Vol 16 No 7, July 2007
1009-1963/2007/16(07)/1878-05

Chinese Physics

© 2007 Chin. Phys. Soc.
and IOP Publishing Ltd

Dynamics of the entanglement witness for three qubits in common environment*

Lu Huai-Xin(逯怀新)†

Department of Physics, Weifang University, Weifang 261061, *China*
Department of Physics, Qufu Normal University, Qufu 273165, *China*

(Received 8 November 2006; revised manuscript received 11 December 2006)

In the measurement-based model of quantum computing, a one-way quantum computer consisting of many qubits can be immersed in a common environment as a simple decoherence mechanism. This paper studies the dynamics of entanglement witness for 3-qubit cluster states in the common environment. The result shows that environment can induce an interesting feature in the time evolution of the entanglement witness: i.e., the periodical collapse and revival of the entanglement dynamics.

Keywords: cluster states, entanglement witness, entanglement dynamics
PACC: 0367

Quantum entanglement is one of the essential features of quantum mechanics and plays also a crucial role in quantum information theory. The creation and measurement of the entanglement have been an interesting and important topic since it was first noticed by Einstein, Podolsky, and Rosen (EPR).[1] In recent years, much research has been devoted to study the properties of the entangled states induced by environment.[2−6] It is well known that in all physical processes there is an associated loss mechanism, known as decoherence,[7] which is one of the biggest obstacles in quantum information processing. A quantum system loses its quantum characteristics if it opens to the environment. However, the environment can also provide an indirect interaction between the subsystems without a direct interaction and thus a means to entangle them. For such an indirectly interacting system, it is interesting to know the dynamics of quantum entanglement.

It has been shown that two qubits without a direct coupling but interacting with the environment could be entangled[2,5] when evolving according to a Markovian, completely positive reduced dynamics.[6] The phase-induced collapse and revival of entanglement for two qubits have been investigated in Ref.[3]. This work concerning the two-qubit entanglement has not only led to significant implication in our understanding of quantum entanglement and quantum dynamics, but also to an effective approach for creating entanglement applicable to quantum computation. Nevertheless, entanglement of many qubits[8] are needed for quantum tasks such as quantum computation based on cluster states.[9] The cluster states of many qubits are an essential physical resource for the so-called one-way quantum computation.[10−13] Based on such a resource, single-qubit measurements on a large set of entangled quantum system are sufficient to undertake quantum computation.

In this paper, we aim at studying entanglement dynamics of a 3-qubit cluster state (which is actually also a Greenberger–Horne–Zeilinger state for the three-qubit case, cluster states of more qubits are not equivalent to the corresponding Greenberger–Horne–Zeilinger states) in the common environment. In the measurement-based model of quantum computing, a one-way quantum computer consisting of many qubits could be immersed in a common environment. The situation under consideration could be exactly the case where the one-way quantum computer interacts with the common environment as a simple decoherence mechanism of one-way quantum computing. In the following, we first solve the time evolution of the system and the environment. Then, by using the entanglement witness operator[14] made up of stabilizer operators,[15,16] we study the dynamics of the quantum entanglement for the 3-qubit cluster state. It is worthwhile to point out that, while we merely consider the three-qubit case as an illustrative example in the

*Project supported by the Natural Science Foundation of Shandong Province, China (Grant No Y2006 A05).
†E-mail: luhuaixin@wfu.edu.cn

present paper, our analysis is equally applicable to the case of multiple qubits.

Consider the following model for a 3-qubit system interacting with a common environment of a single harmonic oscillator:[17,18]

$$H = \hbar\omega a^{+}a + \hbar g \sum_{i=1}^{3} \sigma_{iz}\left(a^{+} + a\right), \tag{1}$$

where ω and $a^{+}(a)$ is frequency and creation (annihilation) operator of the harmonic oscillator respectively, σ_{iz} is the z-component of the Pauli matrices of the ith qubit, and g is a coupling constant. In order to conveniently obtain the time evolution of the total states for the system and environment, we take the following unitary transformation:[19−21]

$$\begin{aligned} &H' = UHU^{-1}, \\ &U = \mathrm{e}^{\frac{g}{\omega}\sum_{i=1}^{3}\sigma_{iz}\left(a^{+}-a\right)}. \end{aligned} \tag{2}$$

With the help of the unitary transformation (2), the Hamiltonian (1) can be rewritten as

$$\begin{aligned} H' &= UHU^{-1} \\ &= \hbar\omega a^{+}a - \frac{1}{3}I_{\mathrm{eff}} \sum_{i\neq j=1}^{3} \sigma_{iz}\sigma_{jz} - I_{\mathrm{eff}}. \end{aligned} \tag{3}$$

where $I_{\mathrm{eff}} = \dfrac{3\hbar g^2}{\omega}$ is the indirect three-qubit coupling induced by the harmonic oscillator. Under the unitary transformation (2), the time evolution of the total state is given by

$$\left|\Psi(t)\right\rangle = \mathrm{e}^{-\frac{\mathrm{i}Ht}{\hbar}}\left|\Psi\left(0\right)\right\rangle = U^{-1}\mathrm{e}^{-\frac{\mathrm{i}H't}{\hbar}}U\left|\Psi\left(0\right)\right\rangle. \tag{4}$$

