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Unified symmetry of Vacco dynamical systems

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Based on the total time derivative along the trajectory of the time, we study the unified symmetry of Vacco dynamical systems. The definition and the criterion of the unified symmetry for the system are given. Three kinds of conserved quantities, i.e. the Noether conserved quantity, the generalized Hojman conserved quantity and the Mei conserved quantity, are deduced from the unified symmetry. An example is presented to illustrate the results.

Keywords: Vacco dynamical system, unified symmetry, conserved quantity
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1. Introduction

The research on symmetries and conserved quantities of a mechanical system has very important significance in mathematics, mechanics, and physics. The modern methods to find conserved quantities for constrained mechanical systems mainly include Noether symmetry,[1] Lie symmetry[2] and Mei symmetry[3,4] (the form invariance), which can respectively lead to the Noether conserved quantity,[1,5,6] the Hojman conserved quantity[7,8] and the Mei conserved quantity.[9] A lot of results[5−16] about symmetry and conserved quantity have been achieved. Recently, Mei *et al*[17] have given a unified symmetry of a Lagrangian system, which could lead to the three kinds of conserved quantities mentioned above. Wu[18] have studied the Lie- form invariance of holonomic mechanical systems, from which two kinds of conserved quantities are found. Since then, some results of unified symmetry have been obtained.[19−26] These researches have developed the symmetrical theory to a certain degree and paved the way for finding more conserved quantities of mechanical systems.

From 1982 to 1983, Chetaev proposed a new mathematical model which could be used to study the un-accumulated constrained mechanical systems. In 1990, Guo obtained a new equation which has the same form as that of Vacco dynamical equation. Many people have been involved in the study of the new equation and obtained a series of results.[27−31] Zhang[32] gave the Lie symmetry for the Vacco dynamic systems. Gu and Zhang presented the Mei symmetry, Lie symmetry and Noether symmetry of a Vacco system.[33] In this paper, we study the unified symmetry of Vacco dynamical systems.

The rest of this paper is organized as follows. The definition and the criterion of the unified symmetry for the system are given in Section 2. Three kinds of conserved quantities, i.e. the Noether conserved quantity, the generalized Hojman conserved quantity and the Mei conserved quantity, which can be deduced from the unified symmetry, are presented in Section 3. An illustrative example is given in Section 4.

2. Definition and criterion of unified symmetry of Vacco dynamical mechanical systems

Suppose that the configuration of a mechanical system composed of N particles is determined by n generalized coordinates $q_s (s = 1, \ldots, n)$. Letting its motion be subjected to the g ideal bilateral Vacco-nonholonomic constraints

$$\phi_\beta(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = 0, \quad (\beta = 1, \ldots, g), \tag{1}$$

the equations of motion for the Vacco dynamical system can be written as

$$E_s(L) = Q_s - \lambda_\beta \left[\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \phi_\beta}{\partial \dot{q}_s} - \frac{\partial \phi_\beta}{\partial q_s} \right] - \dot{\lambda}_\beta \frac{\partial \phi_\beta}{\partial \dot{q}_s}, \quad (s = 1, \ldots, n), \tag{2}$$

where $L(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = T - V$ is the Lagrangian of the Vacco dynamical system, $Q_s = Q_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ are the

generalized non-potential forces, λ_β are the constraint multipliers and E_s are the Euler operators

$$E_s = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}_s} - \frac{\partial}{\partial q_s}, \quad (s = 1, \ldots, n). \tag{3}$$

Before integrating Eqs.(2), we can obtain the constraints multipliers

$$\lambda_\beta = \lambda_\beta(t), \quad (\beta = 1, \ldots, g). \tag{4}$$

Introducing the generalized Lagrangian of the system

$$L' = L + \lambda_\beta \phi_\beta, \quad (\beta = 1, \ldots, g), \tag{5}$$

then Eqs.(2) can be written as

$$E_s(L') = Q_s, \quad (s = 1, \ldots, n). \tag{6}$$

Supposing that the system is non-singular, i.e.

$$\det(h_{sk}) = \det\left(\frac{\partial^2 L'}{\partial \dot{q}_s \partial \dot{q}_k}\right) \neq 0, \tag{7}$$

and expanding Eqs.(6), we can obtain all the generalized accelerations

$$\ddot{q}_s = \alpha_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \quad (s = 1, \ldots, n). \tag{8}$$

2.1. Definition[17]

If a symmetry is a Noether one as well as a Lie one and is also a form invariance at one time, then the symmetry is called a unified symmetry.

Introduce the following infinitesimal transformations of time and generalized coordinates:

$$\begin{aligned} &t^* = t + \varepsilon\xi_0(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \quad \text{and} \\ &q_s^*(t^*) = q_s(t) + \varepsilon\xi_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \quad (s = 1, \ldots, n), \end{aligned} \tag{9}$$

where ε is the infinitesimal parameter, $\xi_0(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ and $\xi_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ are the infinitesimal transformations.

