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Synchronization in a unified fractional-order chaotic system

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In this paper, the synchronization in a unified fractional-order chaotic system is investigated by two methods. One is the frequency-domain method that is analysed by using the Laplace transform theory. The other is the time-domain method that is analysed by using the Lyapunov stability theory. Finally, the numerical simulations are used to illustrate the effectiveness of the proposed synchronization methods.

Keywords: chaos, synchronization, fractional-order system, fractional calculus
PACC: 0545

1. Introduction

The dynamics of fractional-order systems has increasingly attracted attention in recent years. Many physical phenomena, including certain types of electrical noises, relaxation behaviours of polarized impedances in dielectrics and interfaces, transmission lines, cardiac rhythm, and spectral densities of music are known to exhibit fractional-order (or called noninteger-order) dynamics.[1,2] The theory of derivatives of non-integer order was presented in Leibniz's note to L'Hopital, dated 30 September 1695, in which the meaning of derivative of order one half was discussed.[3] Although the fractional calculus has a 300-year-old history, it has not been applied to physics and engineering long, especially to the study of chaos and its synchronization. Therefore, in the last decade, many investigations have been devoted to the chaotic dynamics of fractional-order systems.[4−14]

At the same time, as the twin of chaos, synchronization of chaotic systems with fractional orders has attracted more and more attention due to its potential applications in secure communication and control processing. Recently, in Ref.[15], chaos synchronization of fractional-order chaotic systems has been studied only through numerical simulations. In Refs.[16–18], Lu applied the stability theory of fractional-order systems to the synchronization of a class of chaotic systems which are commensurate order. In Ref.[19], Deng and Li had investigated the synchronization of the chaotic fractional Lü system, and then they studied chaos synchronization of the fractional Chen system.[20] In this paper, we investigate the synchronization behaviours in a unified fractional-order chaotic system by two methods which were mentioned in Ref.[21] as special cases of a general formalism for integer-order system. One is the frequency-domain method that is analysed by using the Laplace transform theory. The other is the time-domain method that is analysed by using the Lyapunov stability theory. Finally, numerical simulation results are presented to illustrate the effectiveness of synchronization.

2. Synchronization of unified fractional-order chaotic system

The fractional order unified system reads

$$\begin{cases} \dfrac{\mathrm{d}^{\alpha_1}x}{\mathrm{d}t^{\alpha_1}} = a(y-x), \\ \dfrac{\mathrm{d}^{\alpha_2}y}{\mathrm{d}t^{\alpha_2}} = bx - xz + cy, \\ \dfrac{\mathrm{d}^{\alpha_3}z}{\mathrm{d}t^{\alpha_3}} = xy - dz, \end{cases} \tag{1}$$

where $\dfrac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}}$ denotes the Riemann–Liouville fractional-order derivative. When $\alpha = (\alpha_1, \alpha_2, \alpha_3) = (1, 1, 1)$, and the parameters $(a, b, c, d) = (10, 28, -1, 8/3)$, $(a, b, c, d) = (35, -7, 28, 3)$, and $(a, b, c, d) = (36, 0, 20, 3)$, the above system becomes the famous Lorenz system, Chen system, and Lü system respectively. In Ref.[22], Grigorenko and Grigorenko have

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found that chaos exists in a fractional-order Lorenz with an order being less than 3. In Ref.[19] the chaos dynamics and synchronization of fractional-order Lü system have been discussed by Deng and Li. Moreover, in Ref.[12], Lu and Chen have found that the chaos exists in the fractional-order Chen system with order being as low as 0.3, which is much smaller than those reported in Refs.[13,14].

