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Impulsive generalized synchronization of chaotic system*

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In this paper, with a given manifold $y = H(x)$, we have constructed a response system for a continuous-time chaotic system as a drive system, and used impulsive control theory to demonstrate theoretically that this response system can achieve impulsive generalized synchronization (GS) with the drive system. Our theoretical result is supported by numerical examples.

Keywords: synchronization manifold, impulsive generalized synchronization, impulsive control, impulsive differential equation

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1. Introduction

The synchronization of chaotic systems has received considerable attention, especially in recent years. And there have been presented many chaotic synchronization methods, such as drive-response synchronization method,[1] variable structure control method,[2] and mutually coupled synchronization method[3] and so on. Most of them are devoted to the study of identical synchronizations (ISs).

However, the IS occurs at a certain point in the system parameter space, and often is difficult to achieve except under ideal conditions. Recently a generalization concept, called generalized synchronization (GS), has been studied.[4−7] Because there always exist parameter mismatches and distortions in the physical world, the applications of GSs may be more extensive and practical than those of ISs. The chaotic communication based on the GS has been studied.[8−11] Lu and Xi[12] have developed a general approach by constructing a response system to implement the linear generalized synchronization with a drive continuous-time chaotic system.

The impulsive synchronization has been presented[13] and studied.[14−21] The impulsive control gives an efficient method to deal with the systems which cannot endure any continuing disturbances. Additionally, in a synchronization process, the response system receives the information about the active variables of the driving system only at discrete amounts of time. This drastically reduces the amount of information transmitted from the driving system to the response system, which makes this method more efficient and useful in applications.

In this paper with a given manifold $y = H(x)$, we construct a response system for a continuous-time chaotic system as a drive system, and use the impulsive control theory to demonstrate theoretically that this response system can achieve the impulsive GS with the drive system. Our theoretical result is supported by numerical examples.

The rest of the paper is organized as follows. In Section 2 is given the basic theory of impulsive differential equations. An impulsive generalized synchronization scheme is developed in Section 3. Numerical examples are given in Section 4 to illustrate the effectiveness of the main result. Finally, some concluding remarks are presented in Section 5.

2. Basic theory of impulsive differential equations

An impulsive differential equation is described by[14]

$$\begin{cases} \dot{x} = f(t,x), & t \neq \tau_i, \\ \Delta x = x(\tau_i^+) - x(\tau_i) = U(i,x), & t = \tau_i,\ i = 1, 2, \ldots, \\ x(t_0^+) = x_0, & t_0 \geq 0, \end{cases} \tag{1}$$

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where $f : R_+ \times R^n \mapsto R^n$ is a continuous function, $x \in R^n$ is the state variable, and a discrete instant set $\{\tau_i\}$ satisfies: $0 < \tau_1 < \tau_2 < \cdots < \tau_i < \tau_{i+1} < \cdots$, $\tau_i \to \infty$ as $i \to \infty$.

We use the following definitions and theorems:[14]

Definition 1 Let $V : R_+ \times R^n \mapsto R_+$, then V is said to belong to class υ_0 if

(1) V is continuous in $(\tau_{i-1}, \tau_i] \times R^n$ and for each $x \in R^n$, $i = 1, 2, \ldots,$

$$\lim_{(t,y)\to(\tau_i^+,x)} V(t,y) = V(\tau_i^+, x);$$

(2) V is locally Lipschitzian in x.

