listed below:

1. If $a_0 + a_1 f + a_2 f^2 + a_3 f^3 + a_4 f^4$ has four different real roots ranked as $b_1 > b_2 > b_3 > b_4$, that is,

$$a_4 = 1, \quad a_3 = -\sum_{i=1}^{4} b_i, \quad a_2 = \sum_{i>j=1}^{4} b_i b_j, \quad \text{and} \quad a_1 = \sum_{i>j>k=1}^{4} b_i b_j b_k, \quad a_0 = b_1 b_2 b_4 b_3, \tag{19}$$

then expression (18) has general solutions

$$f = \pm \frac{b_2(b_1 - b_3) - b_1(b_2 - b_3)\ \mathrm{sn}^2(k\theta, m)}{b_1 - b_3 - (b_2 - b_3)\ \mathrm{sn}^2(k\theta, m)},$$

$$\theta = s(x - \lambda t), \tag{20}$$

where

$$k = \frac{1}{2}\sqrt{(b_2 - b_4)(b_1 - b_3)}, \quad \text{and} \quad m = \frac{(b_1 - b_4)(b_2 - b_3)}{(b_2 - b_4)(b_1 - b_3)}. \tag{21}$$

2. If we have double roots in case 1, i.e. $b_3 = b_4$ or $b_1 = b_2$, then $m = 1$, the Jacobian elliptic function sn reduces to the tanh function

$$f = \pm \frac{b_2(b_1 - b_3) - b_1(b_2 - b_3)\ \tanh^2(k\theta)}{b_1 - b_3 - (b_2 - b_3)\ \tanh^2(k\theta)}.$$

3. If the four coefficients of expression (18) are given by

$$a_0 = 1, \quad a_1 = a_3 = 0, \quad a_2 = -2lb_2, \quad \text{and} \quad a_4 = b_2^2, \tag{22}$$

i.e. $f_x^2 = s^2(1 - 2lb_2 f^2 + b_2 f^4)$, and $f_t = -\lambda f_x$, then

$$f = \frac{\sqrt{b_2(l - \sqrt{l^2 - 1})}}{b_2}\ \mathrm{sn}(k\theta, m), \tag{23}$$

where

$$k = \sqrt{b_2(l + \sqrt{l^2 - 1})}, \quad \text{and} \quad m = \sqrt{2l^2 - 1 - 2l\sqrt{l^2 - 1}}. \tag{24}$$

4. If we take $l = 1$ and $b_2 = 1$ in case 3, then $f = \tanh(k\theta)$ with $k = 1$.

5. If we take $l = -1$,and $b_2 = 1$ in case 3, then $f = \tan(k\theta)$ with $k = 1$.

The particular forms of f for cases 1 and 2 are carefully chosen so as to generate physically meaningful solutions of the EB system with some particular coefficients a, b, c and d.

On the other hand, there are three kinds of truncations of the extended Painlevé expansion,[26] i.e.

$$\eta = \sum_{j=0}^{m} \eta_j f^{j-m}, \quad \text{and} \quad w = \sum_{j=0}^{n} w_j f^{j-n},$$
$$\eta = \sum_{j=0}^{m} \eta_j f^{j}, \quad \text{and} \quad w = \sum_{j=0}^{n} w_j f^{j},$$
$$\eta = \sum_{j=0}^{2m} \eta_j f^{j-m}, \quad \text{and} \quad w = \sum_{j=0}^{2n} w_j f^{j-n},$$

where m and n are determined by balancing the highest order linear term with the nonlinear terms for the case where $(a, b) \neq (0, 0)$. However, from the last section we know that the EB system possesses one branch in the standard Painlevé analysis except for case C(iii). Therefore one cannot obtain different exact solutions by the three kinds of truncations,[27] that is, these expansions are equivalent except for case C(iii). For simplicity, we directly use the second kind of function expansion with the expansion function f determined by expression (18) to search for the exact solitary wave solutions. Thus, in order to find the travelling wave solutions of the EB equation, we adopt the following expansions:

$$\eta = \sum_{i=0}^{m} \eta_i f^i, \quad \text{and} \quad w = \sum_{i=0}^{n} w_i f^i, \tag{25}$$

where $f = f(\theta) = f(s(x - \lambda t))$ satisfies expression (18). Taking the step similar to the first step of the Painlevé analysis in the last section, substituting expression (25) into Eq.(1) and balancing the highest order linear term with the nonlinear terms for the case where $(a, b) \neq (0, 0)$, we obtain

$$\begin{array}{ll} \text{(I)}\ b=c=d=0 & \Rightarrow m=2,\quad n=1,\\ \left.\begin{array}{l}\text{(II}_1\text{)}\ c=d=0,\ b\neq 0\quad \text{or}\\ \text{(II}_2\text{)}\ a=c=0,\ b,d\neq 0\end{array}\right\} & \Rightarrow m=4,\quad n=2,\\ \text{(III) otherwise} & \Rightarrow m=n=2. \end{array} \tag{26}$$

If $(a,b)=(0,0)$, it is not possible to find positive integers for m and n.

