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Discrete doubly periodic and solitary wave solutions for the semi-discrete coupled mKdV equations*

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In this paper, the improved Jacobian elliptic function expansion approach is extended and applied to constructing discrete solutions of the semi-discrete coupled modified Korteweg de Vries (mKdV) equations with the aid of the symbolic computation system Maple. Some new discrete Jacobian doubly periodic solutions are obtained. When the modulus $m \to 1$, these doubly periodic solutions degenerate into the corresponding solitary wave solutions, including kink-type, bell-type and other types of excitations.

Keywords: semi-discrete coupled mKdV equations, extended Jacobian elliptic function expansion approach, discrete doubly periodic solutions, discrete solitary wave solutions

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1. Introduction

Since the work of Fermi, Pasta, and Ulam in the 1950s,[1] differential-difference equations (DDEs) which describe many important phenomena and dynamical processes in many different fields, such as particle vibrations in lattices, currents in electrical networks, pulses in biological chains, nonlinear fibre arrays, nonlinear charge and excitation transport in biological macromolecules, elastic energy transfer in harmonic crystals and so on, have been a focus of many nonlinear studies.[2−6] In the theory of lattice soliton, there are such systematic methods to obtain exact solutions of DDEs as the inverse scattering transformation,[7] Bäcklund transformation,[8] the symmetry reduction,[9] bilinear method,[10] the Wronskian or Casorati determinant technique,[11] numerical method,[12] improved Jacobian elliptic function expansion approach,[13] and so on.[14−19] Recently, Tsuchida *et al*[20] have extended the inverse scattering method to the study of DDEs, Qian *et al*[21] have modified the multilinear variable separation approach to solving a special DDE. More recently, Baldwin *et al*[22] have presented an algorithm to find exact travelling wave solutions of DDEs in terms of tanh function and found kink-type solutions in many spatially discrete nonlinear models such as the celebrated Toda lattice[23] and Ablowitz–Ladik lattice,[24] etc.

In this paper, we extend the improved Jacobian elliptic function expansion approach[13] by means of a new general ansatz, which is more powerful than those methods mentioned above[14−16,22] in terms of uniformly constructing some new discrete exact solutions of DDEs. By use of the approach, we can obtain not only the single Jacobian elliptic function doubly periodic solutions, but also the rational formal Jacobian elliptic function doubly periodic solutions and corresponding degenerative solutions. For illustration, we apply the approach to exploring the semi-discrete coupled modified Korteweg de Vries (mKdV) equations[20]

$$u_{it}(n) = \left(1 + \sum_{j,k=1}^{M} c_{jk} u_j(n) u_k(n)\right) \times (u_i(n+1) - u_i(n-1)), \quad i = 1, 2, \ldots, M \tag{1}$$

and successfully construct some new formal discrete Jacobian doubly periodic solutions. When the modulus of Jacobian elliptic function $m \to 1$, these doubly periodic solutions degenerate into the corresponding solitary wave solutions, including kink-type, bell-type and other types of excitations.

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2. The extended Jacobian elliptic function expansion approach to nonlinear DDEs

Given a system comprising M polynomial DDEs,

$$\Delta(u_{n+\boldsymbol{p}_1}(x),\ldots,u_{n+\boldsymbol{p}_k}(x),u'_{n+\boldsymbol{p}_1}(x),\ldots,u'_{n+\boldsymbol{p}_k}(x),\ldots,u^{(r)}_{n+\boldsymbol{p}_1}(x),\ldots,u^{(r)}_{n+\boldsymbol{p}_k}(x))=0, \tag{2}$$

where the dependent variable $\boldsymbol{u}$ has M components u_i, the continuous variable $\boldsymbol{x}$ has N components x_i, the discrete variable $\boldsymbol{n}$ has Q components n_j, the k shift vectors $\boldsymbol{p}_i \in Z^Q$, and $\boldsymbol{u}^{(r)}(\boldsymbol{x})$ denotes the collection of mixed derivative terms of order r. We suppose that any arbitrary coefficients parameterizing the system are strictly positive and denoted by lower-case Greek letters.