Definition 2 For $(t,x) \in (\tau_{i-1}, \tau_i] \times R^n$, we define

$$D^+V(t,x) = \lim_{h\to 0} \sup \frac{1}{h}[V(t+h, x + hf(t,x)) - V(t,x)].$$

Definition 3 Letting $V \in \upsilon_0$ and assuming that

$$\begin{cases} D^+V(t,x) \le g(t, V(t,x)), & t \ne \tau_i, \\ V(t, x + U(i,x)) \le \psi_i(V(t,x)), & t = \tau_i, \end{cases} \tag{2}$$

where $g : R_+ \times R_+ \mapsto R$ is continuous and $\psi_i : R_+ \mapsto R_+$ is non-descending, then the system

$$\begin{aligned} &\dot{w} = g(t,w), \quad t \ne \tau_i, \quad w(\tau_i^+) = \psi_i(w(\tau_i)), \\ &w(\tau_0^+) = w_0 \ge 0, \end{aligned} \tag{3}$$

is the comparison system of Eq.(1).

Definition 4 $S_\rho = \{x \in R^n \,|\, ||\,x\,|| < \rho\}$, where $||\cdot||$ denotes the Euclidean norm on R^n.

Definition 5 A function α is said to belong to class κ if $\alpha \in C[R_+, R_+]$, $\alpha(0) = 0$ and $\alpha(x)$ is strictly ascending with x.

Theorem 1 Assume that the following three conditions:

(1) $V : R_+ \times S_\rho \mapsto R_+, \rho > 0, V \in \upsilon_0$, $D^+V(t,x) \le g(t, V(t,x))$, $t \ne \tau_i$;

(2) There exists a $\rho_0 > 0$ such that $x \in S_{\rho_0}$ implies that $x + U(i,x) \in S_{\rho_0}$ for all i and $V(t, x + U(i,x)) \le \phi_i(V(t,x))$, $t = \tau_i, x \in S_{\rho_0}$;

(3) $\beta(||\,x\,||) \le V(t,x) \le \alpha(||\,x\,||)$ on $R_+ \times S_\rho$, where $\alpha(||\,x\,||)$, $\beta(||\,x\,||) \in \kappa$

are satisfied, then the stability properties of the trivial solution of the comparison system (3) will imply the corresponding stability properties of the trivial solution of system (1).

Theorem 2 Let $g(t,w) = \dot{\lambda}(t)w$, $\lambda \in C^1[R_+, R_+]$, $\psi_i(w) = d_i w$, $d_i \ge 0$ for all i, then the origin of system (1) will be asymptotically stable if the conditions $\lambda(\tau_{i+1}) + \ln(\gamma d_i) \le \lambda(\tau_i)$, for all i where $\gamma > 1$, and $\dot{\lambda}(t) \ge 0$ are satisfied.

3. Impulsive generalized synchronization with a synchronization manifold $y = H(x)$

Consider the following two dynamical systems:

$$\begin{aligned} &\dot{x} = f(x), \\ &\dot{y} = g(y, h(x)), \end{aligned} \tag{4}$$

where $x \in R^n$ is the state of the driving system, $y \in R^n$ is the state of the response system, $h(x) \in R^m$ is a coupling between the driving and response systems for driving the response system.

Definition 6 (generalized synchronization[11])
The two systems in expression (4) are said to be in a GS state , if there exist a transformation $H : R^n \to R^m$, a manifold $M = \{(x,y) | \, y = H(x)\}$, and a set $B = B_x \times B_y \subset R^n \times R^m$ with $M \subset B$ such that all trajectories of systems (4) with initial conditions in basin B approach M as time goes to infinity.

Theorem 3[11] The GS occurs in systems (4) if and only if for all $(x_0, y_0) \in B$ the response system $\dot{y} = g(y, h(x))$ is asymptotically stable.

Now, we study the impulsive GS with a given synchronization manifold.

Given a drive system

$$\dot{x} = Ax + \varphi(x) \tag{5}$$

and a synchronization manifold $y = H(x)$ with a continuous inverse $x = H^{-1}(y)$, we construct a response system as follows:

$$\begin{cases} \dot{y} = (DH^{-1}((y))^{-1} AH^{-1}(y) + (DH^{-1}(y))^{-1}\varphi(H^{-1}(y)), \\ \Delta y|_{t=\tau_i} = -(DH^{-1}(y))^{-1}B(x - H^{-1}(y)), \\ y(t_0^+) = y_0, \quad t_0 < \tau_1, \quad i = 1,\ 2, \ldots, \end{cases} \tag{6}$$

where $DH^{-1}(y)$ is the Jacobian matrix of the vector function $H^{-1}(y)$.