Substituting expressions (25), (26) and (18) into Eq.(1) and letting each coefficient of the powers of f vanish, we can solve η_i and w_j. Omitting the details of solving the over-determined equations, we list the final results below:

Case I $m=2,\ n=1,\ b=c=d=0$ and $a=\dfrac{1}{3}$, the EB system turns into the integrable Kaup–Boussinesq system.

We have

$$w_0=\lambda+\frac{as^2a_3}{w_1},\quad w_1=\pm 2s\sqrt{aa_4},$$

$$\eta_0=\frac{1}{4a_4}(as^2a_3^2-4a_4-4as^2a_4a_2),$$
$$\eta_1=-as^2a_3,\quad \text{and}\quad \eta_2=-2as^2a_4. \tag{27}$$

Substituting expressions (27), (20), (21) and (19) into expression (25) gives a periodic solution of the EB equation. When $b_4=b_3$, this periodic solution reduces to a single soliton solution. Moreover, under the zero boundary condition at infinity, we have

$$\lambda=\pm[1+\sqrt{a}s(b_2-b_3)],\quad \text{and}\quad \sqrt{a}s(b_1-b_2)=2,$$

and the soliton solution can be written as

$$w=\pm\frac{2A(A+2)\mathrm{sech}^2\theta}{A+2-A\tanh^2\theta},\qquad \eta=2A(A+2)\mathrm{sech}^2\theta\frac{A+2+A\tanh^2\theta}{(A+2-A\tanh^2\theta)^2},$$
$$A=\sqrt{a}s(b_2-b_3),\qquad \theta=ks(x-\lambda t),\quad \text{and}\quad k=\frac{1}{2}\sqrt{(b_1-b_3)(b_2-b_3)}.$$

Since $a=1/3$ and s is chosen to be $\sqrt{3}/2$, then $\sqrt{a}s=1/2$. For a right-going soliton, $\lambda>0$, then $A=\sqrt{a}s(b_2-b_3)=\lambda-1$ and $b_1-b_3=b_1-b_2+b_2-b_3=4+2(\lambda-1)=2(\lambda+1)$. The wave number k is determined to be $\sqrt{\lambda^2-1}$. The soliton solution is now written in a one-parameter form

$$w=\frac{2(\lambda^2-1)\mathrm{sech}^2\theta}{(\lambda+1)-(\lambda-1)\tanh^2\theta}=\frac{2(\lambda^2-1)}{\lambda+\cosh 2\theta},\qquad \theta=\frac{\sqrt{3}}{2}\sqrt{\lambda^2-1}\,(x-\lambda t),$$
$$\eta=2(\lambda^2-1)\mathrm{sech}^2\theta\frac{\lambda+1+(\lambda-1)\tanh^2\theta}{[(\lambda+1)-(\lambda-1)\tanh^2\theta]^2}=2(\lambda^2-1)\frac{1+\lambda\cosh 2\theta}{(\lambda+\cosh 2\theta)^2}, \tag{28}$$

which agrees with the formula provided by Zhang and Li.[13]

The periodic solution corresponding to the expansion functions (23), (24) and (22) can be easily obtained. However, when the periodic wave solution reduces to a solitary wave solution, the zero boundary condition cannot be satisfied.

Case II $m=4,\ n=2$ and $c=0$, we find the solution as follows:

$$\begin{aligned} w_0=&\ \frac{\lambda}{12a_4}(15s^2ba_3^2-80s^2ba_2a_4+12a_4)+\frac{a}{\lambda(d-4b)},\\ w_1=&\ -10b\lambda s^2a_3,\quad w_2=-20b\lambda s^2a_4,\\ \eta_0=&\ \frac{bs^4\lambda^2}{18a_4^2}(3d-5b)(432a_0a_4^3+51a_3^2a_4a_2-9a_3^4-64a_2^2a_4^2)\\ &-\frac{48ba_4+80bas^2a_4a_2-15bs^2a_3^2a-12da_4}{12a_4(d-4b)}+\frac{a^2}{\lambda^2(d-4b)^2},\\ \eta_1=&\ \frac{5bs^4a_3}{6a_4}\lambda^2(3d-5b)(16a_4a_2-3a_3^2)+\frac{10abs^2a_3}{d-4b},\\ \eta_2=&\ \frac{5bs^4\lambda^2}{3}(3d-5b)(16a_4a_2+3a_3^2)+\frac{20abs^2a_4}{d-4b},\\ \eta_3=&\ 40b\lambda^2s^4a_3a_4(3d-5b),\quad \eta_4=40b\lambda^2s^4a_4^2(3d-5b). \end{aligned} \tag{29}$$

We also find that the coefficients of $a_i (i = 1, 2, 3, 4)$ are restricted by

$$a_1 = \frac{a_3(4a_4a_2 - a_3^2)}{8a_4^2},$$

which is equivalent to

$$b_4 = b_3 - b_1 + b_2 \tag{30}$$

for coefficients of $b_i (i = 1, 2, 3, 4)$. Substituting expression(30) into expression(19) and eliminating b_4 give

$$a_0 = b_1b_2b_3(b_1 - b_3 + b_2), \quad a_1 = -(b_2 + b_1)(b_1b_2 + b_1b_3 + b_2b_3 - b_3^2),$$

$$a_2 = 3b_1b_2 + b_2b_3 + b_2^2 + b_1b_3 + b_1^2 - b_3^2, \quad \textit{and} \quad a_3 = -2(b_2 + b_1).$$

