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The open-plus-closed loop (OPCL) method for chaotic systems with multiple strange attractors*

Song Yun-Zhong (宋运忠)†

Complex Networks Laboratory, College of Electrical Engineering and Automation, Henan Polytechnic University, Jiaozuo 454000, China

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Based on the open-plus-closed-loop (OPCL) control method a systematic and comprehensive controller is presented in this paper for a chaotic system, that is, the Newton–Leipnik equation with two strange attractors: the upper attractor (UA) and the lower attractor (LA). Results show that the final structure of the suggested controller for stabilization has a simple linear feedback form. To keep the integrity of the suggested approach, the globality proof of the basins of entrainment is also provided. In virtue of the OPCL technique, three different kinds of chaotic controls of the system are investigated, separately: the original control forcing the chaotic motion to settle down to the origin from an arbitrary position of the phase space; the chaotic intra-attractor control for stabilizing the equilibrium points only belonging to the upper chaotic attractor or the lower chaotic one; and the inter-attractor control for compelling the chaotic oscillation from one basin to another one. Both theoretical analysis and simulation results verify the validity of the proposed means.

Keywords: chaos, OPCL control, the Newton–Leipnik equation attractor
PACC: 0545, 0547

1. Introduction

Chaotic control is a topic occupying an important position and receiving much attention in physics and engineering applications. Many researchers have extensively investigated several issues related to the topic; various effective schemes have been developed, such as linear state feedback,[1] the more general synthesis Lyapunov-type method,[2] the feedback linearization technique,[3] adaptive control,[4] the bang–bang controller,[5] prediction feedback control via historical information,[6] the inverse system method,[7] Linear Matrix Inequalities (LMI) based control,[8] etc.

Recently, the concept of the open-plus-closed-loop (OPCL) method has aroused the new interest in nonlinear controls.

Lemma 1[9] Let $g(t)$ be an arbitrary smooth function of t and $x(t)$ be a solution of

$$\frac{\mathrm{d}x}{\mathrm{d}t} = F(x) + S(t)K(g,x), \tag{1}$$

where $x \in R^n$, $S(t) = 0$ if $t < t_0$, and $S(t) = 1$ if $t \geq t_0$, while $F(x)$ is everywhere Lipshitz. There exists a function $K(g,x)$, linear in $x(t)$, such that none of the basins of entrainment which is associated with $g(t)$ and t_0, defined by

$$\mathrm{BE}(g|t_0) = \left\{ x(t_0) | \lim_{t\to\infty} |x(t) - g(t)| = 0 \right\}, \tag{2}$$

is empty sets.

This can be explained using an interaction, $K(g,x)$, which is the sum of Huble's open-loop interaction and a linear closed-loop interaction, namely

$$K(g,x) = \frac{\mathrm{d}g}{\mathrm{d}t} - F(g) + C(g)(g(t) - x(t)), \tag{3a}$$

where the matrix $C(g)$ varies with time according to

$$C(g) = \frac{\mathrm{d}F(g)}{\mathrm{d}g} - A. \tag{3b}$$

and $A = (a_{ij})$ is an arbitrary constant matrix whose eigenvalues all have negative real parts. Interaction (3) is referred to as the OPCL interaction.

Setting $x(t) = g(t) + u(t)$, where $u(t_0)$ is arbitrarily small, substituting it into Eq.(1), and when $t \geq t_0$, expanding $F(g(t) + u(t))$ for small $u(t)$ yield the variational equations (i.e. the linear-approximation equations)

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\mathrm{d}F(g)}{\mathrm{d}t}u(t) - C(g)u(t) = Au(t). \tag{4}$$

Since all eigenvalues of A have negative real parts, the asymptotic stability, expression (2), of Eq.(4) is es-

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†E-mail: songhpu@126.com

tablished. It then follows that this asymptotic stability also holds for the nonlinear system (1), and hence BE(g) is not an empty set

This is a remarkably general theorem, regardless of whether $g(t)$ is a solution of Eq.(1), without any control action and $g(t)$ restricted to any region of the phase space. In retrospect, it is clear that this generality comes simply from the interaction (3) that includes the cancellation of the first two terms of a Taylor expansion of $F(u+g)$, for small u, plus a stabilizing linear feedback action. What is not established by the theorem is the range of the basin of entrainment, BE(g) in the phase space, and it is this issue that is essential to this method.