The invariance of Vacco-nonholonomic constraints (1) under the infinitesimal transformations (9) leads to the following restrictive equations:

$$X^{(1)}(\phi_\beta) = 0, \quad (\beta = 1, \ldots, g). \tag{10}$$

For Vacco dynamical systems (2), the Noether identity takes the form

$$L'\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0 + X^{(1)}(L') + Q_s(\xi_s - \dot{q}_s\xi_0) + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}G_N = 0. \tag{11}$$

The determining equations of the Lie symmetry are

$$X^{(2)}\left\{\bar{E}_s(L')\right\} = X^{(1)}(Q_s). \tag{12}$$

The criterion equations of the Mei symmetry are

$$\bar{E}_s\{X^{(1)}(L')\} = X^{(1)}(Q_s), \tag{13}$$

where

$$\begin{aligned} X^{(1)} &= \xi_0\frac{\partial}{\partial t} + \xi_s\frac{\partial}{\partial q_s} + \left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s - \dot{q}_s\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0\right)\frac{\partial}{\partial \dot{q}_s}, \\ X^{(2)} &= X^{(1)} + \left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s - 2\ddot{q}_s\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0 - \dot{q}_s\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0\right)\frac{\partial}{\partial \ddot{q}_s}, \\ \frac{\bar{\mathrm{d}}}{\mathrm{d}t} &= \frac{\partial}{\partial t} + \dot{q}_s\frac{\partial}{\partial q_s} + \alpha_s\frac{\partial}{\partial \dot{q}_s}, \\ \bar{E}_s &= \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}_s} - \frac{\partial}{\partial q_s}. \end{aligned} \tag{14}$$

Based on the total time derivative along the trajectory of the time, by using the above definition, we can obtain a criterion as follows.

2.2. Criterion

For the Vacco dynamical system, if there exists such a gauge function $G_N = G_N(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ that the infinitesimal generators ξ_0 and ξ_s satisfy the following condition:

$$\begin{aligned} &\left[L'\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0 + X^{(1)}(L') + Q_s(\xi_s - \dot{q}_s\xi_0) + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}G_N\right]^2 \\ &+\{X^{(2)}[\bar{E}_s(L')] - X^{(1)}(Q_s)\}^2 + \left\{\bar{E}_s\left[X^{(1)}(L')\right] - X^{(1)}(Q_s)\right\}^2 = 0, \end{aligned} \tag{15}$$

and

$$X^{(1)}(\phi_\beta)=0, \quad (\beta=1,\ldots,g), \tag{16}$$

then the symmetry is called the unified symmetry for the Vacco dynamical system.

3. The conserved quantities deduced from the unified symmetry

Under some conditions, the unified symmetry of Vacco dynamical system (2) can lead to the Noether conserved quantity, the generalized Hojman conserved quantity, and the Mei conserved quantity.

Proposition 1 For Vacco dynamical system (2), the unified symmetry leads to the Noether conserved quantity

$$I_N = L'\xi_0 + \frac{\partial L'}{\partial \dot{q}_s}(\xi_s - \dot{q}_s\xi_0) + G_N = \text{const.} \tag{17}$$

Proof Because the unified symmetry of Vacco dynamical system (2) is certainly a Noether symmetry, there exists such a gauge function $G_N = G_N(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ that the identity (11) holds. By using the Noether theorem, the system has the conserved quantity (17).

Proposition 2 For Vacco dynamical system (2), if there exists such a function $G_{\rm H} = G_{\rm H}(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ that the generators of the infinitesimal transformations ξ_0 and ξ_s satisfy

$$\frac{\partial \alpha_s}{\partial \dot{q}_s} + \frac{\bar{\rm d}}{{\rm d}t}\ln G_{\rm H} = 0 \tag{18}$$

then the unified symmetry leads to the generalized Hojman conserved quantity

$$I_{\rm H} = \frac{1}{G_{\rm H}}\frac{\partial}{\partial t}(G_{\rm H}\xi_0) + \frac{1}{G_{\rm H}}\frac{\partial}{\partial q_s}(G_{\rm H}\xi_s) + \frac{1}{G_{\rm H}}\frac{\partial}{\partial \dot{q}_s}\left[G_{\rm H}\left(\frac{\bar{\rm d}}{{\rm d}t}\xi_s - \dot{q}_s\frac{\bar{\rm d}}{{\rm d}t}\xi_0\right)\right] - \frac{\bar{\rm d}}{{\rm d}t}\xi_0 = \text{const.} \tag{19}$$

Proof Based on the definition and the criterion of the unified symmetry of Vacco dynamical system (2), the unified symmetry must be a Lie symmetry, so we have the determining equations of the Lie symmetry

$$\frac{\bar{\rm d}}{{\rm d}t}\frac{\bar{\rm d}}{{\rm d}t}\xi_s - \dot{q}_s\frac{\bar{\rm d}}{{\rm d}t}\frac{\bar{\rm d}}{{\rm d}t}\xi_0 - 2\alpha_s\frac{\bar{\rm d}}{{\rm d}t}\xi_0 = X^{(1)}(\alpha_s). \tag{20}$$

Using Eqs.(18) and (20), we can verify proposition 2 by using a method similar to that in Ref.[8].