Substituting these relationships between a_i and b_i into expression (29), we can obtain a periodic travelling wave solution corresponding to the expansion function (20) with expressions (21) and (19). However, this periodic solution cannot be reduced to a solitary wave when $m \to 1$. In order to have $m = 1$, we let $b_3 = b_4$, but the constraint (30) requires that $b_1 = b_2$. If $b_1 = b_2$, the expansion function f becomes a constant b_2. Thus the solutions of w and η are also constants.

Corresponding to the expansion function (23) with expressions (24) and (22), we have

$$\begin{aligned}
&w_0 = \frac{\lambda}{3}(40s^2lbb_2 + 3) - \frac{a}{\lambda(4b - d)}, \quad w_1 = 0, \quad w_2 = -20b\lambda s^2b_2^2,\\
&\eta_0 = \frac{8}{9}bs^4(5b - 3d)b_2^2(16l^2 - 27)\lambda^2 + \frac{40bas^2lb_2 + 3d - 12b}{3(4b - d)} + \frac{a^2}{\lambda^2(4b - d)^2},\\
&\eta_1 = 0, \quad \eta_2 = \frac{160}{3}bs^4(5b - 3d)lb_2^3\lambda^2 - \frac{20abs^2b_2^2}{4b - d},\\
&\eta_3 = 0, \quad \text{and} \quad \eta_4 = -40b\lambda^2s^4b_2^4(5b - 3d).
\end{aligned} \tag{31}$$

We have obtained a solution corresponding to the expansion function (23) with expressions (24) and (22). Especially, if $l = 1$ and $b_2 = 1$, we have $f = \tanh(k\theta)$ and this periodic travelling wave solution is reduced to a single solitary wave. Moreover, under the zero boundary condition at infinity, s and λ are determined to be fixed values as follows:

$$s^2 = \frac{3}{20b} - \frac{3a}{20b\lambda^2(4b - d)}, \quad \text{and}$$

$$\lambda^4 = \frac{100b^2 - 25bd + 45ab - 12ad)\lambda^2}{2(4b - d)(5b - 3d)} - \frac{3a^2(10b - d)}{(4b - d)^2(5b - 3d)}.$$

Then the solitary wave solution is given by

$$w = -\frac{3(a - 4b\lambda^2 + d\lambda^2)}{\lambda(4b - d)}\text{sech}^2[s(x - \lambda t)], \quad \text{and}$$

$$\eta = -\frac{3\text{sech}^2[s(x - \lambda t)]}{4\lambda^2(4b - d)^4}\{[30a^2 - 15\lambda^2(4b - d)(4b + a - d)]\tanh^2[s(x - \lambda t)] + \lambda^2(4b - d)(20b - 5d + a) - 6a^2\}.$$

For a particular case where $a = c = 0$ and $b = d = \frac{1}{6}$, the solution becomes

$$w = 3\lambda\ \text{sech}^2\left(\frac{3}{\sqrt{10}}(x - \lambda t)\right), \quad \text{where} \quad \lambda = \pm\frac{5}{2}, \quad \text{and} \tag{32}$$

$$\eta = \frac{15}{4}\text{sech}^2\left(\frac{3}{\sqrt{10}}(x - \lambda t)\right)\left(3\tanh^2\left(\frac{3}{\sqrt{10}}(x - \lambda t)\right) - 1\right), \tag{33}$$

which is used by Bona and Chen[5] to test their numerical codes.

Case III $m = n = 2$, and $c = 0$ ($\eta_4 = \eta_3 = 0$), we have

$$\begin{aligned} w_0 &= \frac{\lambda}{4a_4}(3ds^2a_3^2 - 16ds^2a_2a_4 + 4a_4) + \frac{a}{\lambda(b-2d)}, \\ w_1 &= -6d\lambda s^2a_3, \quad w_2 = -12d\lambda s^2a_4, \\ \eta_0 &= \frac{16s^2a_2a_4ad + 8da_4 - 4ba_4 - 3ds^2a_3^2a}{4a_4(b-2d)} + \frac{a^2}{\lambda^2(b-2d)^2}, \\ \eta_1 &= \frac{6ds^2a_3a}{b-2d}, \quad \eta_2 = \frac{12ads^2a_4}{b-2d}, \quad \text{and} \quad a_1 = \frac{a_3(4a_4a_2 - a_3^2)}{8a_4^2}. \end{aligned} \tag{34}$$

Corresponding to the expansion function (20) with expressions (21) and (19), we have the same restriction on $b_i (i = 1, 2, 3, 4)$ in expression (30). When $m \to 1$, the expansion function becomes a constant as indicated previously.