2.1. Synchronization of fractional-order unified system via Pecora–Carroll method

In this section, based on the Laplace transform theory of fractional-order, we apply the most popular method, which was introduced by Pecora and Carroll (PC),[23] to constructing synchronizing systems. Consider an autonomous n-dimensional fractional-order dynamical system

$$\frac{\mathrm{d}^{\alpha}x}{\mathrm{d}t^{\alpha}} = f(x), \tag{2}$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n$ is the vector of fractional orders, and divide the system, arbitrarily, into two subsystems $[x = (u, v)]$ with the corresponding $[\alpha = (\beta_1, \beta_2)]$

$$\begin{cases} \dfrac{\mathrm{d}^{\beta_1}u}{\mathrm{d}t^{\beta_1}} = g(u, v), \\ \dfrac{\mathrm{d}^{\beta_2}v}{\mathrm{d}t^{\beta_2}} = h(u, v), \end{cases} \tag{3}$$

where $u = (x_1, x_2, \ldots, x_m)$, $g = (f_1, f_2, \ldots, f_m)$, $v = (x_{m+1}, x_{m+2}, \ldots, x_n)$, $h = (f_{m+1}, f_{m+2}, \ldots, f_n)$, $\beta_1 = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ and $\beta_2 = (\alpha_{m+1}, \alpha_{m+2}, \ldots, \alpha_n)$. Now creating a new subsystem $\tilde{v}$ identical to the v system, substituting the set of variables u for the corresponding $\tilde{v}$ into function h, and augmenting Eqs.(3) with this new system, gives

$$\begin{cases} \dfrac{\mathrm{d}^{\beta_1}u}{\mathrm{d}t^{\beta_1}} = g(u, v), \\ \dfrac{\mathrm{d}^{\beta_2}v}{\mathrm{d}t^{\beta_2}} = h(u, v), \\ \dfrac{\mathrm{d}^{\beta_2}\tilde{v}}{\mathrm{d}t^{\beta_2}} = h(u, \tilde{v}). \end{cases} \tag{4}$$

When the difference $\Delta v = \tilde{v} - v \to 0$ as $t \to \infty$, the subsystem components $\tilde{v}$ and v will synchronize with each other.

Therefore, we construct a PC drive–response configuration with a drive system given by the fractional-order unified system (with three state variables denoted by the subscript m) and with a response system given by the subspace containing the (x, z) variables denoted by the subscript s. The chaotic signal y_{m} is chosen to drive the response subsystem. Then, the drive and response systems can be described respectively by

$$\begin{cases} \dfrac{\mathrm{d}^{\alpha_1}x_{\mathrm{m}}}{\mathrm{d}t^{\alpha_1}} = -a(x_{\mathrm{m}} - y_{\mathrm{m}}), \\ \dfrac{\mathrm{d}^{\alpha_2}y_{\mathrm{m}}}{\mathrm{d}t^{\alpha_2}} = bx_{\mathrm{m}} - x_{\mathrm{m}}z_{\mathrm{m}} + cy_{\mathrm{m}}, \\ \dfrac{\mathrm{d}^{\alpha_3}z_{\mathrm{m}}}{\mathrm{d}t^{\alpha_3}} = x_{\mathrm{m}}y_{\mathrm{m}} - dz_{\mathrm{m}}, \end{cases} \tag{5}$$

$$\begin{cases} \dfrac{\mathrm{d}^{\alpha_1}x_{\mathrm{s}}}{\mathrm{d}t^{\alpha_1}} = -a(x_{\mathrm{s}} - y_{\mathrm{m}}), \\ \dfrac{\mathrm{d}^{\alpha_3}z_{\mathrm{s}}}{\mathrm{d}t^{\alpha_3}} = x_{\mathrm{s}}y_{\mathrm{m}} - dz_{\mathrm{s}}. \end{cases} \tag{6}$$

Define the synchronization errors as $e_x = x_{\mathrm{s}} - x_{\mathrm{m}}$ and $e_z = z_{\mathrm{s}} - z_{\mathrm{m}}$. Subtracting Eq.(6) from Eq.(5), we obtain the error system as follows:

$$\begin{cases} \dfrac{\mathrm{d}^{\alpha_1}e_x}{\mathrm{d}t^{\alpha_1}} = -ae_x, \\ \dfrac{\mathrm{d}^{\alpha_3}e_z}{\mathrm{d}t^{\alpha_3}} = y_{\mathrm{m}}e - de_z. \end{cases} \tag{7}$$