Theorem 4 Assume $\varphi(x) - \varphi(y) = M_{x,y}(x - y)$, $||M_{x,y}|| \leq L$.[22] Let d be the largest eigenvalue of $(I + B^{\mathrm{T}})(I + B)$, where B is Hermitian, the spectral radius of $I + B$, $\rho(I + B) \leq 1$, and let q be the largest eigenvalue of $A + A^{\mathrm{T}}$ and the impulses be equi-spaced with an interval Δ. If $0 \leq (q + 2L) \leq -\dfrac{1}{\Delta}\ln(\xi d)$, $\xi > 1$, $0 < \xi d < 1$, then the impulsive GS between systems (5) and (6) is asymptotically stable and the synchronization manifold is $M = \{(x, y) | y = H(x)\}$.

Proof Letting $e = x - H^{-1}(y)$, then the error system is given by

$$\begin{cases} e = \dot{x} - DH^{-1}(y)\dot{y} = A(x - H^{-1}(y)) + \varphi(x) - \varphi(H^{-1}(y)) = Ae + M_{x,H^{-1}(y)}e, \\ \Delta e|_{t=\tau_i} = Be, \qquad i = 1,\ 2,\ \ldots, \end{cases} \tag{7}$$

Let $V(t, e) = e^{\mathrm{T}}e$, for $t \neq \tau_i$,

$$\begin{aligned} D^+V(t, e) &= e^{\mathrm{T}}Ae + e^{\mathrm{T}}A^{\mathrm{T}}e + e^{\mathrm{T}}M_{x,H^{-1}(y)}e \\ &\quad + e^{\mathrm{T}}M^{\mathrm{T}}_{x,H^{-1}(y)}e \leq qe^{\mathrm{T}}e + 2Le^{\mathrm{T}}e \\ &= (q + 2L)\, e^{\mathrm{T}}e = (q + 2L)\, V(t, e). \end{aligned}$$

Hence condition 1 of Theorem 1 is satisfied when $g(t, w) = (q + 2L)w$.

B is Hermitian, so is $I + B$. By using the Euclidean norm we have $\rho(I + B) = \|I + B\|$.

Given any $\rho_0 > 0$ and $e \in S_{\rho_0}$, we have

$$\begin{aligned} ||\, e + U(i, e)\,|| &= ||\, e + Be\,|| \leq ||\, I + B\,||\, ||\, e\,|| \\ &= \rho(I + B)||\, e\,|| \leq ||\, e\,||. \end{aligned}$$

Thus we know $e + U(i, e) \in S_{\rho_0}$.

For $t = \tau_i$, we have $V(\tau_i, e+Be) = (e+Be)^{\mathrm{T}}(e+Be) = e^{\mathrm{T}}(I + B^{\mathrm{T}})(I + B)\, e \leq dV(\tau_i, e)$.

Hence condition 2 of Theorem 1 is satisfied when $\phi_i = dw$. We can see that condition 3 of Theorem 1 is also satisfied. It follows from Theorem 1 that the asymptotic stability of the impulsively controlled system (6) is implied by that of the following comparison system

$$\dot{\omega} = (q + 2L)\omega, \quad t \neq \tau_i,$$

$$\omega(\tau_i) = d\omega(\tau_i), \quad \omega(t_0^+) = \omega_0 \geq 0$$

Since $0 \leq (q + 2L) \leq -\dfrac{1}{\Delta}\ln(\xi d)$, $\xi > 1$, $0 < \xi d < 1$, we have $\int_{\tau_i}^{\tau_{i+1}} (q + 2L)\mathrm{d}t + \ln(\xi d) \leq 0$. It follows from Theorem 2 that the origin of Eq.(7) is asymptotically stable; that is, the impulsive GS between (5) and (6) is asymptotically stable and the synchronization manifold is $M = \{(x, y) | y = H(x)\}$.