Step 1 When we seek after the travelling wave solutions of Eq.(2), the first step is to introduce the wave transformation

$$\boldsymbol{u}_{\boldsymbol{n}+\boldsymbol{p}_s}(\boldsymbol{x})=u_{n+p_s}(\xi_n)(s=1,2,\ldots,k),\quad \text{and}\quad \xi_n=\sum_{i=1}^{Q}d_in_i+\sum_{j=1}^{N}c_jx_j+\delta, \tag{3}$$

where the coefficients d_i and c_j and the phase δ are all constants to be determined later. In this way, Eq.(2) becomes

$$\Delta(u_{n+p_1}(\xi_n),\ldots,u_{n+p_k}(\xi_n),u'_{n+p_1}(\xi_n),\ldots,u'_{n+p_k}(\xi_n),\ldots,u^{(r)}_{n+p_1}(\xi_n),\ldots,u^{(r)}_{n+p_k}(\xi_n))=0. \tag{4}$$

Step 2 We assume that the travelling wave solutions of Eq.(4) can be expanded as follows:

$$u_n(\xi_n)=a_0+\sum_{j=1}^{l}\frac{f_n^{j-1}(\xi_n)}{(\mu f_n(\xi_n)+1)^j}(a_jf_n(\xi_n)+b_jg_n(\xi_n)+c_jh_n(\xi_n)) \tag{5}$$

with

$$f_n(\xi_n)=\mathrm{sn}\xi_n,\quad g_n(\xi_n)=\mathrm{cn}\xi_n,\quad \text{and}\quad h_n(\xi_n)=\mathrm{dn}\xi_n, \tag{6}$$

where a_0,a_j,b_j and $c_j(j=1,\ldots,l)$ and μ are all constants to be determined later, l is given according to the homogeneous balance principle, $\mathrm{sn}\xi$, $\mathrm{cn}\xi$ and $\mathrm{dn}\xi$ are the Jacobian elliptic sine function, the Jacobian elliptic cosine function, and the Jacobian elliptic function of the third kind, respectively. Other Jacobian elliptic functions denoted by Glaisher symbols[25] are generated by these three kinds of functions, namely

$$\mathrm{ns}\xi_n=\frac{1}{\mathrm{sn}\xi_n},\ \ \mathrm{nc}\xi_n=\frac{1}{\mathrm{cn}\xi_n},\ \ \mathrm{nd}\xi_n=\frac{1}{\mathrm{dn}\xi_n},\ \ \mathrm{sc}\xi_n=\frac{1}{\mathrm{cs}\xi_n}=\frac{\mathrm{sn}\xi_n}{\mathrm{cn}\xi_n},$$

$$\mathrm{sd}\xi_n=\frac{1}{\mathrm{ds}\xi_n}=\frac{\mathrm{sn}\xi_n}{\mathrm{dn}\xi_n},\ \ \mathrm{cd}\xi_n=\frac{1}{\mathrm{dc}\xi_n}=\frac{\mathrm{cn}\xi_n}{\mathrm{dn}\xi_n}, \tag{7}$$

which have the closed relations

$$\mathrm{sn}^2\xi_n+\mathrm{cn}^2\xi_n=1,\ \ \mathrm{dn}^2\xi_n=1-m^2\mathrm{sn}^2\xi_n,\ \ \mathrm{ns}^2\xi_n=m^2+\mathrm{ds}^2\xi_n,\ \ \mathrm{ns}^2\xi_n=1+\mathrm{cs}^2\xi_n,$$

$$\mathrm{sc}^2\xi_n+1=\mathrm{nc}^2\xi_n,\ \ m^2\mathrm{sd}^2\xi_n+1=\mathrm{nd}^2\xi_n,\ \ \mathrm{dc}^2\xi_n=m^2+(1-m^2)\mathrm{nc}^2\xi_n, \tag{8}$$

and

$$\begin{aligned}
&\mathrm{sn}'\xi_n=\mathrm{cn}\xi_n\mathrm{dn}\xi_n,\ \ \mathrm{cn}'\xi_n=-\mathrm{sn}\xi_n\mathrm{dn}\xi_n,\ \ \mathrm{dn}'\xi_n=-m^2\mathrm{sn}\xi_n\mathrm{cn}\xi_n,\ \ \mathrm{ns}'\xi_n=-\mathrm{cs}\xi_n\mathrm{ds}\xi_n,\\
&\mathrm{cs}'\xi_n=-\mathrm{ns}\xi_n\mathrm{ds}\xi_n,\ \ \mathrm{ds}'\xi_n=-\mathrm{ns}\xi_n\mathrm{cs}\xi_n,\ \ \mathrm{sc}'\xi_n=\mathrm{nc}\xi_n\mathrm{dc}\xi_n,\ \ \mathrm{dc}'\xi_n=(1-m^2)\mathrm{sc}\xi_n\mathrm{nc}\xi_n,\\
&\mathrm{nc}'\xi_n=\mathrm{sc}\xi_n\mathrm{dc}\xi_n,\ \mathrm{sd}'\xi_n=\mathrm{cd}\xi_n\mathrm{nd}\xi_n,\ \mathrm{nd}'\xi_n=m^2\mathrm{sd}\xi_n\mathrm{cd}\xi_n,\ \mathrm{cd}'\xi_n=-(1-m^2)\mathrm{sd}\xi_n\mathrm{nd}\xi_n
\end{aligned} \tag{9}$$