Taking the Laplace transform on both sides of Eqs.(7), and notations $E_j(s) = \mathcal{L}(e_j(t)), j = x, y, z$ and using $\mathcal{L}(\mathrm{d}^{\alpha_i}e_j/\mathrm{d}t^{\alpha_i}) = s^{\alpha_i}E_j(s) - s^{\alpha_i - 1}e_j(0), i = 1, 2, 3$, yield

$$\begin{cases} s^{\alpha_1}E_x(s) - s^{\alpha_1 - 1}e_x(0) = -aE_x(s), \\ s^{\alpha_3}E_z(s) - s^{\alpha_3 - 1}e_z(0) = \mathcal{L}(y_{\mathrm{m}}e_x) - dE_z(s). \end{cases} \tag{8}$$

Then,

$$\begin{cases} E_x(s) = \dfrac{\tilde{e}_x(0)}{s^{\alpha_1} + a}, \\ E_z(s) = \dfrac{\tilde{e}_z(0) + \mathcal{L}(y_{\mathrm{m}}e_x)}{s^{\alpha_3} + d}, \end{cases} \tag{9}$$

where $\tilde{e}_x(0) = s^{\alpha_1 - 1}e_x(0)$ and $\tilde{e}_z(0) = s^{\alpha_3 - 1}e_z(0)$. By using the final-value theorem of the Laplace transformation, we can obtain

$$\lim_{t\to\infty} e_x(t) = \lim_{s\to 0} sE_x(s) = \lim_{s\to 0} s\frac{\tilde{e}_x(0)}{s^{\alpha_1} + a} = 0, \tag{10}$$

$$\begin{aligned} \lim_{t\to\infty} e_z(t) &= \lim_{s\to 0} sE_z(s) \\ &= \lim_{s\to 0} s\frac{\tilde{e}_z(0) + \mathcal{L}(y_{\mathrm{m}}e_x)}{s^{\alpha_3} + d} \\ &= \frac{1}{d}\lim_{s\to 0} s\mathcal{L}(y_{\mathrm{m}}e_x). \end{aligned} \tag{11}$$

Because of the attractiveness of the chaotic attractor, $y_{\rm m}$ is bound, i.e. $|y_{\rm m}| \leq M|$ for a positive integer M, hence

$$\begin{aligned}|\mathcal{L}(y_{\rm m}e_x)| &= \left|\int_0^\infty y_{\rm m}e_x{\rm e}^{-st}{\rm d}t\right| \\ &\leq \int_0^\infty |y_{\rm m}e_x{\rm e}^{-st}|{\rm d}t \\ &\leq M\int_0^\infty |e_x{\rm e}^{-st}|{\rm d}t. \end{aligned} \tag{12}$$

On the other hand, from Eq.(7), we can obtain the analytical solution of e_x in the form

$$e_x(t) = e_x(0)E_\alpha(-at^\alpha), \tag{13}$$

where E_α is the one-parameter Mittag–Leffler function[24] defined by

$$E_\alpha(x) = \sum_{k=0}^{\infty}\frac{x^k}{\Gamma(\alpha k+1)}, \quad (\alpha > 0). \tag{14}$$

For $\alpha = 1$, E_α reduces to a common exponent function. Feller conjectured and Pollard (1948) proved that $E_\alpha(-x)$ was completely monotonic for $x \geq 0$ if $0 \leq \alpha \leq 1$, i.e.

$$(-1)^n\frac{{\rm d}^nE_\alpha(-x)}{{\rm d}x^n} \geq 0 \quad x \geq 0, \quad 0 \leq \alpha \leq 1. \tag{15}$$

Inserting expression (13) into expression (12) gives

$$\begin{aligned}|\mathcal{L}(y_{\rm m}e_x)| &\leq M\int_0^\infty |e_x{\rm e}^{-st}|{\rm d}t \\ &= M\int_0^\infty |e_x(0)E_\alpha(-at^\alpha){\rm e}^{-st}|{\rm d}t \\ &= M\int_0^\infty |e_x(0)|E_\alpha(-at^\alpha){\rm e}^{-st}{\rm d}t, \end{aligned} \tag{16}$$

where $at^\alpha \geq 0$, and $0 \leq \alpha \leq 1$.