4. Numerical examples

In these examples, we study the impulsive GS with weak couplings as follows:

$$C = \begin{bmatrix} \theta & 0 & 0 \\ 0 & -0.1 & 0 \\ 0 & 0 & -0.1 \end{bmatrix},$$

with $\theta = -0.2$, $\xi = 1.1$, $\Delta = -\dfrac{\ln(\xi d_1)}{q + 2L}$.

Example 1 (Linear GS)

Let the synchronization manifold $y = H(x) = Px + Q$, where $P \in R^{n\times n}$, and $Q \in R^{n\times 1}$ are constant matrices, P is invertible. The drive system is a Lorenz oscillator expressed as $\dot{x} = Ax + \varphi(x)$, where $A = \begin{pmatrix} -a & a & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{pmatrix}$, and $\varphi(x) = \begin{pmatrix} 0 \\ -x_1x_3 \\ x_1x_2 \end{pmatrix}$ with parameters $a = 10$, $b = \dfrac{8}{3}$, and $r = 28$.

In this case we construct a response system as given in Eq.(6)

$$\begin{cases} \dot{y} = PAP^{-1}(y-Q) + P\phi_1(P^{-1}(y-Q)), & t \neq \tau_i \\ \Delta y| = -PC(x - P^{-1}(y-Q)), & t = \tau_i, \quad i = 1, 2, \dots, \\ y(t_0^+) = y_0, & t_0 < \tau_1. \end{cases} \tag{8}$$

Choose $P = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and $Q = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

The numerical results are shown in Figs.1 and 2 with $L = 50$, and $\Delta = 0.002$. In Fig.1(a) we plot the attractor of Lorenz chaotic system and the response system (8) without impulsive control; in Fig.1(b) we plot the attractor of Lorenz chaotic system and the response system (8) with impulsive control.

Example 2 (Nonlinear GS)

Let the synchronization manifold $y = H(x)$ be $y_1 = x_1$, $y_2 = x_2 + x_3 - x_3^2$, and $y_3 = x_3$. The drive system is a Chua's oscillator expressed as $\dot{x} = Ax + \varphi(x)$, where $A = \begin{pmatrix} -\alpha & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & \beta \end{pmatrix}$, $\varphi(x) = \begin{pmatrix} -\alpha f \\ 0 \\ 0 \end{pmatrix}$, and $f = bx_1 + \frac{1}{2}(a-b)(|x_1+1| - |x_1-1|)$, with parameters $\alpha = 10$, $\beta = 14.87$, $\gamma = 0$, $a = -1.27$, and $b = -0.68$. The response system is given in Eq.(6), with $x = H^{-1}(y)$: $\begin{cases} x_1 = y_1, \\ x_2 = y_2 - y_3 + y_3^2, \\ x_3 = y_3. \end{cases}$ and

$$DH^{-1}(y) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 + 2y_3 \\ 0 & 0 & 1 \end{bmatrix}.$$

The numerical results are shown in Figs.3 and 4 with $L = |a\alpha|$, and $\Delta = 0.0029$.