Corresponding to the expansion function (23) with expressions (24) and (22), we have

$$\begin{aligned} w_0 &= (8ds^2lb_2 + 1)\lambda + \frac{a}{\lambda(b-2d)}, \\ w_1 &= 0, \quad w_2 = -12d\lambda s^2b_2^2, \\ \eta_0 &= -\frac{8as^2ldb_2 + b - 2d}{b-2d} + \frac{a^2}{\lambda^2(b-2d)^2}, \\ \eta_1 &= 0, \quad \text{and} \quad \eta_2 = \frac{12ads^2b_2^2}{b-2d}. \end{aligned} \tag{35}$$

Similar to case II, if $l = 1$ and $b_2 = 1$, then $f = \tanh(k\theta)$ and the periodic solution is reduced to a solitary wave. Applying the zero boundary condition at infinity gives the fixed values of λ and s as follows:

$$s^2 = \frac{b + a - 2d}{8ad}, \quad \text{and}$$
$$\lambda^2 = \frac{2a^2}{(b-2d)(b-a-2d)}.$$

The solitary wave solution is

$$w = \frac{3a(b + a - 2d)}{\lambda(b-2d)(a+2d-b)}\operatorname{sech}^2[s(x - \lambda t)],$$

$$\eta = -\frac{3(b + a - 2d)}{2(b-2d)}\operatorname{sech}^2[s(x - \lambda t)].$$

Case IV: $m = n = 2$ and $c \neq 0$ ($\eta_4 = \eta_3 = 0$), we have

$$\begin{aligned} w_0 &= -\frac{a_3^2w_2}{16a_4^2} - \frac{\lambda}{2c}(8s^2a_2dc + 4s^2a_2bc - 2c + b) + \frac{6s^2a_4}{cw_2}(1 + 4a_2cs^2)(ca - d\lambda^2b), \\ w_1 &= \frac{w_2a_3}{2a_4}, \quad w_2 = -3s^2a_4(2\lambda d + b\lambda \pm \sqrt{\lambda^2(b-2d)^2 + 8ca}), \\ \eta_0 &= \frac{1}{16a_4c^2}[8a_4c(a-2c) + 4ba_4\lambda^2(b-2d) - c(16a_2a_4 - 3a_3^2)(2ca + b^2\lambda^2)s^2] \\ &\quad + \frac{3\lambda s^2}{4c^2w_2}[cbs^2(16a_4a_2 - 3a_3^2) - 4a_4b + 8da_4](ca - d\lambda^2b), \\ \eta_1 &= -\frac{w_2a_3(12d\lambda s^2a_4 + w_2)}{24a_4^2cs^2}, \quad \eta_2 = \frac{b\lambda w_2}{2c} - \frac{6}{c}s^2a_4(ac - db\lambda^2), \quad \text{and} \\ a_1 &= \frac{a_3(4a_4a_2 - a_3^2)}{8a_4^2}. \end{aligned} \tag{36}$$

Similar to the case above, corresponding to the expansion function (20) with expressions (21) and (19), we have the same relation (30) and when $m \to 1$ the expansion function is reduced to a constant.

Corresponding to the expansion function (23) with expressions (24) and (22), we have

$$\begin{aligned} w_0 &= \frac{w_2(1 - 8b_2cs^2)}{12cs^2b_2^2} + \frac{\lambda(d+c)}{c}, \quad w_1 = 0, \\ w_2 &= -3b_2^2s^2(2d\lambda + b\lambda \pm \sqrt{\lambda^2(b-2d)^2 + 8ac}), \\ \eta_0 &= -\frac{w_2\lambda(8b_2cs^2b - 2d + b)}{24s^2b_2^2c^2} - \frac{d}{2c^2}(8b_2cs^2b - 2d + b)\lambda^2 + \frac{8cb_2s^2a - 2c + a}{2c}, \\ \eta_1 &= 0, \quad \textit{and} \quad \eta_2 = \frac{bw_2\lambda}{2c} + \frac{6}{c}s^2b_2^2(\lambda^2bd - ac). \end{aligned} \tag{37}$$

If we take $l = 1$ and $b_2 = 1$, then $f = \tanh(k\theta)$ and the periodic travelling wave solution is reduced to a solitary wave. Moreover, to satisfy the zero boundary condition, two parameters are fixed as follows:

$$\lambda^2 = \frac{2(a-c)^2}{(c-b+2d)(a-b+2d)}, \quad \text{and}$$
$$s^2 = \frac{b+a-2d-2c}{4(2ad-bc+ac)}.$$

The final solitary solution has the form

$$w = w_0 \text{sech}^2(s(x-\lambda t)), \quad \text{and}$$
$$\eta = \eta_0 \text{sech}^2(s(x-\lambda t)).$$

All of the travelling wave solutions obtained in this section are mathematically appealing because they are exact and explicit. Some solitary wave solutions have been presented by Chen[11] and some periodic wave solutions seem new to us. These solutions are of mathematical interest and can be used to test convergence and accuracy of a numerical code, although most of these exact solutions are not meaningful in physics. For example, the solution shown by the expressions (32) and (33) that is used by Bona and Chen[5] to test their numerical code has no physical significance, because the wave speed is 5/2 and wave amplitude is 15/4. It is well known that the maximum amplitude of a solitary wave on a layer of water with uniform depth (scaled to be 1) is 0.83320 based on the solution of the Euler equations (see e.g. Refs.[3, 28]), and the maximum wave speed is 1.2942. They are far beyond the range of the physical validity.