Since the integral part of expression (16) can be considered as the Laplace transform of e_x with different initial values, we obtain

$$\begin{aligned}|\mathcal{L}(y_{\rm m}e_x)| &\leq M\int_0^\infty |e_x(0)|E_\alpha(-at^\alpha){\rm e}^{-st}{\rm d}t \\ &= M\frac{\bar{e}_x(0)}{s^{\alpha_1}+a}, \end{aligned} \tag{17}$$

where $\bar{e}_x(0) = s^{\alpha_1-1}|e_x(0)|$. Therefore, expression (11) can be written as

$$\begin{aligned}\lim_{t\to\infty} e_z(t) &= \lim_{s\to 0} sE_z(s) \\ &= \frac{1}{d}\lim_{s\to 0} s\mathcal{L}(y_{\rm m}e_x) \\ &\leq \frac{1}{d}\lim_{s\to 0} s|\mathcal{L}(y_{\rm m}e_x)| \\ &\leq \frac{M}{d}\lim_{s\to 0} s\frac{\bar{e}_x(0)}{s^{\alpha_1}+a} \\ &= 0. \end{aligned} \tag{18}$$

Consequently, the asymptotic synchronization between the drive system (6) and the response system (7) with $y_{\rm m}$-drive is realized.

2.2. Synchronization of fractional-order unified system via one-way coupling method

In this section, based on the idea coupling method, which is outlined in Refs.[25,26], a new systematic scheme to synchronize the unified fractional-order chaotic systems is developed. The approach is simple but comprehensive and theoretically rigorous. It is applicable for not only commensurate fractional-order systems but also incommensurate fractional-order systems. Before proposing our method, we give a useful definition concerning the algebraic observability condition.

Consider a fractional-order nonlinear system described by the following dynamic equations:

$$\begin{cases} x^\alpha(t) = f(x), \\ y(t) = h(x), \end{cases} \tag{19}$$

where $x = (x_1, x_2, \ldots, x_n)^{\rm T} \in \mathbb{R}^n$ is the state vector, $\alpha \in \mathbb{R}^n, f : \mathbb{R}^n \mapsto \mathbb{R}$ is continuously differentiable and satisfies $f(0) = 0, y \in \mathbb{R}^n$ is a smooth non-singular output and h: $\mathbb{R}^n \mapsto \mathbb{R}$

Definition 1 System (19) is said to be algebraically observable if the vector valued

$$x(t) = \Phi(y, y^{\alpha_1}, y^{\alpha_2}, \ldots, y^{\alpha_m})(t), \tag{20}$$

where $0 < \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_m$, and Φ is defined as $\Phi : \mathbb{R}^{m+1} \mapsto \mathbb{R}^n$ for a positive integer m. The above condition will be called the algebraic observability condition.

Since the fractional-order differential is a linear operator and the unified fractional-order system satisfies the algebraic observability condition that is to say,

$$\begin{aligned} y &= \frac{x^{(\alpha_1)}+ax}{a}, \\ z &= \frac{1}{x}\left(bx - y^{(\alpha_2)} + cy\right) \\ &= b + c - \frac{x^{(\alpha_1+\alpha_2)} + ax^{(\alpha_2)} - cx^{(\alpha_1)}}{ax}, \end{aligned} \tag{21}$$

we use the variable $x_{\rm m}$ as the output of system (5) and the drive signal. Simultaneously, $y_{\rm m}$ and $z_{\rm m}$ can be reconstructed by using signal $x_{\rm m}$. Thus, we construct the response system as follows:

$$\begin{cases} \dfrac{\mathrm{d}x_\mathrm{s}}{\mathrm{d}t} = -a(x_\mathrm{s} - y_\mathrm{s}) - \dfrac{\mathrm{d}^{\alpha_1} x_\mathrm{m}}{\mathrm{d}t^{\alpha_1}} + \dfrac{\mathrm{d}x_\mathrm{m}}{\mathrm{d}t} + k_1(x_\mathrm{s} - x_\mathrm{m}), \\ \dfrac{\mathrm{d}y_\mathrm{s}}{\mathrm{d}t} = bx_\mathrm{s} - x_\mathrm{s}z_\mathrm{s} + cy_\mathrm{s} - \dfrac{\mathrm{d}^{\alpha_2} y_\mathrm{m}}{\mathrm{d}t^{\alpha_2}} + \dfrac{\mathrm{d}y_\mathrm{m}}{\mathrm{d}t} + k_2(y_\mathrm{s} - y_\mathrm{m}), \\ \dfrac{\mathrm{d}z_\mathrm{s}}{\mathrm{d}t} = x_\mathrm{s}y_\mathrm{s} - \mathrm{d}z_\mathrm{s} - \dfrac{\mathrm{d}^{\alpha_3} z_\mathrm{m}}{\mathrm{d}t^{\alpha_3}} + \dfrac{\mathrm{d}z_\mathrm{m}}{\mathrm{d}t} + k_3(z_\mathrm{s} - z_\mathrm{m}). \end{cases} \tag{22}$$

Defining the synchronization errors $e_x = x_\mathrm{s} - x_\mathrm{m}$, $e_y = y_\mathrm{s} - y_\mathrm{m}$, and $e_z = z_\mathrm{s} - z_\mathrm{m}$, the fractional-order synchronization error system can be obtained by subtracting system (5) from system (22) as follows:

$$\begin{cases} \dfrac{\mathrm{d}e_x}{\mathrm{d}t} = -a(e_x - e_y) + k_1 e_x, \\ \dfrac{\mathrm{d}e_y}{\mathrm{d}t} = be_x + ce_y - x_\mathrm{s}z_\mathrm{s} + x_\mathrm{m}z_\mathrm{m} + k_2 e_y, \\ \dfrac{\mathrm{d}e_z}{\mathrm{d}t} = -de_y + x_\mathrm{s}y_\mathrm{s} - x_\mathrm{m}y_\mathrm{m} + k_3 e_z. \end{cases} \tag{23}$$

Obviously, the fractional-order synchronization problem has been converted into an equivalent problem of stabilizing the integer-order system (23) through a suitable choice of feedback gains k_1, k_2, k_3. Here, let $k_1 = 0$, and $k_3 = 0$ for simplifying calculation, then we will obtain a sufficient condition of k_2 which can synchronize the system (5) and system (22) with each other.

Theorem 1 The two systems (5) and (22) are synchronized with each other if the feedback control k_2 is chosen such that

$$k_2 < -\frac{(M^2 + 3d(b+M))^2}{3dM^2} - c, \tag{24}$$

where the $M(> |x_\mathrm{m}|, |y_\mathrm{m}|, \text{ and } |z_\mathrm{m}|)$ is a positive real number, since the chaos trajectory is bounded.

Proof In order to achieve the complete synchronization between the drive and response systems (5) and (22), we must prove that the zero solution of the error dynamical system (23) is asymptotically stable. We define a Lyapunov function as

$$V(e_x, e_y, e_z) = \frac{1}{2}\left(\frac{M^2}{3ad}e_x + e_y + e_z\right), \quad (a > 0,\ d > 0). \tag{25}$$

Then V is a positive definite matrix, and its derivative, with error system (23) considered, is given by

$$\begin{aligned} \dot{V} &= -\frac{M^2}{3d}e_x^2 + (c + k_2)e_y^2 - de_z^2 + \left(\frac{M^2}{3d} + b - z_\mathrm{m}\right)e_x e_y + y_\mathrm{m} e_x e_z \\ &< -\frac{M^2}{3d}e_x^2 + (c + k_2)e_y^2 - de_z^2 + \left(\frac{M^2}{3d} + b + M\right)|e_x e_y| + M|e_x e_z| \\ &= -[|e_x| \ \ |e_y| \ \ |e_z|] \begin{bmatrix} \dfrac{M^2}{3d} & -\dfrac{1}{2}\left(\dfrac{M^2}{3d} + b + M\right) & -\dfrac{M}{2} \\ -\dfrac{1}{2}\left(\dfrac{M^2}{3d} + b + M\right) & -(c + k_2) & 0 \\ -\dfrac{M}{2} & 0 & d \end{bmatrix} \begin{bmatrix} |e_x| \\ |e_y| \\ |e_z| \end{bmatrix} \\ &= -E^\mathrm{T} P E, \end{aligned} \tag{26}$$