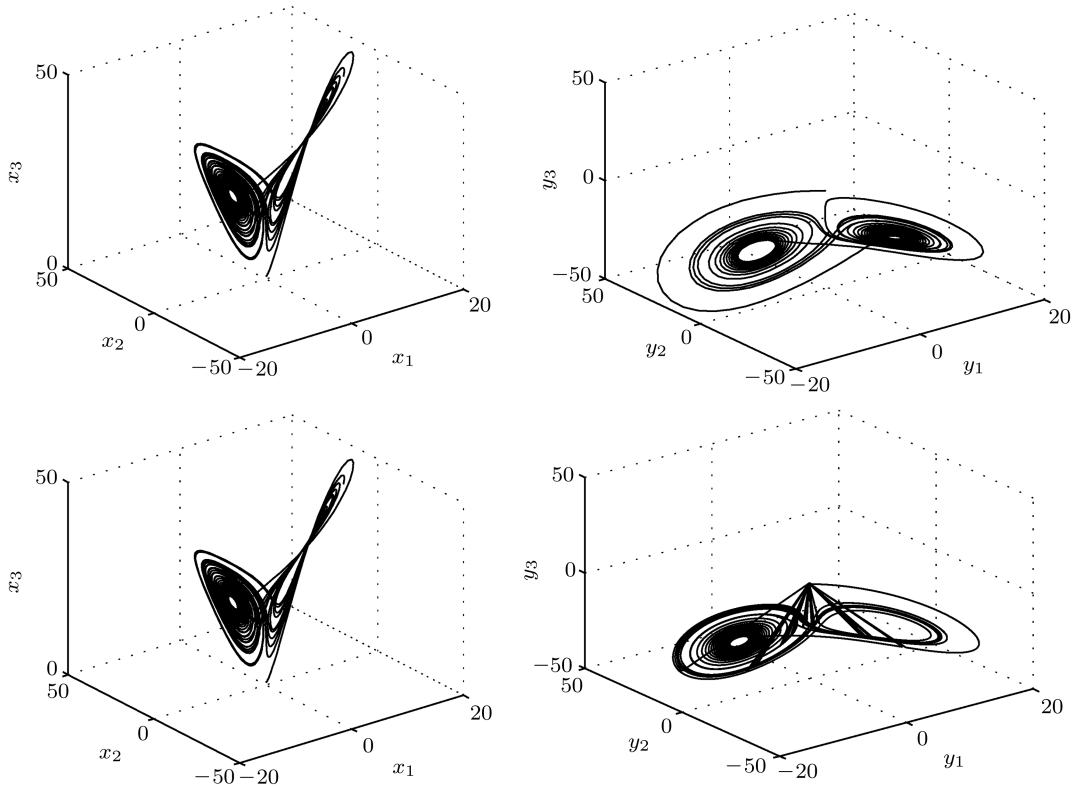


Fig.1. (a) Lorenz chaotic attractor and the response system (8) without impulsive control (above row); (b) Lorenz chaotic attractor and the response system (8) with impulsive control (below row).