In order to study a solitary wave solution with physical significance of the EB system, we turn to a perturbation method and search for an approximate formula and investigate the influences of the four parameters a, b, c and d on the wave profile, speed–amplitude and wave number–amplitude relations.

4. Asymptotic solitary wave solution

Our perturbation scheme is based on the same assumption as that for the EB system to be derived, i.e. the amplitude $\alpha \ll 1$, long-wave parameter $\epsilon = k^2 \ll 1$ and $\alpha = O(\epsilon)$. The method adopted here is the one which is used by Wu[3] to find the asymptotic solitary wave solution of full water wave problem.

For the two-dimensional waves progressing in a uniform medium, with their shapes being unchanged, the flow is characterized as the form of $f(x,t) = f(\theta)$, where $\theta = k(x-\lambda t)$, with λ being the undermined constant wave speed. In terms of θ, the expressions in Eq.(1) reduce to an ordinary differential equation which can be integrated once, yielding

$$\begin{aligned} &-\lambda\eta + w + w\eta + ak^2 w_{\theta\theta} + bk^2\lambda\eta_{\theta\theta} = 0, \\ &-\lambda w + \eta + \frac{1}{2}w^2 + ck^2\eta_{\theta\theta} + dk^2\lambda w_{\theta\theta} = 0. \end{aligned} \tag{38}$$

Since $\alpha = O(\epsilon)$, we introduce the following expansions for η and w:

$$\eta = \sum_{n=1}^{N} \epsilon^n \eta_n, \quad \text{and} \quad w = \sum_{n=1}^{N} \epsilon^n w_n, \tag{39}$$

for some $N(< \infty)$. We further assume the following expansion for the wave velocity λ and the phase function $\theta = k(x-\lambda t)$ to be

$$\lambda = \sum_{n=0}^{N} \epsilon^n \lambda_n, \quad \text{and} \quad k = \sum_{n=1}^{N} k_0(1+\epsilon^n k_n). \tag{40}$$

It is convenient to determine η_n and w_n as functions of θ. Substituting these expansions into Eq.(38), we obtain the following hierarchy for various orders in ϵ.

$O(\epsilon)$—for the leading order, we have

$$\begin{aligned} &\lambda_0\eta_1 - w_1 = 0, \\ &\eta_1 - \lambda_0 w_1 = 0. \end{aligned} \tag{41}$$

In order to let η_1 and w_1 be solvable, the determinant of coefficients must vanish; so

$$\lambda_0^2 = 1, \quad \text{and} \quad \eta_1 = \lambda_0 w_1. \tag{42}$$

In the following, we shall work with $\lambda_0 = 1$ for the right-going waves. The leading order wave profile $w_1 = \eta_1$ will be solved after performing the next step of solution.

$O(\epsilon^2)$—for the second order, the perturbation equations are found to be

$$w_2 - \eta_2 - \lambda_1\eta_1 + w_1\eta_1 + ak_0^2 w_{1\theta\theta} + bk_0^2\eta_{1\theta\theta} = 0, \tag{43}$$

$$\eta_2 - w_2 - \lambda_1 w_1 + \frac{w_1^2}{2} + ck_0^2\eta_{1\theta\theta} + dk_0^2 w_{1\theta\theta} = 0. \tag{44}$$

Adding these two equations with $w_1 = \eta_1$ and $a+b+c+d = 1/3$, gives

$$\eta_{1\theta\theta} = \frac{3\eta_1}{2k_0^2}(4\lambda_1 - 3\eta_1). \tag{45}$$

This equation can be integrated once, giving

$$\eta_{1\theta}^2 = \frac{3\eta_1^2}{k_0^2}(2\lambda_1 - \eta_1), \tag{46}$$

where the integration constant is taken as zero so as to ensure that η falls off exponentially towards infinity for the solitary wave. In addition, from the boundary condition that at the wave crest, $\eta_1 = \alpha$ and $\eta_{1\theta} = 0$, we have

$$\eta_1 = \alpha\ \mathrm{sech}^2\theta, \qquad \lambda_1 = \frac{\alpha}{2}, \quad \text{and} \quad k_0^2 = \frac{3\alpha}{4}. \tag{47}$$

This is the solitary solution of the KdV equation (2). Then from Eq.(43), we have

$$w_2 = \eta_2 + \lambda_1\eta_1 - \eta_1^2 - (a+b)k_0^2\eta_{1\theta\theta}, \tag{48}$$

which will be used in the next step.