with the modulus m $(0 < m < 1)$. In addition, we know that

$$\begin{aligned}
\mathrm{sn}(\xi_1 \pm \xi_2) &= \frac{1}{\mathrm{ns}(\xi_1 \pm \xi_2)} = \frac{\mathrm{sn}\xi_1\mathrm{cn}\xi_2\mathrm{dn}\xi_2 \pm \mathrm{sn}\xi_2 cn\xi_1\mathrm{dn}\xi_1}{1 - m^2\mathrm{sn}^2\xi_1\mathrm{sn}^2\xi_2},\\
\mathrm{cn}(\xi_1 \pm \xi_2) &= \frac{1}{\mathrm{nc}(\xi_1 \pm \xi_2)} = \frac{\mathrm{cn}\xi_1\mathrm{cn}\xi_2 \mp \mathrm{sn}\xi_1\mathrm{dn}\xi_1\mathrm{sn}\xi_2\mathrm{dn}\xi_2}{1 - m^2\mathrm{sn}^2\xi_1\mathrm{sn}^2\xi_2},\\
\mathrm{dn}(\xi_1 \pm \xi_2) &= \frac{1}{\mathrm{nd}(\xi_1 \pm \xi_2)} = \frac{\mathrm{dn}\xi_1\mathrm{dn}\xi_2 \mp m^2\mathrm{sn}\xi_1\mathrm{cn}\xi_1\mathrm{sn}\xi_2\mathrm{cn}\xi_2}{1 - m^2\mathrm{sn}^2\xi_1\mathrm{sn}^2\xi_2},\\
\mathrm{cs}(\xi_1 \pm \xi_2) &= \frac{1}{\mathrm{sc}(\xi_1 \pm \xi_2)} = \frac{\mathrm{cn}\xi_1\mathrm{cn}\xi_2 \mp \mathrm{sn}\xi_1\mathrm{dn}\xi_1\mathrm{sn}\xi_2\mathrm{dn}\xi_2}{\mathrm{sn}\xi_1\mathrm{cn}\xi_2\mathrm{dn}\xi_2 \pm \mathrm{sn}\xi_2\mathrm{cn}\xi_1\mathrm{dn}\xi_1},\\
\mathrm{ds}(\xi_1 \pm \xi_2) &= \frac{1}{\mathrm{sd}(\xi_1 \pm \xi_2)} = \frac{\mathrm{dn}\xi_1\mathrm{dn}\xi_2 \mp m^2\mathrm{sn}\xi_1\mathrm{cn}\xi_1\mathrm{sn}\xi_2\mathrm{cn}\xi_2}{\mathrm{sn}\xi_1\mathrm{cn}\xi_2\mathrm{dn}\xi_2 \pm \mathrm{sn}\xi_2\mathrm{cn}\xi_1\mathrm{dn}\xi_1},\\
\mathrm{cd}(\xi_1 \pm \xi_2) &= \frac{1}{\mathrm{dc}(\xi_1 \pm \xi_2)} = \frac{\mathrm{cn}\xi_1\mathrm{cn}\xi_2 \mp \mathrm{sn}\xi_1\mathrm{dn}\xi_1\mathrm{sn}\xi_2\mathrm{dn}\xi_2}{\mathrm{dn}\xi_1\mathrm{dn}\xi_2 \mp m^2\mathrm{sn}\xi_1\mathrm{cn}\xi_1\mathrm{sn}\xi_2\mathrm{cn}\xi_2}.
\end{aligned} \tag{10}$$

Then, we can obtain

$$u_{n+p_s}(\xi_n) = a_0 + \sum_{j=1}^{l} \frac{f_{n+p_s}^{j-1}(\xi_n)}{(\mu f_{n+p_s}(\xi_n) + 1)^j}(a_j f_{n+p_s}(\xi_n) + b_j g_{n+p_s}(\xi_n) + c_j h_{n+p_s}(\xi_n)) \tag{11}$$