In this paper we use the OPCL technique to control a chaotic system: the Newton–Leipnik system, in which there coexist two strange attractors. The control condition for stabilizing the unstable equilibria as well as the dynamics of this chaotic system transferring between arbitrary attractors related to the Newton–Leipnik system are touched upon. The resulting controller structure proposed here is in a very simple linear feedback form. Using the OPCL method it is also very easy to prove the stability of a closed-loop system.

The rest of this paper is organized as follows. In Section 2 the OPCL control of the Newton–Leipnik system is described in detail. Namely, a procedure to design an OPCL controller is presented. In Section 3, the numerical results are given. Finally, the conclusions drawn from the present study are presented in Section 4.

2. OPCL control of the Newton–Leipnik system

The Newton–Leipnik system was obtained by modifying Euler's rigid body equations with the addition of a linear feedback in 1981.[10] The differential equations of the Newton–Leipnik chaotic system are as follows:[10]

$$\begin{aligned}\dot{x}_1 &= -\gamma x_1 + x_2 + 10x_2x_3,\\ \dot{x}_2 &= -x_1 - 0.4x_2 + 5x_1x_3,\\ \dot{x}_3 &= \sigma x_3 - 5x_1x_2,\end{aligned} \tag{5}$$

where x_1, x_2 and x_3 represent the angular velocities about principal axes, which are fixed in the body, γ and σ are the positive parameters related to the eigenvalues of inertia moments of the body. The Newton–Leipnik system model shows a double strange attractor for the nominal parameters $\gamma = 0.4$ and $\sigma = 0.175$. For the initial states $x_0^{(1)} = (0.349, 0, -0.160)^{\mathrm{T}}$ and $x_0^{(2)} = (0.349, 0, -0.180)^{\mathrm{T}}$, we obtain two strange attractors (see Fig.1).

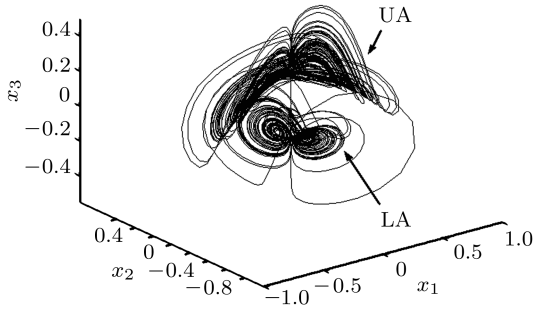


Fig.1. The double strange attractors of the Newton–Leipnik system.

As discussed in the previous work on the Newton–Leipnik system,[10−12] we refer to the attractor A1 in the upper space as the upper attractor (UA) and the attractor A2 in the lower space as the lower attractor (LA). The equilibrium points of the system (5) with the nominal parameters $\gamma = 0.4$ and $\sigma = 0.175$ are

$$\begin{aligned}x_{\mathrm{e}}^{(1)} &= (0, 0, 0)^{\mathrm{T}},\\ x_{\mathrm{e}}^{(2)} &= (0.2390, 0.0308, 0.2103)^{\mathrm{T}},\\ x_{\mathrm{e}}^{(3)} &= (-0.2390, -0.0308, 0.2103)^{\mathrm{T}},\\ x_{\mathrm{e}}^{(4)} &= (0.0315, -0.1224, -0.1103)^{\mathrm{T}},\\ x_{\mathrm{e}}^{(5)} &= (-0.0315, 0.1224, -0.1103)^{\mathrm{T}}.\end{aligned}$$

All these equilibrium points are unstable. Note that the points $x_{\mathrm{e}}^{(2)}$ and $x_{\mathrm{e}}^{(3)}$ lie in the eye-centres of UA, while the points $x_{\mathrm{e}}^{(4)}$ and $x_{\mathrm{e}}^{(5)}$are situated in the eye-centres of LA.

Theorem 1 The basins of entrainment of the Newton–Leipnik chaotic system are global.