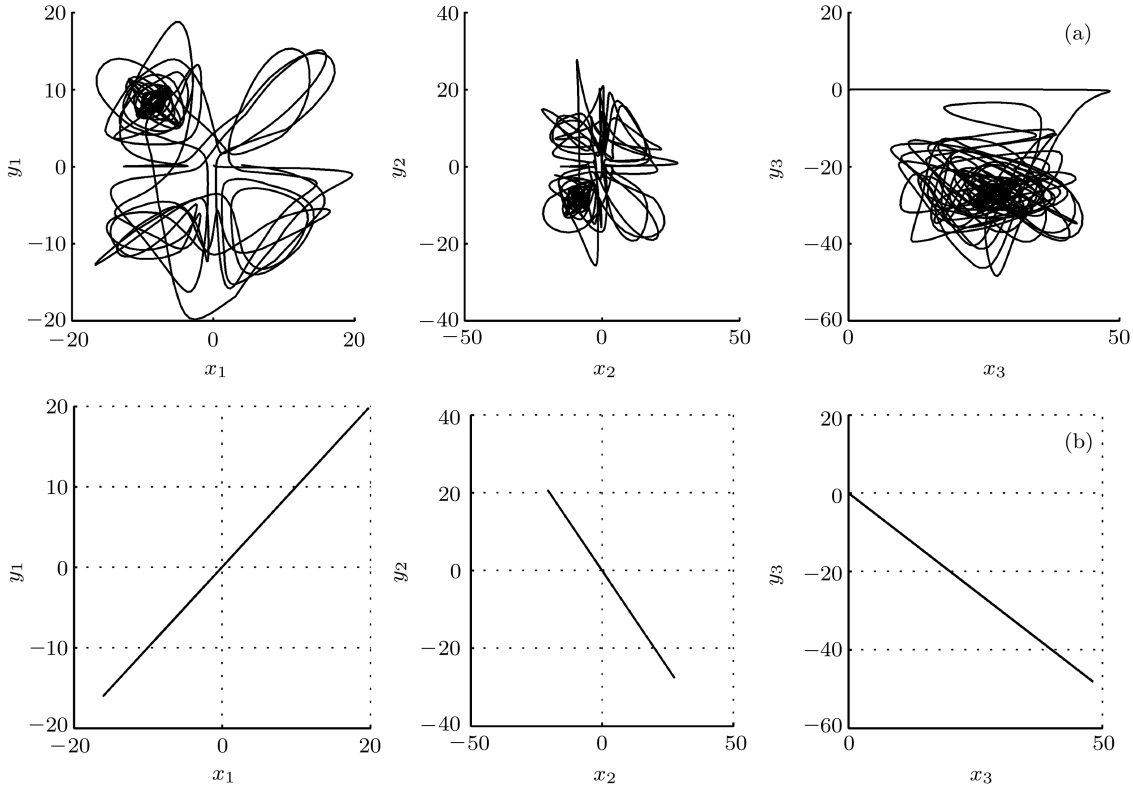


Fig.2. (a) Relationships between Lorenz chaotic attractor and the response system (8) without impulsive control; (b) Relationships between Lorenz chaotic attractor and the impulsively controlled response system (8).