For a given initial state, it is easy to obtain the time-dependent state from Eq.(4). Now, suppose that an initial state is $\left|\Psi\left(0\right)\right\rangle = \left|\psi\left(0\right)\right\rangle \otimes \left|0\right\rangle_{\mathrm{E}}$, where $\left|0\right\rangle_{\mathrm{E}}$ is the initial state for the environment, and system's initial state is a cluster state[9]

$$\left|\psi\left(0\right)\right\rangle = \frac{1}{\sqrt{2}}\left(\left|+\right\rangle_1\left|0\right\rangle_2\left|+\right\rangle_3 + \left|-\right\rangle_1\left|1\right\rangle_2\left|-\right\rangle_3\right). \tag{5}$$

This cluster state is the eigenstate of the following stabilizer operators:[15]

$$S_k\left|\psi\left(0\right)\right\rangle = \left|\psi\left(0\right)\right\rangle, \tag{6}$$

$$\begin{aligned} &S_1 = \sigma_x^{(1)}\sigma_z^{(2)}, \\ &S_2 = \sigma_z^{(1)}\sigma_x^{(2)}\sigma_z^{(3)}, \\ &S_3 = \sigma_z^{(2)}\sigma_x^{(3)}. \end{aligned} \tag{7}$$

Substituting the initial state $\left|\Psi\left(0\right)\right\rangle$ into Eq.(4), we obtain the total state at time t as

$$\begin{aligned} \left|\Psi\left(t\right)\right\rangle = \frac{1}{2\sqrt{2}}\mathrm{e}^{-\mathrm{i}\alpha^2\sin(\omega t)}\Big\{ &\mathrm{e}^{9\mathrm{i}\theta t - 8\mathrm{i}\alpha^2\sin\omega}\left[\left|000\right\rangle \otimes \left|3\alpha\left(t\right)\right\rangle_{\mathrm{E}} + \left|111\right\rangle \otimes \left|-3\alpha\left(t\right)\right\rangle_{\mathrm{E}}\right] \\ &+\mathrm{e}^{\mathrm{i}\theta t}[(\left|0\right\rangle_1\left|0\right\rangle_2\left|1\right\rangle_3 + \left|1\right\rangle_1\left|0\right\rangle_2\left|0\right\rangle_3 + \left|0\right\rangle_1\left|1\right\rangle_2\left|0\right\rangle_3) \otimes \left|\alpha\left(t\right)\right\rangle_{\mathrm{E}} \\ &+ (\left|1\right\rangle_1\left|0\right\rangle_2\left|1\right\rangle_3 - \left|0\right\rangle_1\left|1\right\rangle_2\left|1\right\rangle_3 - \left|1\right\rangle_1\left|1\right\rangle_2\left|0\right\rangle_3 \otimes \left|-\alpha\left(t\right)\right\rangle_{\mathrm{E}})]\Big\}. \end{aligned} \tag{8}$$

Where

$$\alpha\left(t\right) = \alpha\left(\mathrm{e}^{-\mathrm{i}\omega t} - 1\right) = \frac{g}{\omega}\left(\mathrm{e}^{-\mathrm{i}\omega t} - 1\right), \tag{9}$$

$$\theta = \frac{I_{\mathrm{eff}}}{3\hbar} = \frac{g^2}{\omega}, \quad \alpha = \frac{g}{\omega}. \tag{10}$$

In order to investigate the entanglement for the 3-qubit state in Eq.(8), we apply the stabilizer witness operators[14] to detect the entanglement for the entangled states. To do this, let us first recall the definition of the entanglement witness.[22] For a given state ρ, its entanglement can be measured by entanglement witness operator W which is defined as [22]

$$\mathrm{Tr}\left(W\rho\right) = \begin{cases} \geq 0 & \text{for all separable states,} \\ < 0 & \text{for entangled states.} \end{cases} \tag{11}$$