$O(\epsilon^3)$—for the third order, we find from the original expansions the equations as follows:

$$w_3 = \eta_3 - k_0^2(2bk_1 + 2ak_1 + b\lambda_1)\eta_{1\theta\theta} - ak_0^2 w_{2\theta\theta} - bk_0^2\eta_{2\theta\theta} - \eta_1(\eta_2 + w_2 - \lambda_2) + \lambda_1\eta_2, \tag{49}$$

and

$$4k_0^2\eta_{2\theta\theta} - 12(\alpha - 3\eta_1)\eta_2 = 3A_1\eta_1^3 + 3A_2\eta_1^2 + 3A_3\eta_1, \tag{50}$$

where

$$\begin{aligned} A_1 &= 60c - 270bc - 270ab - 12 + 54a + 114b - 270b^2 - 270ac, \\ A_2 &= 12k_1 + \alpha(270ab + 270ac + 270bc - 114b + 16 + 270b^2 - 66a - 66c), \\ A_3 &= \alpha^2(12c - 36ab - 36bc - 3 - 36ac + 12b + 12a - 36b^2) - 8\alpha k_1 + 8\lambda_2. \end{aligned} \tag{51}$$

It is found from Eq.(45) that $\eta_{1\theta}$ is a solution of the homogeneous equation of Eq.(50), i.e.

$$4k_0^2\eta_{1\theta\theta\theta} - 12(\alpha - 3\eta_1)\eta_{1\theta} = 0, \tag{52}$$

so Eq.(50) can be integrated once by Lagrange's scheme, giving

$$\eta_{1\theta}\eta_{2\theta} - \eta_2\eta_{1\theta\theta} = \frac{3A_1}{16}\eta_1^4 + \frac{A_2}{4}\eta_1^3 + \frac{3A_3}{8}\eta_1^2. \tag{53}$$

Then from the boundary conditions, we have

$$\begin{aligned} \eta_2 &= \frac{\alpha^2}{8}A_1\mathrm{sech}^2\theta\tanh^2\theta, \\ k_1 &= -\frac{\alpha}{24}(135bc - 51a - 57b + 14 + 135b^2 - 42c + 135ab + 135ac), \\ \lambda_2 &= -\frac{\alpha^2}{24}(27bc + 5 + 27ac - 15a - 21b + 27b^2 - 6c + 27ab). \end{aligned} \tag{54}$$

With Eq.(48), it can be obtained that

$$w_2 = \frac{\alpha^2}{6}\mathrm{sech}^2\theta[A_1 + 3 - 18a - 18b + (27a + 27b - A_1 - 6)\mathrm{sech}^2\theta]. \tag{55}$$

Solution can be pursued to higher order by following the same procedure. When all this is carried through, the final result is

$$\eta = \alpha\mathrm{sech}^2\theta + \frac{\alpha^2}{8}A_1\mathrm{sech}^2\theta\tanh^2\theta + \alpha^3\mathrm{sech}^2\theta(B_1 + B_2\mathrm{sech}^2\theta)\tanh^2\theta, \tag{56}$$

$$w = \alpha\mathrm{sech}^2\theta + \frac{\alpha^2}{6}\mathrm{sech}^2\theta[A_1 + 3 - 18a - 18b + (27a + 27b - A_1 - 6)\mathrm{sech}^2\theta] + \alpha^3\mathrm{sech}^2\theta(B_3 + B_4\mathrm{sech}^2\theta + B_5\mathrm{sech}^4\theta), \tag{57}$$

$$\lambda = 1 + \frac{\alpha}{2} - \frac{\alpha^2}{24}(27bc + 5 + 27ac - 15a - 21b + 27b^2 - 6c + 27ab) + \alpha^3\lambda_3 + \cdots, \tag{58}$$

$$k = \frac{\sqrt{3\alpha}}{2}\left[1 - \frac{\alpha}{24}(135bc - 51a - 57b + 14 + 135b^2 - 42c + 135ab + 135ac) + \alpha^2 k_2 + \cdots\right], \tag{59}$$

where $B_i,\ (i = 1, \ldots, 5)$, λ_3 and k_2 are given in Appendix A.

Equations (56), (58) and (59) give us a picture about how the model parameters a, b and c ($d = 1/3 - a - b - c$) affect the wave profile, speed–amplitude relation and wave number–amplitude relation.

The second order approximate solitary wave solution of the Euler equations, provided by Laitone[16], is given as follows:

$$\eta = \alpha\mathrm{sech}^2\theta - \alpha^2\frac{3}{4}\mathrm{sech}^2\theta\tanh^2\theta + O(\alpha^3), \tag{60}$$

$$\lambda = 1 + \frac{1}{2}\alpha - \frac{3}{20}\alpha^2 + O(\alpha^3), \tag{61}$$

$$k = \frac{\sqrt{3\alpha}}{2}\left(1 - \frac{5}{8}\alpha + O(\alpha^2)\right). \tag{62}$$

Since the EB model has three free parameters a, b and c, we can match the asymptotic solution of the EB system, i.e. Eqs.(56), (58) and (59), with Laitone's solution shown in the expressions (60), (61) and (62) by solving the following equations:

$$-\frac{3}{4}(10c - 45bc - 45ab - 2 + 9a + 19b - 45b^2 - 45ac) = -\frac{3}{4}, \tag{63}$$

$$-\frac{1}{24}(27bc + 5 + 27ac - 15a - 21b + 27b^2 - 6c + 27ab) = -\frac{3}{20}, \tag{64}$$

$$-\frac{1}{24}(135bc - 51a - 57b + 14 + 135b^2 - 42c + 135ab + 135ac) = -\frac{5}{8}. \tag{65}$$