where $E^\mathrm{T} = [|e_x| \ \ |e_y| \ \ |e_z|]$, and

$$P = \begin{bmatrix} \dfrac{M^2}{3d} & -\dfrac{1}{2}\left(\dfrac{M^2}{3d} + b + M\right) & -\dfrac{M}{2} \\ -\dfrac{1}{2}\left(\dfrac{M^2}{3d} + b + M\right) & -(c + k_2) & 0 \\ -\dfrac{M}{2} & 0 & d \end{bmatrix}.$$

The matrix P is a positive definite matrix if $k_2 < -\dfrac{(M^2 + 3d(b+M))^2}{3dM^2} - c$. Then, $\dot{V}$ is a negative definite matrix, which implies that the origin of error system (23) is asymptotically stable, thereby the two systems (5) and (22) are synchronized with each other.

3. Numerical simulations

In this section numerical simulations are given to confirm the effectiveness of the Pecora–Carroll and one-way coupling methods of synchronizing the fractional-order Lorenz system. But the standard definition of fractional differintegral does not allow the direct implementation of fractional operators in time–domain simulations. An efficient method to circumvent this problem is to approximate fractional operators by using standard integer-order operators. In Ref.[1], an effective algorithm was developed to approximate fractional order transfer functions. Basically, the idea is to approximate the system behaviour in frequency domain. By utilizing the frequency domain techniques based on the Bode diagrams, one can obtain a linear approximation of the fractional-order integrator, the order of which depends on the desired bandwidth and discrepancy between the actual and the approximate magnitude Bode diagrams. This approximate approach has been adopted in Refs.[11–13, 15–18]. In Table 1 of Ref.[4], the approximations for $\dfrac{1}{s^{\alpha}}$ with $\alpha = 0.1 - 0.9$ in steps of 0.1 were given, with errors of approximately 2 dB. We mainly use these approximations in the following simulations.

Letting $\alpha = (0.95, 0.95, 0.9)$ and $(a, b, c, d) = (10, 28, 5, 8/3)$ in system (1), its chaotic figures can be shown in Fig.1.