Proposition 3 For Vacco dynamical system (2), if there exists such a function $G_{\rm M} = G_{\rm M}(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ that the generators of the infinitesimal transformations ξ_0 and ξ_s satisfy

$$X^{(1)}(L')\frac{\bar{\rm d}}{{\rm d}t}\xi_0 + X^{(1)}\left\{X^{(1)}(L')\right\} + X^{(1)}(Q_s)(\xi_s - \dot{q}_s\xi_0) + \frac{\bar{\rm d}}{{\rm d}t}G_{\rm M} = 0, \tag{21}$$

then the unified symmetry leads to the generalized Mei conserved quantity

$$I_{\rm M} = X^{(1)}(L')\xi_0 + \frac{\partial X^{(1)}(L')}{\partial \dot{q}_s}(\xi_s - \dot{q}_s\xi_0) + G_{\rm M} = \text{const.} \tag{22}$$

Proof Based on the definition and the criterion of the unified symmetry of Vacco dynamical system (2), the unified symmetry must be a Mei symmetry, so the criterion equations (13) hold. Using Eqs.(13) and (21), we can prove that the system has the conserved quantity (22).

4. Illustrative example

Consider a dynamic system with Lagrangian

$$L = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2) \tag{23}$$

under the nonpotential generalized forces:

$$Q_1 = Q_2 = Q_3 = 0, \tag{24}$$

thus the motion of the particles is subjected to a Vacco-nonholonomic constraint

$$\phi = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2) - C = 0. \tag{25}$$

Now we try to study the unified symmetry and conserved quantities of the system.

The differential equations of the system can be expressed as

$$\begin{aligned} &(1+\lambda)\ddot{q}_1 + \dot{\lambda}\dot{q}_1 = 0, \\ &(1+\lambda)\ddot{q}_2 + \dot{\lambda}\dot{q}_2 = 0, \\ &(1+\lambda)\ddot{q}_3 + \dot{\lambda}\dot{q}_3 = 0. \end{aligned} \tag{26}$$

From Eqs.(25) and (26), we have

$$\lambda = \lambda_0 = \text{const}. \tag{27}$$

The generalized Lagrangian is

$$L' = \frac{1+\lambda_0}{2}(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2) - \lambda_0 C. \tag{28}$$

Taking the generators of the infinitesimal transformations as

$$\xi_0 = \dot{q}_1 + \dot{q}_2 + \dot{q}_3, \quad \xi_1 = \dot{q}_1, \quad \xi_2 = \dot{q}_2, \quad \xi_3 = \dot{q}_3, \tag{29}$$

it is easy to verify that the corresponding symmetry is a unified symmetry of Vacco dynamical system(23) and (25).

From Noether identity (11), we have

$$G_{\rm N} = t\dot{q}_1 - q_1 + \dot{q}_2. \tag{30}$$

Using Eqs.(29) and (30), according to proposition 1, we obtain

$$I_{\rm N} = C(\dot{q}_1 + \dot{q}_2 + \dot{q}_3) + t\dot{q}_1 - q_1 + \dot{q}_2 = \text{const}, \tag{31}$$

which is the Noether conserved quantity deduced from the unified symmetry (29).

Equation (18) gives

$$\frac{\bar{\rm d}}{{\rm d}t}\ln G_{\rm H} = 0, \tag{32}$$

which has the solution

$$G_{\rm H} = t\dot{q}_1 - q_1 + \dot{q}_2. \tag{33}$$

Substituting Eqs.(29) and (33) into Eq.(19), we have

$$I_{\rm H} = \frac{\dot{q}_1(\dot{q}_1 + \dot{q}_2 + \dot{q}_3) - \dot{q}_1}{t\dot{q}_1 - q_1 + \dot{q}_2} = \text{const}, \tag{34}$$

which is the generalized Hojman conserved quantity deduced from the unified symmetry (29).

From Eq.(21), we have

$$\frac{\bar{\rm d}}{{\rm d}t}\ln G_{\rm M} = 0, \tag{35}$$

which has the solution

$$G_{\rm M} = t\dot{q}_1 - q_1 + \dot{q}_2. \tag{36}$$

Substituting Eqs.(29) and (36) into Eq.(22), we have

$$I_{\rm M} = t\dot{q}_1 - q_1 + \dot{q}_2 = \text{const}, \tag{37}$$

which is the Mei conserved quantity deduced from the unified symmetry (29).

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