We find out an optimal set of parameters

$$a = -\frac{29}{300}, \quad b = \frac{9}{50}, \quad c = -\frac{7}{50}, \quad \text{and} \quad d = \frac{1}{3} - a - b - c = \frac{39}{100}, \tag{66}$$

which implies that the depth parameter $\gamma = 1/\sqrt{2}$. This result shows that the EB model with depth $\gamma = 1/\sqrt{2}$ is able to match Laitone's second order solution. It provides a better prediction than any other leading order model, such as KdV and Kaup–Boussinesq equations, in the sense of the property of a single solitary wave.

We now provide the asymptotic solutions of a solitary wave for a few particular cases.

Case 1 $a = \frac{1}{3}$ and $b = c = 0$, then $d = 0$ and $\gamma = 1$—surface speed system, Kaup–Boussinesq system. The asymptotic solitary solution is

$$\eta = \alpha\mathrm{sech}^2\theta + \frac{3}{4}\alpha^2\mathrm{sech}^2\theta\tanh^2\theta + \frac{\alpha^3}{16}\mathrm{sech}^2\theta\tanh^2\theta(2 - 5\mathrm{sech}^2\theta),$$

$$w = \alpha\mathrm{sech}^2\theta + \frac{\alpha^2}{4}\mathrm{sech}^2\theta\tanh^2\theta - \frac{\alpha^3}{16}\mathrm{sech}^4\theta\tanh^2\theta,$$

$$\lambda = 1 + \frac{1}{2}\alpha, \quad \text{and} \quad k = \frac{\sqrt{3\alpha}}{2}\left(1 + \frac{\alpha}{8} - \frac{\alpha^2}{128}\right),$$

which agrees with the expansion of the exact solution (28), provided by Zhang and Li.[13]

Case 2 $a=0,\ b=\dfrac{1}{6}$ and $c=0$, then $d=\dfrac{1}{6}$ and $\gamma=\dfrac{\sqrt{6}}{3}$, the case that is studied by Bona and Chen.[5] The asymptotic solitary solution is

$$\eta=\alpha\mathrm{sech}^2\theta-\frac{\alpha^2}{16}\mathrm{sech}^2\theta\tanh^2\theta+\frac{\alpha^3}{256}\mathrm{sech}^2\theta\tanh^2\theta(31\mathrm{sech}^2\theta+6),$$
$$w=\alpha\mathrm{sech}^2\theta-\frac{\alpha^2}{16}\mathrm{sech}^2\theta(4-3\tanh^2\theta)+\frac{\alpha^3}{256}\mathrm{sech}^2\theta(6-15\mathrm{sech}^2\theta+41\mathrm{sech}^4\theta),$$
$$\lambda=1+\frac{\alpha}{2}-\frac{3}{32}\alpha^2+\frac{5}{128}\alpha^3,\qquad\text{and}\qquad k=\frac{\sqrt{3\alpha}}{2}\left(1-\frac{11}{32}\alpha+\frac{407}{2048}\alpha^2\right).$$

Case 3 $a=b=c=0$ the $d=1/3$ and $\gamma=\dfrac{1}{\sqrt{3}}$—depth-mean speed system. The asymptotic solitary solution is

$$\eta=\alpha\mathrm{sech}^2\theta-\frac{3}{2}\alpha^2\mathrm{sech}^2\theta\tanh^2\theta-\frac{\alpha^3}{60}\mathrm{sech}^2\theta\tanh^2\theta(143\mathrm{sech}^4\theta-120),$$
$$w=\alpha\mathrm{sech}^2\theta-\frac{\alpha^2}{2}\mathrm{sech}^2\theta(1+\tanh^2\theta)-\frac{\alpha^3}{120}\mathrm{sech}^2\theta(125-136\mathrm{sech}^2\theta+46\mathrm{sech}^4\theta),$$
$$\lambda=1+\frac{\alpha}{2}-\frac{5\alpha^2}{24}+\frac{79}{720}\alpha^3,\quad\text{and}\quad k=\frac{\sqrt{3\alpha}}{2}\left(1-\frac{7\alpha}{12}+\frac{241}{480}\alpha^2\right).$$

The wave speed-amplitude relation agrees with the expansion of the exact formula

$$\lambda=\sqrt{\frac{6(1+\alpha)^2}{\alpha^2(3+2\alpha)}[(1+\alpha)\ln(1+\alpha)-\alpha]}$$

provided by Teng and Wu.[29]