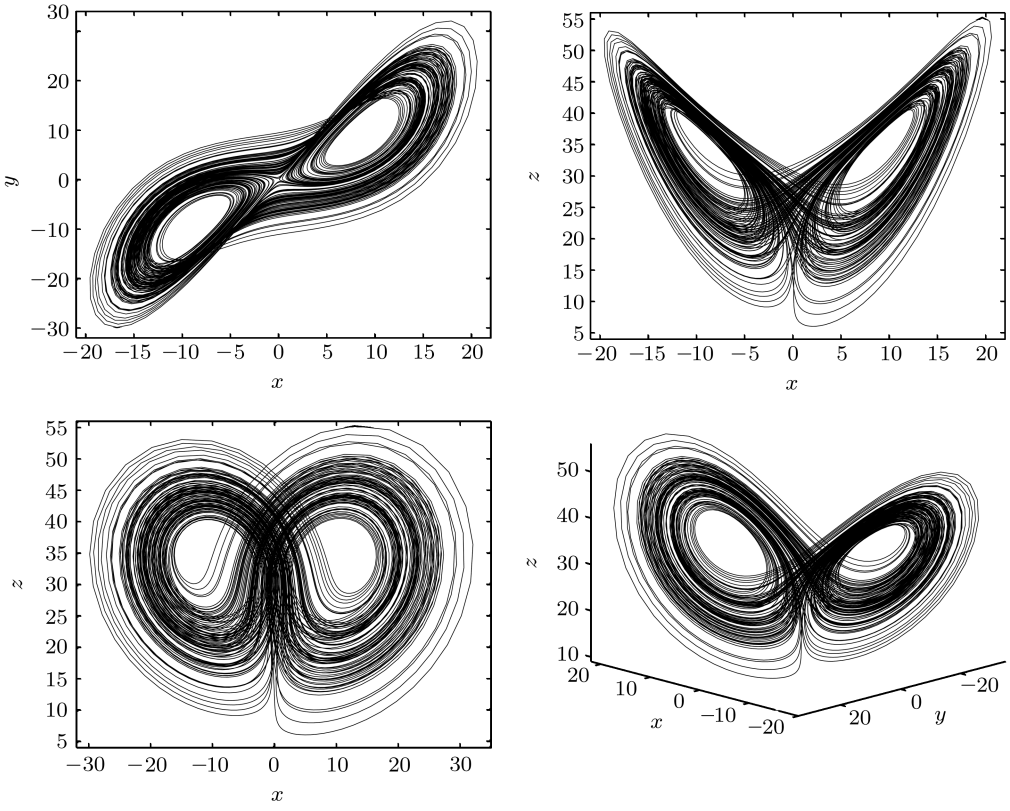


Fig.1. Attractor of Lorenz system with fractional order, $\alpha = (0.95, 0.95, 0.9)$ and $(a, b, c, d) = (10, 28, 5, 8/3)$.

Figure 2 displays the time response of the states $x_{\rm s}$ and $z_{\rm s}$ to the states $x_{\rm m}$ and $z_{\rm m}$ while $y_{\rm m}$ is used as the drive signal, just like that of the Pecora–Carroll method mentioned above. Here, the initial states of the drive and the response systems (5) and (6) are (0.2,0.2,0.2) and (20,30) respectively. The errors e_x and e_z between the response variables and their drive counterparts agree well with those from the theoretical analyse.

The numerical simulations of the drive and the response systems (5) and (22) are presented to show the effect of chaos synchronization by using one-way coupling method. The initial states of the drive and the response systems (5) and (22) are $(0.2, 0.2, 0.2)$ and $(8, -7, 10)$ respectively. Figure 3 shows the time response of the states $(x_{\rm m}, y_{\rm m}, z_{\rm m})$ for the drive system (5) and the time response of the states $(x_{\rm s}, y_{\rm s}, z_{\rm s})$ for the response system (22), where the feedback control gain $(k_1, k_2, k_3) = (0, -650, 0)$ is calculated by using theorem 1 and activated at $t = 5$s.