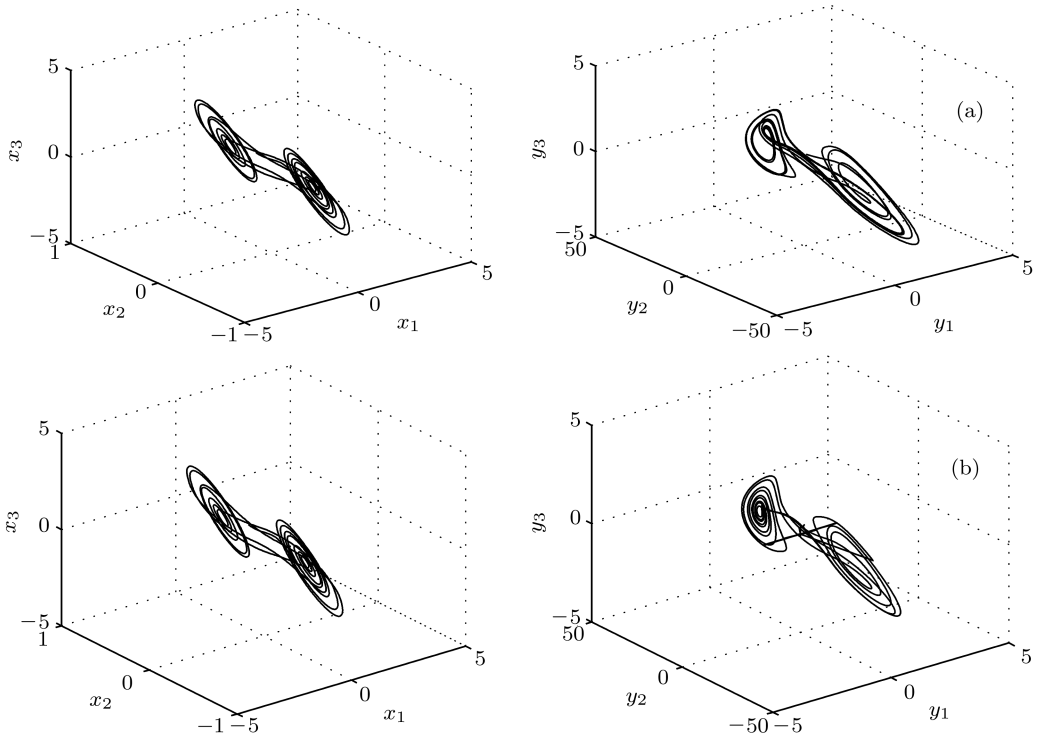


Fig.3. (a) Attractor of Chua's oscillator and the response system (6) without impulsive control (above row); (b) Attractor of Chua's oscillator and the response system (6) with impulsive control (below row).

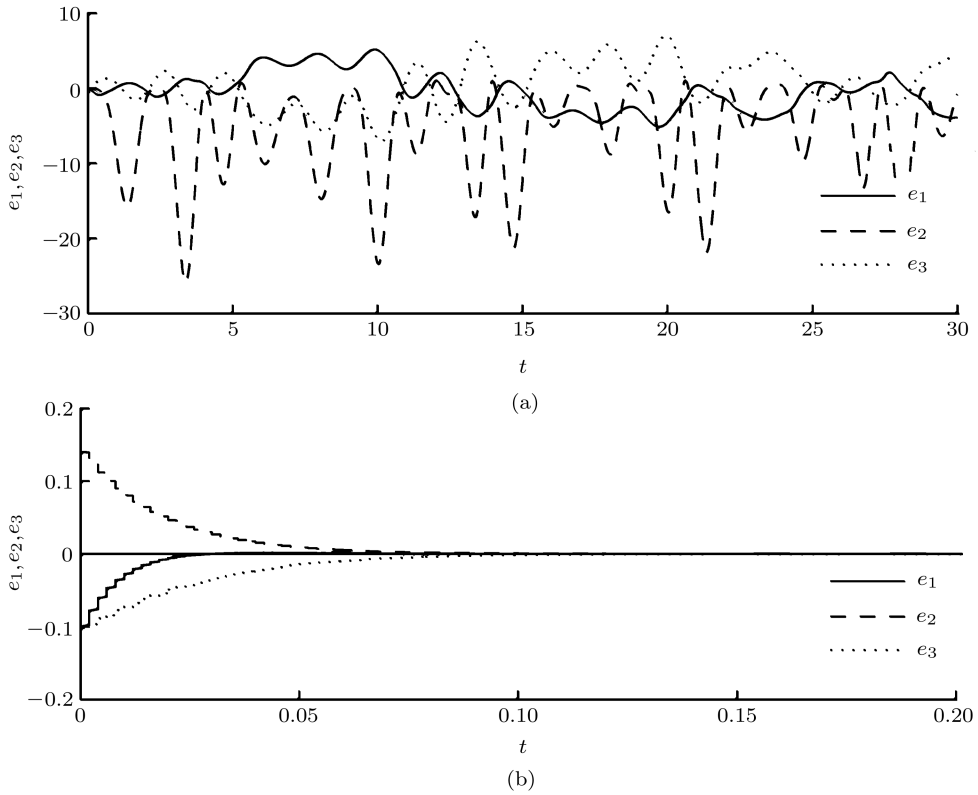


Fig.4. GS errors without (a) and with (b) impulsive control respectively, where $e_1 = x_1 - y_1, e_2 = x_2 - (y_2 - y_3 + y_3^2)$, and $e_3 = x_3 - y_3$.

5. Conclusion

In this paper, we have investigated the issue on the impulsive generalized synchronization of chaotic systems. We have constructed a response system and used impulsive control theory to demonstrate theoretically that this response system can achieve the impulsive generalized synchronization with the drive system, with the synchronization manifold being a given manifold. Examples, including linear impulsive GS and nonlinear impulsive GS, along with computer simulation results, provide us with a chance to visualize the satisfactory control performance.

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