Case 4 $a=-\dfrac{1}{6}$ and $b=c=0$, then $d=1/2$ and $\gamma=0$—bottom speed system. The asymptotic solitary solution is

$$\eta=\alpha\mathrm{sech}^2\theta-\frac{21}{8}\alpha^2\mathrm{sech}^2\theta\tanh^2\theta-\frac{\alpha^3}{320}\mathrm{sech}^2\theta\tanh^2\theta(2177\mathrm{sech}^4\theta-2110),$$
$$w=\alpha\mathrm{sech}^2\theta-\frac{\alpha^2}{8}\mathrm{sech}^2\theta(6+7\tanh^2\theta)-\frac{\alpha^3}{320}\mathrm{sech}^2\theta(1030-1107\mathrm{sech}^2\theta+357\mathrm{sech}^4\theta),$$
$$\lambda=1+\frac{\alpha}{2}-\frac{5\alpha^2}{16}+\frac{27}{80}\alpha^3,\quad\text{and}\quad k=\frac{\sqrt{3\alpha}}{2}\left(1-\frac{15\alpha}{16}+\frac{3659}{2560}\alpha^2\right).$$

5. Conclusions

We have studied the EB equation. Our study covers the Painlevé analysis, and exact travelling (periodic and solitary) wave solutions. Our results show that the EB equation is in general not Painlevé integrable. The exact explicit solutions obtained with a function expansion approach are of mathematical interest and can be used to test numerical codes, even though most of them are not meaningful in physics.

In order to study the effect of the model parameters a, b, c and d on the property of a solitary wave, we turn to the perturbation method of Wu[3] and find an asymptotic formula. We have made a comparison between our formula and Laitone's[16] second order solitary wave solution of the Euler equations, showing that the EB equation with depth parameter $\gamma=1/\sqrt{2}$ is an optimal one because it is able to match Laitone's solution of a single solitary wave. This salient feature distinguishes between the EB equation and all other leading order models including KdV and Kaup–Boussinesq equations. The changes of higher order terms such as η_{xxt} with $-w_{xxx}$ and η_{xxx} with $-w_{xxt}$ in the leading order models seem small in the physical essence. But their influences on the property of a solitary wave are not negligible. This important issue needs to be considered carefully by the model developers and users.

Appendix The coefficients B_i, λ_3 and k_2

$$
\begin{aligned}
B_1 = & -\frac{1}{8}[-6075b^4+(5130-12150a-12150c)b^3-16+162a-6075a^2c^2\\
& +(8640a-24300ac-6075a^2-6075c^2-1713+8370c)b^2+3240ac^2-1614ac\\
& +(-12150ac^2+11880ac-1974c+266-2064a+3240c^2-12150a^2c+3510a^2)b\\
& +160c-360c^2+3510a^2c-351a^2],\\
B_2 = & -\frac{1}{240}(9a+9b-2)(125550abc-20430ab-20430ac+62775ac^2+62775ab^2+1083a\\
& -8910c^2-286+62775bc^2-45810bc+125550b^2c+62775b^3-36900b^2+3348c+6471b),\\
B_3 = & \frac{1}{24}[18225b^4+(36450a-12150+36450c)b^3+25-234a+18225a^2c^2+477a^2+1080c^2\\
& +(72900ac-19440a+18225a^2+18225c^2+3339-21870c)b^2-9720ac^2+3618ac-384c\\
& +(36450ac^2-29160ac+4698c-450+3816a-9720c^2+36450a^2c-7290a^2)b-7290a^2c],\\
B_4 = & \frac{1}{240}[-747225b^4+(465750-1494450a-1494450c)b^3-272+2490a-747225a^2c^2\\
& +(609120a-2988900ac-747225a^2-747225c^2-101619+768690c)b^2+302940ac^2\\
& +(912060ac-1494450ac^2-127242c+9006-73836a+302940c^2-1494450a^2c+143370a^2)b\\
& -65682ac+6996c-28620c^2+143370a^2c-5157a^2],\\
B_5 = & \frac{1}{240}[564975b^4+(1129950a-336150+1129950c)b^3+92-780a+564975a^2c^2\\
& +(2259900ac+564975a^2+564975c^2-398520a+64539-541890c)b^2-205740ac^2\\
& +(1129950ac^2-604260ac+78552c-4356+33246a-205740c^2+1129950a^2c-62370a^2)b\\
& +27792ac-3096c+17820c^2-62370a^2c+1647a^2],\\
\lambda_3 = & -\frac{1}{720}[36450b^4+(72900a-30780+72900c)b^3-79+735a+36450a^2c^2\\
& +(145800ac+36450a^2+36450c^2-38880a+7767-38880c)b^2-8100ac^2\\
& +(72900ac^2-46980ac+4581c-303+2853a-8100c^2+72900a^2c-8100a^2)b\\
& +981ac+42c+180c^2-8100a^2c-1494a^2],\\
k_2 = & \frac{1}{1920}[66825b^4+(133650a-39150+133650c)b^3+964-8100a+66825a^2c^2\\
& +(267300ac+66825a^2+66825c^2-106920a+18333-98010c)b^2-58860ac^2\\
& +(133650ac^2-165780ac+35784c-5772+43002a-58860c^2+133650a^2c-67770a^2)b\\
& +43344ac-5592c+8100c^2-67770a^2c+15489a^2].
\end{aligned}
$$

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