with

$$\begin{aligned}
f_{n+p_s}(\xi_n) &= \mathrm{sn}(\xi_n + \varphi_s) = \frac{\mathrm{sn}\xi_n\mathrm{cn}\varphi_s\mathrm{dn}\varphi_s + \mathrm{sn}\varphi_s cn\xi_n\mathrm{dn}\xi_n}{1 - m^2\mathrm{sn}^2\xi_n\mathrm{sn}^2\varphi_s},\\
g_{n+p_s}(\xi_n) &= \mathrm{cn}(\xi_n + \varphi_s) = \frac{\mathrm{cn}\xi_n\mathrm{cn}\varphi_s - \mathrm{sn}\xi_n\mathrm{dn}\xi_n\mathrm{sn}\varphi_s\mathrm{dn}\varphi_s}{1 - m^2\mathrm{sn}^2\xi_n\mathrm{sn}^2\varphi_s},\\
h_{n+p_s}(\xi_n) &= \mathrm{dn}(\xi_n + \varphi_s) = \frac{\mathrm{dn}\xi_n\mathrm{dn}\varphi_s - m^2\mathrm{sn}\xi_n\mathrm{cn}\xi_n\mathrm{sn}\varphi_s\mathrm{cn}\varphi_s}{1 - m^2\mathrm{sn}^2\xi_n\mathrm{sn}^2\varphi_s},
\end{aligned} \tag{12}$$

where φ_s satisfying

$$\varphi_s = p_{s1}d_1 + p_{s2}d_2 + \cdots + p_{sQ}d_Q. \tag{13}$$

Meanwhile, it is important to note that $u_{n+p_s}(\xi_n)$ is a function of ξ_n but not ξ_{n+p_s}.

Step 3 Substituting the ansatzes (5) and (11) along with Eqs.(6) and (12) into Eq.(4), clearing the denominator and setting the coefficients of power terms like Jacobian elliptic functions to zero, a system of algebraic equations with a_0, a_j, b_j and c_j $(j = 1, \ldots, l)$ and μ are obtained. From these equations, we can obtain the corresponding undetermined coefficients.

Step 4 Substituting the obtained constants a_0, a_j, b_j and $c_j (j = 1, \ldots, l)$ and μ into ansatz (5) along with Eq.(6), a series of explicit and exact Jacobian doubly periodic solutions of DDE (2) are derived. The solitary wave solutions are also derived, when the modulus of Jacobian elliptic function $m \to 1$.

Remark The ansatz proposed here is more general than the ansatzes in the improved Jacobian elliptic function expansion approach[13] and other existing Jacobian elliptic function expansion approaches,[26−29] which can be recovered by our proposed method. We can replace the Jacobian elliptic functions $\mathrm{sn}\xi$, $\mathrm{cn}\xi$, $\mathrm{dn}\xi$ in the ansatz (5) with other triples of Jacobian elliptic functions such as $\mathrm{ns}\xi$, $\mathrm{cs}\xi$, $\mathrm{ds}\xi$; $\mathrm{sc}\xi$, $\mathrm{nc}\xi$, $\mathrm{dc}\xi$; $\mathrm{sd}\xi$, $\mathrm{cd}\xi$, $\mathrm{nd}\xi$. Similarly, other new Jacobian doubly periodic solutions and solitary wave solutions can also be obtained for some equations. Here we omit them.

3. Exact solutions of the semi-discrete coupled mKdV equations

In order to seek after exact solutions of Eqs.(1), we make the transformation

$$u_i(n) = u_{in}(\xi_n), \quad u_j(n) = u_{jn}(\xi_n), \quad u_k(n) = u_{kn}(\xi_n), \quad \xi_n = d_1 n + ct + \delta, \tag{14}$$

$$u_i(n+1) = u_{i(n+1)}(\xi_n), \quad \text{and} \quad u_i(n-1) = u_{i(n-1)}(\xi_n). \tag{15}$$

And then Eqs.(1) become

$$cu'_{in} = \left(1 + \sum_{j,k=1}^{M} c_{jk} u_{jn} u_{kn}\right)(u_{i(n+1)} - u_{i(n-1)}), \quad i = 1, 2, \ldots, M, \tag{16}$$

where $u_{in} \equiv u_{in}(\xi_n)$, $u_{jn} \equiv u_{jn}(\xi_n)$, $u_{kn} \equiv u_{kn}(\xi_n)$, $u_{i(n+1)} \equiv u_{i(n+1)}(\xi_n)$, and $u_{i(n-1)} \equiv u_{i(n-1)}(\xi_n)$.

In the following, we consider two cases: $M = 1$ and $M = 2$.