Proof To study this question, we substitute

$$x(t) = g(t) + u(t) \tag{6}$$

into Eq.(1) (with $S(t) = 1$), and expand $F(g+u)$ to obtain the nonlinear generalization of Eq.(4),

$$\begin{aligned}\frac{\mathrm{d}u_i}{\mathrm{d}t} = \; & a_{ij}u_j + \frac{1}{2}\frac{\partial^2 F_i(g,t)}{\partial g_j \partial g_k}u_j u_k\\ & + \frac{1}{3!}\frac{\partial^3 F_i(g,t)}{\partial g_j \partial g_k \partial g_l}u_j u_k u_l + \ldots,\end{aligned} \tag{7}$$

where $A = (a_{ij})$ is the matrix in Eq.(4), repeated indices refer to the summation over $(1, \ldots, n)$, and the higher order derives are added, required by the form of $F(x)$. In this paper $F(x)$ is a second-order polynomial (See Eq.(5)), so there are no additional terms except the first two ones. Also, for simplicity, we restrict our controls to

$$a_{ij} = \delta_{ij}, a_{ii} < 0. \tag{8}$$

As introduced before, the Newton–Leipnik system is characterized by

$$\begin{aligned} F_1 &= -\gamma x_1 + x_2 + 10 x_2 x_3, \\ F_2 &= -x_1 - 0.4 x_2 + 5 x_1 x_3, \\ F_3 &= \sigma x_3 - 5 x_1 x_2, \end{aligned} \tag{9}$$

so that

$$\begin{aligned} \frac{\mathrm{d}u_1}{\mathrm{d}t} &= a_{11} u_1 + 10 u_2 u_3, \\ \frac{\mathrm{d}u_2}{\mathrm{d}t} &= a_{22} u_2 + 5 u_1 u_3, \\ \frac{\mathrm{d}u_3}{\mathrm{d}t} &= a_{33} u_3 - 5 u_1 u_2, \end{aligned} \tag{10}$$

which do not contain $g(t)$. This is due to the fact that the polynomials in Eq.(10) contain no more than second-order terms. If we choose the Lyapunov energy function as

$$V = \frac{1}{2}(u_1^2 + u_2^2 + 3u_3^2), \tag{11}$$

since

$$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}t} &= \sum_{i=1}^{3} \frac{\partial V}{\partial u_i} \frac{\mathrm{d}u_i}{\mathrm{d}t} \\ &= \frac{\partial V}{\partial u_1} \frac{\mathrm{d}u_1}{\mathrm{d}t} + \frac{\partial V}{\partial u_2} \frac{\mathrm{d}u_2}{\mathrm{d}t} + \frac{\partial V}{\partial u_3} \frac{\mathrm{d}u_3}{\mathrm{d}t} \\ &= \frac{1}{2}(2u_1(a_{11}u_1 + 10u_2u_3) + 2u_2(a_{22}u_2 + 5u_1u_3) \\ &\quad + 6u_3(a_{33}u_3 - 5u_1u_2)) \\ &= a_{11}u_1^2 + a_{22}u_2^2 + 3a_{33}u_3^2 \le 0 \end{aligned} \tag{12}$$

and by using expression (8), i.e. all $a_{ii} < 0$ (nonzero), then $\|u\|$ must approach zero. Hence, for the Newton–Leipnik chaotic system BE(g) we have the following result: The basin of entrainment BE(g) is global for all values of $g(t)$ and $\|u\| \sim \exp(at)$, where $a = \max(a_{11}, a_{22}, a_{33}) < 0$, which is a remarkably strong result.

Now we use the aforementioned OPCL method to realize the control objectives. Rewrite Eqs.(1), (3a) and (3b), and (4) as follows:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = F(x) + S(t)K(g, x), \tag{1′}$$

$$K(g, x) = \frac{\mathrm{d}g}{\mathrm{d}t} - F(g) + C(g)(g(t) - x(t)), \tag{3a′}$$

$$C(g) = \frac{\mathrm{d}F(g)}{\mathrm{d}g} - A, \tag{3b′}$$

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\mathrm{d}F(g)}{\mathrm{d}t} u(t) - C(g)u(t) = Au(t). \tag{4′}$$

For stabilizing $x_{\mathrm{e}}^{(\mathrm{i})}, i \in \{1, 2, 3, 4, 5\}$, there must be

$$\begin{aligned} & g(t) = x_{\mathrm{e}}^{(i)}, \\ & \frac{\mathrm{d}g}{\mathrm{d}t} = \frac{\mathrm{d}\left(x_{\mathrm{e}}^{(i)}\right)}{\mathrm{d}t} = 0, \\ & F(g) = F\left(x_{\mathrm{e}}^{(i)}\right) = 0, \\ & C(g) = \frac{\mathrm{d}F(g)}{\mathrm{d}g} - A = -A. \end{aligned}$$