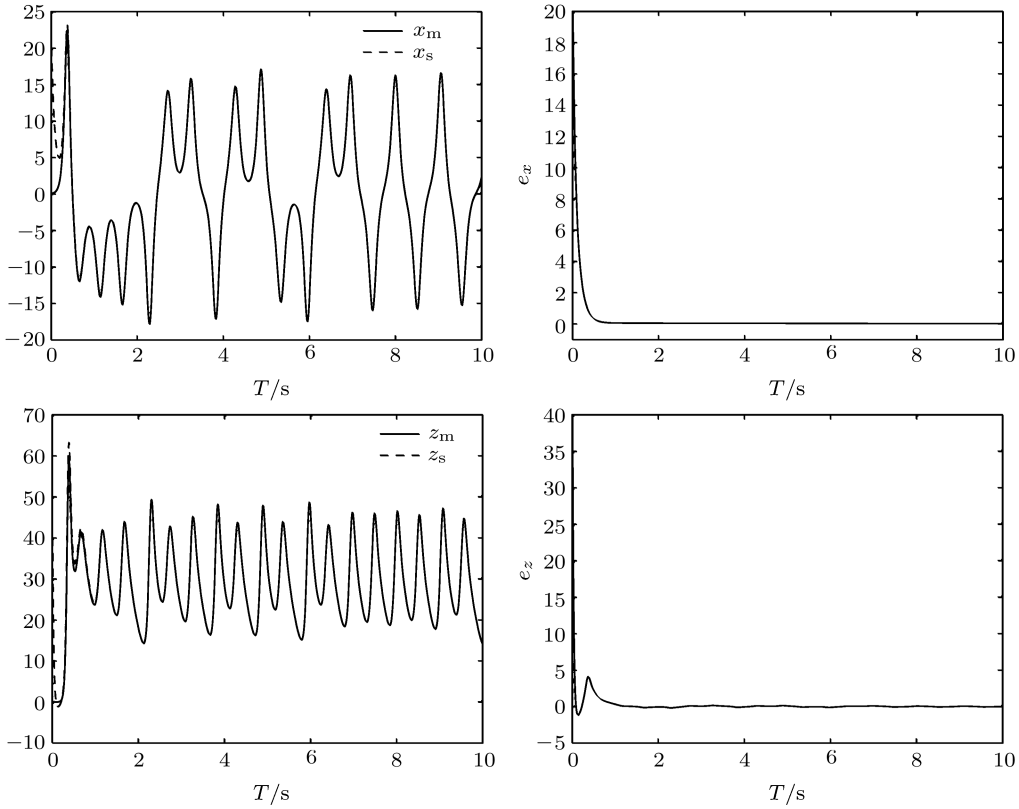


Fig.2. Chaos synchronization between systems (5) and (6) via Pecora–Carroll method.

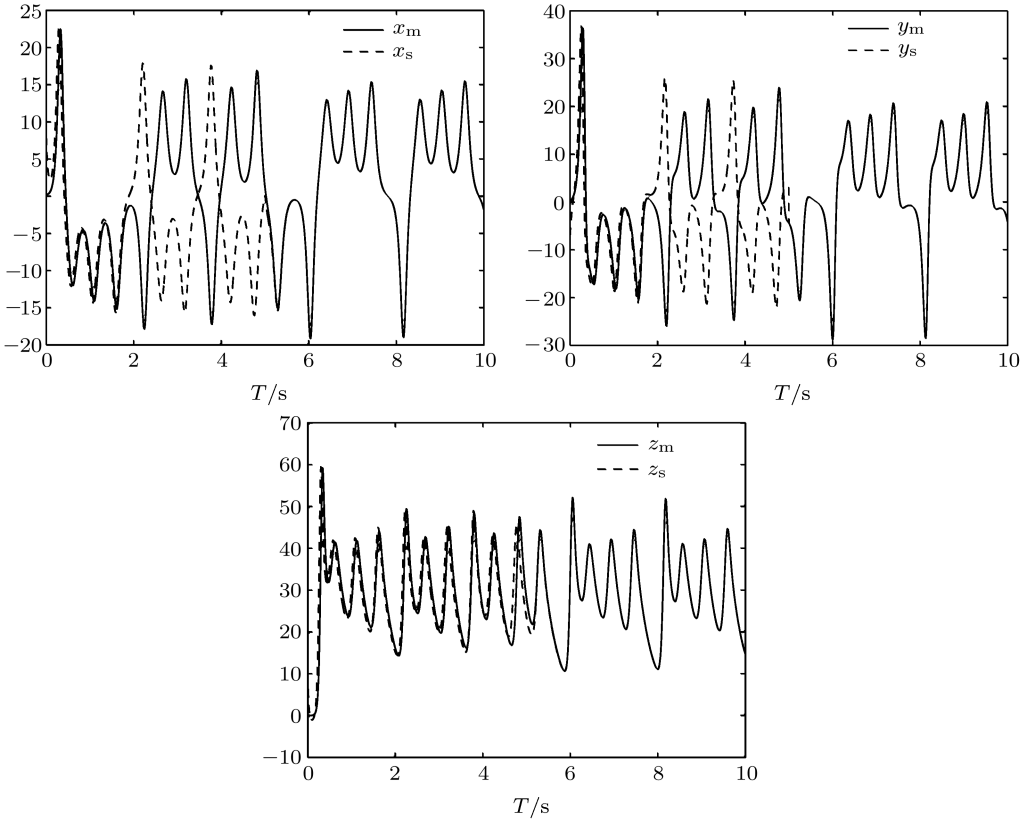


Fig.3. Chaos synchronization between systems (5) and (22) via one-way coupling method.

4. Conclusions

This paper addresses the chaos synchronization of a unified fractional system by using the Pecora–Carroll and one-way coupling methods. The conditions sufficient for the occurrence of chaos synchronization between the fractional-order system and the integer-order system are derived by using the Lyapunov stability theory. Numerical simulations are performed to verify these results.

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