(1) When $M = 1$, Eqs.(16) reduce to

$$cu'_{1n} = (1 + c_{11} u_{1n}^2)(u_{1(n+1)} - u_{1(n-1)}). \tag{17}$$

We expand the solutions of Eq.(17) in the form of ansatzes (5) and (11). In this case, $p_s = 1$ or -1. According to the homogeneous balance principle, we obtain l=1. Therefore, we have

$$u_{1n} = a_0 + \frac{a_1 \mathrm{sn}\xi_n}{\mu \mathrm{sn}\xi_n + 1} + \frac{b_1 \mathrm{cn}\xi_n}{\mu \mathrm{sn}\xi_n + 1} + \frac{c_1 \mathrm{dn}\xi_n}{\mu \mathrm{sn}\xi_n + 1}, \tag{18}$$

$$u_{1(n+1)} = a_0 + \frac{a_1 \mathrm{sn}(\xi_n + d_1)}{\mu \mathrm{sn}(\xi_n + d_1) + 1} + \frac{b_1 \mathrm{cn}(\xi_n + d_1)}{\mu \mathrm{sn}(\xi_n + d_1) + 1} + \frac{c_1 \mathrm{dn}(\xi_n + d_1)}{\mu \mathrm{sn}(\xi_n + d_1) + 1}, \tag{19}$$

$$u_{1(n-1)} = a_0 + \frac{a_1 \mathrm{sn}(\xi_n - d_1)}{\mu \mathrm{sn}(\xi_n - d_1) + 1} + \frac{b_1 \mathrm{cn}(\xi_n - d_1)}{\mu \mathrm{sn}(\xi_n - d_1) + 1} + \frac{c_1 \mathrm{dn}(\xi_n - d_1)}{\mu \mathrm{sn}(\xi_n - d_1) + 1}. \tag{20}$$

Substituting Eqs.(18)–(20) into Eq.(17), clearing the denominator and setting the coefficients of power terms like $\mathrm{sn}^r\xi_n \mathrm{cn}^s\xi_n \mathrm{dn}^\lambda\xi_n$ ($r = 0, 1, \ldots, 6; s = 0, 1; \lambda = 0, 1$) to zero, yield a set of algebraic equations in $a_0, a_1, b_1, c_1, \mu, d_1$ and c. With the aid of symbolic computation system Maple, we can obtain the following results:

Case 1

$$a_0 = b_1 = c_1 = \mu = 0, \quad d_1 = d_1, \quad a_1 = \pm\frac{1}{\sqrt{-c_{11}}} m \mathrm{sn}(d_1), \quad \text{and} \quad c = 2\mathrm{sn}(d_1); \tag{21}$$

Case 2

$$a_0 = a_1 = c_1 = \mu = 0, \quad d_1 = d_1, \quad b_1 = \pm\frac{1}{\sqrt{c_{11}}} m \mathrm{sd}(d_1), \quad \text{and} \quad c = 2\mathrm{sd}(d_1); \tag{22}$$

Case 3

$$a_0 = a_1 = b_1 = \mu = 0, \quad d_1 = d_1, \quad c_1 = \pm\frac{1}{\sqrt{c_{11}}} \mathrm{sc}(d_1), \quad \text{and} \quad c = 2\mathrm{sc}(d_1); \tag{23}$$

Case 4

$$b_1 = c_1 = 0, \quad \mu = \mu, \quad d_1 = d_1, \quad a_0 = \pm\frac{(\mu^2 \mathrm{sn}^2(d_1) + \mathrm{cn}(d_1)\mathrm{dn}(d_1) - 1)\mu}{\sqrt{c_{11}(\mu^2 m^2 \mathrm{sn}^4(d_1) - (\mu^4 + m^2)\mathrm{sn}^2(d_1) - \mu^2(\mathrm{cn}^2(d_1)\mathrm{dn}^2(d_1) - 1))}},$$

$$a_1 = \mp\frac{(\mu^4 + m^2)\mathrm{sn}^2(d_1) + 2\mu^2(\mathrm{cn}(d_1)\mathrm{dn}(d_1) - 1)}{\sqrt{c_{11}(\mu^2 m^2 \mathrm{sn}^4(d_1) - (\mu^4 + m^2)\mathrm{sn}^2(d_1) - \mu^2(\mathrm{cn}^2(d_1)\mathrm{dn}^2(d_1) - 1))}}, \quad \text{and}$$

$$c = -\frac{2\mathrm{sn}(d_1)((\mu^4 + m^2)\mathrm{sn}^2(d_1) + 2\mu^2(\mathrm{cn}(d_1)\mathrm{dn}(d_1) - 1))}{\mu^2 m^2 \mathrm{sn}^4(d_1) - (\mu^4 + m^2)\mathrm{sn}^2(d_1) - \mu^2(\mathrm{cn}^2(d_1)\mathrm{dn}^2(d_1) - 1)}. \tag{24}$$