So

$$\begin{aligned} K(g, x) &= \frac{\mathrm{d}g}{\mathrm{d}t} - F(g) + C(g)(g(t) - x(t)) \\ &= -A\left(x_{\mathrm{e}}^{(i)} - x(t)\right). \end{aligned} \tag{13}$$

Then Eq.(1) turns into

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= F(x) + S(t)K(g, x) \\ &= F(x) + S(t)(-A(x_{\mathrm{e}}^{(i)} - x(t))). \end{aligned}$$

In this paper we choose matrix A as

$$\begin{aligned} A &= \mathrm{diag}(a_{11}, a_{22}, a_{33}) \\ &= \mathrm{diag}(-2, -4, -6). \end{aligned}$$

3. Simulation results

Usually, the parameters of the Newton–Leipnik system are chosen as $\gamma = 0.4$, and $\sigma = 0.175$. First, we check the original control. The Newton–Leipnik system starts from $x_0 = [0.5, -1, 2]^{\mathrm{T}}$. According to the OPCL control law, the closed-loop system converges to the origin $[0, 0, 0]^{\mathrm{T}}$; the results are shown in Fig.2.

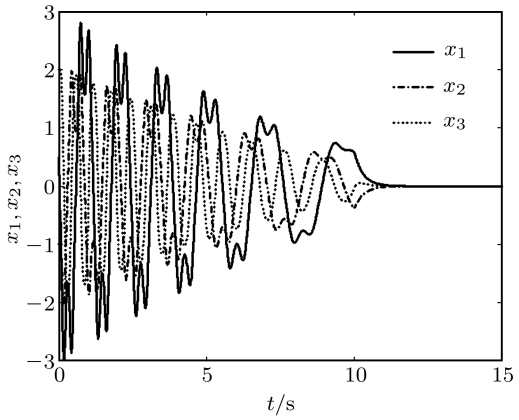


Fig.2. The Newton–Leipnik system driven from $(0.5, -1, 2)^{\mathrm{T}}$ to the origin, with the control action applied from $t = 10\,\mathrm{s}$ on.

We note that the system is free of chaotic transients and reaches the origin eternally. The intra-attractor control is to stabilize the equilibrium points, which are affiliated with the upper attractor (UA) and the lower attractor (LA), separately. Now we manage to force the Newton–Leipnik system into the set-points like $x_{\mathrm{e}}^{(2)}$ and $x_{\mathrm{e}}^{(3)}$, and the system starting from $x_0 = [0.1, -0.1, 0.1]^{\mathrm{T}}$ to $x_{\mathrm{e}}^{(2)}$ is shown in Fig.3.

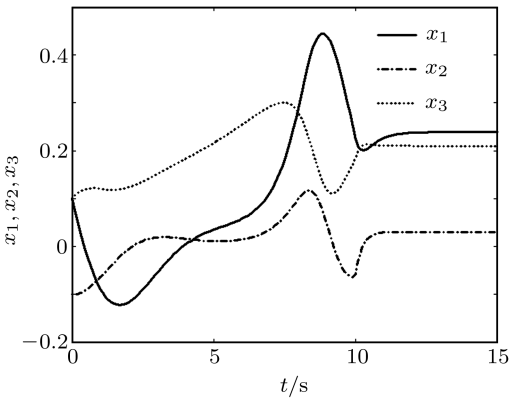


Fig.3. The Newton–Leipnik system driven from $(0.1, -0.1, 0.1)^{\mathrm{T}}$ to $x_{\mathrm{e}}^{(2)}$, with the control action applied from $t = 10\,\mathrm{s}$ on.