From Eqs.(18) and (14) and Cases 1–4, we obtain the following Jacobian doubly periodic solutions for Eqs.(1) with $M = 1$, namely

$$u_{1,1}(n) = \pm\frac{1}{\sqrt{-c_{11}}}m\mathrm{sn}(d_1)\mathrm{sn}(d_1 n + 2\mathrm{sn}(d_1)t + \delta), \tag{25}$$

where $c_{11} < 0, d_1$ and δ are arbitrary constants;

$$u_{1,2}(n) = \pm\frac{1}{\sqrt{c_{11}}}m\mathrm{sd}(d_1)\mathrm{cn}(d_1 n + 2\mathrm{sd}(d_1)t + \delta), \tag{26}$$

where $c_{11} > 0, d_1$ and δ are arbitrary constants;

$$u_{1,3}(n) = \pm\frac{1}{\sqrt{c_{11}}}\mathrm{sc}(d_1)\mathrm{dn}(d_1 n + 2\mathrm{sc}(d_1)t + \delta), \tag{27}$$

where $c_{11} > 0, d_1$ and δ are arbitrary constants; and

$$u_{1,4}(n) = a_0 + \frac{a_1\mathrm{sn}(d_1 n + ct + \delta)}{\mu\mathrm{sn}(d_1 n + ct + \delta) + 1}, \tag{28}$$

where a_0, a_1 and c are determined by Eq.(24), μ, d_1 and δ are arbitrary constants.

When $m \to 1$, the solutions (25) and (26) degenerate into kink-type and bell-type solitary wave solutions

$$u_{1,1'}(n) = \pm\frac{1}{\sqrt{-c_{11}}}\tanh(d_1)\tanh(d_1 n + 2\tanh(d_1)t + \delta), \tag{29}$$

and

$$u_{1,2'}(n) = \pm\frac{1}{\sqrt{c_{11}}}\sinh(d_1)\mathrm{sech}(d_1 n + 2\sinh(d_1)t + \delta), \tag{30}$$

respectively. The solution (28) degenerates into rational solitary wave solution

$$u_{1,4'}(n) = a_0 + \frac{a_1\tanh(d_1 n + ct + \delta)}{\mu\tanh(d_1 n + ct + \delta) + 1}, \tag{31}$$

where

$$a_0 = \pm\frac{(\mu^2\tanh^2(d_1) + \mathrm{sech}^2(d_1) - 1)\mu}{\sqrt{c_{11}(\mu^2\tanh^4(d_1) - (\mu^4+1)\tanh^2(d_1) - \mu^2(\mathrm{sech}^4(d_1) - 1))}},$$

$$a_1 = \mp\frac{(\mu^4+1)\tanh^2(d_1) + 2\mu^2(\mathrm{sech}^2(d_1) - 1)}{\sqrt{c_{11}(\mu^2\tanh^4(d_1) - (\mu^4+1)\tanh^2(d_1) - \mu^2(\mathrm{sech}^4(d_1) - 1))}},$$

$$c = -\frac{2\tanh(d_1)((\mu^4+1)\tanh^2(d_1) + 2\mu^2(\mathrm{sech}^2(d_1) - 1))}{\mu^2\tanh^4(d_1) - (\mu^4+1)\tanh^2(d_1) - \mu^2(\mathrm{sech}^4(d_1) - 1)}.$$

(2) When $M = 2$, Eqs.(16) reduce to

$$\begin{cases} cu'_{1n} = (1 + c_{11}u_{1n}^2 + (c_{12} + c_{21})u_{1n}u_{2n} + c_{22}u_{2n}^2)(u_{1(n+1)} - u_{1(n-1)}), \\ cu'_{2n} = (1 + c_{11}u_{1n}^2 + (c_{12} + c_{21})u_{1n}u_{2n} + c_{22}u_{2n}^2)(u_{2(n+1)} - u_{2(n-1)}). \end{cases} \tag{32}$$

By taking the procedure similar to the above, we can also obtain Jacobian doubly periodic solutions of Eqs.(1) with $M = 2$. These solutions are

$$\begin{cases} u_{1,1}(n) = a_1\mathrm{sn}(d_1 n + ct + \delta), \\ u_{2,1}(n) = A_1\mathrm{sn}(d_1 n + ct + \delta), \end{cases} \tag{33}$$

where

$$a_1 = \frac{-A_1(c_{12} + c_{21}) \pm \sqrt{A_1^2((c_{12} + c_{21})^2 - 4c_{11}c_{22}) - 4c_{11}m^2\mathrm{sn}^2(d_1)}}{2c_{11}}, \quad c = \pm 2\mathrm{sn}(d_1),$$

A_1, d_1 and δ are arbitrary constants;

$$\begin{cases} u_{1,2}(n) = a_0 + a_1 \mathrm{sn}(d_1 n + ct + \delta), \\ u_{2,2}(n) = A_1 \mathrm{sn}(d_1 n + ct + \delta), \end{cases} \tag{34}$$

where

$$a_0 = \pm \frac{\sqrt{A_1^2((c_{12}+c_{21})^2 - 4c_{11}c_{22}) - 4c_{11}m^2\mathrm{sn}^2(d_1)}}{2c_{11}m\mathrm{sn}(d_1)}, \quad a_1 = -\frac{A_1(c_{12}+c_{21})}{2c_{11}},$$
$$c = \frac{A_1^2((c_{12}+c_{21})^2 - 4c_{11}c_{22})}{2c_{11}m^2\mathrm{sn}(d_1)},$$

A_1, d_1 and δ are arbitrary constants;

$$\begin{cases} u_{1,3}(n) = a_0 + b_1 \mathrm{cn}(d_1 n + ct + \delta), \\ u_{2,3}(n) = B_1 \mathrm{cn}(d_1 n + ct + \delta), \end{cases} \tag{35}$$

where

$$a_0 = \pm \frac{\sqrt{B_1^2(4c_{11}c_{22} - (c_{12}+c_{21})^2)\mathrm{dn}^2(d_1) - 4c_{11}m^2\mathrm{sn}^2(d_1)}}{2c_{11}m\mathrm{sn}(d_1)}, \quad b_1 = -\frac{B_1(c_{12}+c_{21})}{2c_{11}},$$
$$c = \frac{B_1^2(4c_{11}c_{22} - (c_{12}+c_{21})^2)\mathrm{dn}(d_1)}{2c_{11}m^2\mathrm{sn}(d_1)},$$

B_1, d_1 and δ are arbitrary constants;

$$\begin{cases} u_{1,4}(n) = a_0 + c_1 \mathrm{dn}(d_1 n + ct + \delta), \\ u_{2,4}(n) = C_1 \mathrm{dn}(d_1 n + ct + \delta), \end{cases} \tag{36}$$

where

$$a_0 = \pm \frac{\sqrt{C_1^2(4c_{11}c_{22} - (c_{12}+c_{21})^2)\mathrm{cn}^2(d_1) - 4c_{11}\mathrm{sn}^2(d_1)}}{2c_{11}\mathrm{sn}(d_1)}, \quad c_1 = -\frac{C_1(c_{12}+c_{21})}{2c_{11}},$$
$$c = \frac{C_1^2(4c_{11}c_{22} - (c_{12}+c_{21})^2)\mathrm{cn}(d_1)}{2c_{11}\mathrm{sn}(d_1)},$$

C_1, d_1 and δ are arbitrary constants; and

$$\begin{cases} u_{1,5}(n) = a_0 + \dfrac{a_1 \mathrm{sn}(d_1 n + ct + \delta)}{\mu \mathrm{sn}(d_1 n + ct + \delta) + 1}, \\ u_{2,5}(n) = \dfrac{A_1 \mathrm{sn}(d_1 n + ct + \delta)}{\mu \mathrm{sn}(d_1 n + ct + \delta) + 1}, \end{cases} \tag{37}$$

where

$$\begin{aligned} a_1 = & -\frac{(c_{12}+c_{21})A_1}{2c_{11}} + \frac{1}{2c_{11}\mathrm{sn}(d_1)\mathrm{cn}(d_1)\mathrm{dn}(d_1)}\{\pm\sqrt{-c_{11}}(m^2\mathrm{sn}^4(d_1) + 2\mathrm{cn}(d_1)\mathrm{dn}(d_1) - 1) \\ & \pm \mathrm{sqrt}[((c_{12}+c_{21})^2 - 4c_{11}c_{22})A_1^2\mathrm{sn}^2(d_1)\mathrm{cn}^2(d_1)\mathrm{dn}^2(d_1) \\ & -c_{11}(m^2\mathrm{sn}^4(d_1)(m^2\mathrm{sn}^4(d_1) + 4\mathrm{cn}(d_1)\mathrm{dn}(d_1) - 2) + (2\mathrm{cn}(d_1)\mathrm{dn}(d_1) - 1)^2)]\}, \end{aligned}$$
$$a_0 = \pm\frac{1}{\sqrt{-c_{11}}}, \quad c = \pm\frac{1}{\sqrt{-c_{11}}}((c_{12}+c_{21})A_1 + 2c_{11}a_1)\mathrm{sc}(d_1)\mathrm{sd}(d_1), \quad \mu = \pm\mathrm{ns}(d_1),$$

$c_{11} < 0, A_1, d_1$ and δ are arbitrary constants.