Similarly, we can drive the Newton–Leipnik system into the set-points like $x_{\mathrm{e}}^{(4)}$ and $x_{\mathrm{e}}^{(5)}$, and the system starting from $x_0 = [0.01, 0.15, -0.2]^{\mathrm{T}}$ to $x_{\mathrm{e}}^{(4)}$ is shown in Fig.4. Now we turn to an inter-attractor control. Starting from an initial state $x_0 = [0.349, 0, -0.160]^{\mathrm{T}}$ belonging to the state basin of the UA, the trajectory settles down on this attractor after the transient. From $t = 10\mathrm{s}$ on, the control action aiming at achieving the equilibrium of the origin is turned on, and at $t = 20\mathrm{s}$ the control action aiming at achieving the equilibrium of $x_{\mathrm{e}}^{(4)}$ that is laying in the LA is turned on; while at $t = 30\mathrm{s}$ all the control actions are set free from the system so that the system returns to chaotic state again, this is because the accumulated error of the computation is enlarged by the Newton–Leipnik system (see Fig.5).

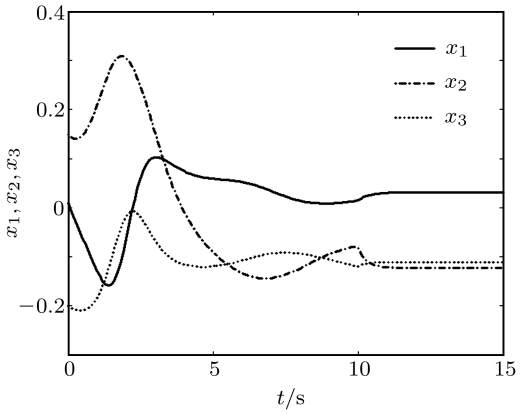


Fig.4. The Newton–Leipnik system driven from $(0.01, 0.15, -0.2)^{\mathrm{T}}$ to $x_{\mathrm{e}}^{(4)}$, with the control action applied from $t = 10\,\mathrm{s}$ on.

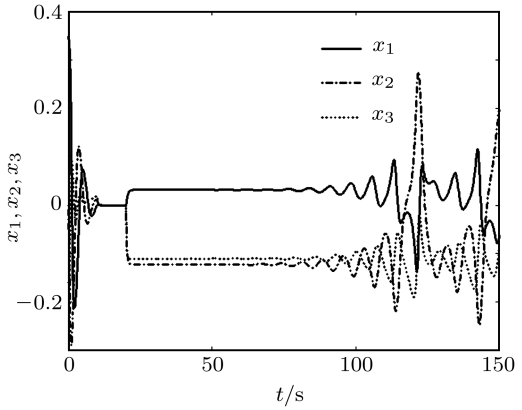


Fig.5. Inter-attractor control taking the system dynamics from UA to the origin at $t = 10\,\mathrm{s}$, and from the origin to the equilibrium $x_{\mathrm{e}}^{(4)}$ of LA at $t = 20\,\mathrm{s}$; and then back to chaotic state at $t = 30\,\mathrm{s}$ again.

4. Conclusions

In this paper, the control of a chaotic system with multiple attractors is demonstrated. The analytical procedure is somewhat complex because we use an OPCL method, but the final structure is easily accepted: a simple linear state feedback control. Using

this special linear controller we can assume the stability of a closed-loop system. Such a control can stabilize a desired motion within one attractor, as well as take the system dynamics from one attractor to another.

Acknowledgments

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References

[1] Chen G and Dong X 1993 *IEEE Trans. Circ. Syst.* **40** 591
[2] Nijmeijer H and Berghuis H 1995 *IEEE Trans. Circ. Syst.* **42** 473
[3] Gallegos J A 1994 *Dynamic Contr.* **4** 277
[4] Zeng Y and Singh S N 1997 *Dynamic Contr.* **7** 143
[5] Vincent T L and Yu J 1991 *Dynamic Contr.* **1** 35
[6] Tang G L and Luo X S 2004 *Acta Phys. Sin.* **53** 115 (in Chinese)
[7] Ren H P and Liu D 2002 *Chin. Phys. Lett.* **19** 1054
[8] Tan W, Wang Y N, Duan F and Li X H 2006 *Chin. Phys.* **15** 2529
[9] Jackson E A and Grosu I 1995 *Physica* D **85** 1
[10] Leipnik R B and Newton T A 1981 *Phys. Lett.* A **86** 63
[11] Hendrik R 2002 *Phys. Lett.* A **300** 182
[12] Song Y Z, Zhao G Z and Qi D L 2006 *Chin. Phys.* **15** 2266