When $m \to 1$, the solutions (33) and (34) degenerate into kink-type solitary wave solutions

$$\begin{cases} u_{1,1'}(n) = a_1 \tanh(d_1 n + ct + \delta), \\ u_{2,1'}(n) = A_1 \tanh(d_1 n + ct + \delta), \end{cases} \tag{38}$$

where

$$a_1=\frac{-A_1(c_{12}+c_{21})\pm\sqrt{A_1^2((c_{12}+c_{21})^2-4c_{11}c_{22})-4c_{11}\tanh^2(d_1)}}{2c_{11}},\quad c=\pm2\tanh(d_1),$$

A_1, d_1 and δ are arbitrary constants; and

$$\begin{cases} u_{1,2'}(n)=a_0+a_1\tanh(d_1n+ct+\delta),\\ u_{2,2'}(n)=A_1\tanh(d_1n+ct+\delta),\end{cases}\tag{39}$$

where

$$a_0=\pm\frac{\sqrt{A_1^2((c_{12}+c_{21})^2-4c_{11}c_{22})-4c_{11}\tanh^2(d_1)}}{2c_{11}\tanh(d_1)},\quad a_1=-\frac{A_1(c_{12}+c_{21})}{2c_{11}},$$
$$c=\frac{A_1^2((c_{12}+c_{21})^2-4c_{11}c_{22})}{2c_{11}\tanh(d_1)},$$

A_1, d_1 and δ are arbitrary constants.

The solution (35) degenerates into bell-type solitary wave solution

$$\begin{cases} u_{1,3'}(n)=a_0+b_1\mathrm{sech}(d_1n+ct+\delta),\\ u_{2,3'}(n)=B_1\mathrm{sech}(d_1n+ct+\delta),\end{cases}\tag{40}$$

where

$$a_0=\pm\frac{\sqrt{B_1^2(4c_{11}c_{22}-(c_{12}+c_{21})^2)\mathrm{sech}^2(d_1)-4c_{11}\tanh^2(d_1)}}{2c_{11}\tanh(d_1)},\quad b_1=-\frac{B_1(c_{12}+c_{21})}{2c_{11}},$$
$$c=\frac{B_1^2(4c_{11}c_{22}-(c_{12}+c_{21})^2)\mathrm{sech}(d_1)}{2c_{11}\tanh(d_1)},$$

B_1, d_1 and δ are arbitrary constants.

The solution (37) degenerates into rational solitary wave solution

$$\begin{cases} u_{1,5'}(n)=a_0+\dfrac{a_1\tanh(d_1n+ct+\delta)}{\mu\tanh(d_1n+ct+\delta)+1},\\ u_{2,5'}(n)=\dfrac{A_1\tanh(d_1n+ct+\delta)}{\mu\tanh(d_1n+ct+\delta)+1},\end{cases}\tag{41}$$

where

$$\begin{aligned} a_1 =& -\frac{(c_{12}+c_{21})A_1}{2c_{11}}+\frac{1}{2c_{11}\tanh(d_1)\mathrm{sech}^2(d_1)}\{\pm\sqrt{-c_{11}}(\tanh^4(d_1)+2\mathrm{sech}^2(d_1)-1)\\ &\pm\mathrm{sqrt}[((c_{12}+c_{21})^2-4c_{11}c_{22})A_1^2\tanh^2(d_1)\mathrm{sech}^4(d_1)\\ &-c_{11}(\tanh^4(d_1)(\tanh^4(d_1)+4\mathrm{sech}^2(d_1)-2)+(2\mathrm{sech}^2(d_1)-1)^2)]\},\end{aligned}$$

$$a_0=\pm\frac{1}{\sqrt{-c_{11}}},\quad c=\pm\frac{1}{\sqrt{-c_{11}}}((c_{12}+c_{21})A_1+2c_{11}a_1)\sinh^2(d_1),\quad \mu=\pm\coth(d_1),$$

$c_{11}<0, A_1, d_1$ and δ are arbitrary constants.

4. Conclusions

Based on a new general ansatz and the symbolic computation, we have extended the improved Jacobian elliptic function expansion approach to constructing discrete solutions of the semi-discrete coupled mKdV equations, and obtained many new discrete Jacobian doubly periodic solutions, kink-type solitary wave solutions, bell-type solitary wave solutions and rational solitary wave solutions. We believe that these solutions will be useful in further numerical analysis of various solutions for the semi-discrete coupled mKdV equations, and this method can also be applied to searching for new solutions of other nonlinear DDEs.

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