Based on the definition of the entanglement witness, one can construct the corresponding witness operator for the given state. Then the entanglement of the state can be determined by the expectation value of the witness operator. In Ref.[14], it was proposed that the entanglement witness (i.e., stabilizer witness) which detects the entanglement close to cluster states can be constructed by the elements of the stabilizer. The stabilizer witness provides us an effective approach for measuring the multipartite entangled states. The stabilizer witness which detects N-qubit entanglement close to a cluster state is defined as[14]

$$\begin{aligned} &W_k^{(\mathrm{C}_N)} = 1 - S_k^{(\mathrm{C}_N)} - S_{k+1}^{(\mathrm{C}_N)} \\ &\text{for} \quad k = 1, 2, \cdots, N-1, \end{aligned} \tag{12}$$

where the cluster state $|\mathrm{C}_N\rangle$ satisfies the eigenvalue equation $S_k^{(\mathrm{C}_N)}|\mathrm{C}_N\rangle = |\mathrm{C}_N\rangle$ with the following stabilizer operators:

$$\begin{aligned} S_1^{(\mathrm{C}_N)} &= \sigma_x^{(1)}\sigma_z^{(2)}, \\ S_k^{(\mathrm{C}_N)} &= \sigma_z^{(k-1)}\sigma_x^{(k)}\sigma_z^{(k+1)}, \quad k = 2, 3, \cdots, N-1, \\ S_N^{(\mathrm{C}_N)} &= \sigma_z^{(N-1)}\sigma_x^{(N)}. \end{aligned} \tag{13}$$

For the case of three qubits in Eq.(12) $W_k^{(\mathrm{C}_N)}$ is

$$W_1^{(\mathrm{C}_3)} = 1 - S_1^{(\mathrm{C}_3)} - S_2^{(\mathrm{C}_3)}. \tag{14}$$

It is easy to see that the entanglement witness in Eq.(14) for the cluster state (6) is equal to -1, i.e.

$$\langle\psi(0)|\, W_1^{(\mathrm{C}_3)}\, |\psi(0)\rangle = -1. \tag{15}$$

From Eq.(14), through some straightforward calculations, we can obtain the following results:

$$\begin{aligned} \langle\Psi(t)|\, S_1^{(\mathrm{C}_3)}\, |\Psi(t)\rangle &= \langle\Psi(t)|\, S_2^{(\mathrm{C}_3)}\, |\Psi(t)\rangle \\ &= \frac{1}{2}\mathrm{e}^{-4\alpha^2(1-\cos(\omega t))}\left(1+\cos\gamma(t)\right), \end{aligned}$$

where

$$\gamma(t) = 8\left(\theta t - \alpha^2\sin(\omega t)\right). \tag{16}$$

Finally, we obtain the entanglement witness in Eq.(8) as follows:

$$\begin{aligned} &\langle\Psi(t)|\, W_1^{(\mathrm{C}_3)}\, |\Psi(t)\rangle \\ &= 1 - \mathrm{e}^{-4\alpha^2(1-\cos(\omega t))}\left(1+\cos\gamma(t)\right). \end{aligned} \tag{17}$$

From the above result in Eq.(17), one can see that the dynamical behaviour of witness for three-qubit cluster state which is induced by the environment is determined by the parameters ω and g. The entanglement witness of the three qubits exhibits the periodical collapse and revival under the condition of $\omega > g$ (see Fig.1). In the case of $\omega \gg g$, we have the result of $\mathrm{e}^{-4\alpha^2(1-\cos(\omega t))} \approx 1$, $\gamma(t) \approx 8\theta t$. Thus the witness (17) becomes

$$\langle\Psi(t)|\, W_1^{(\mathrm{C}_3)}\, |\Psi(t)\rangle = -\cos(8\theta t). \tag{18}$$

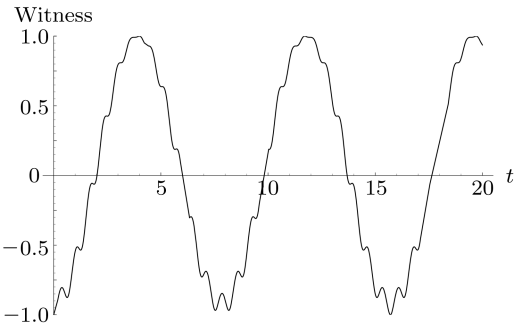


Fig.1. The dynamical behaviour of the entanglement witness for a 3-qubit system. Initially in a cluster state and in the common environment. The periodical collapse and revival of the entanglement dynamics is given under the condition of $\omega > g$.

It is obvious that Eq.(18) also exhibits the periodical collapse and revival of entanglement dynamics for the time evolution of the 3-qubit system. It also means that the environment state is almost always in the vacuum state. Therefore, in this case, the state $|\Psi(t)\rangle$ in Eq.(8) becomes the product state

$$\begin{aligned} |\Psi(t)\rangle &= |\psi(t)\rangle \otimes |0\rangle_{\mathrm{E}} \\ &= \frac{1}{2\sqrt{2}}\left\{\mathrm{e}^{9\mathrm{i}\theta t}\left[|000\rangle + |111\rangle \otimes |0\rangle_{\mathrm{E}}\right] + \mathrm{e}^{\mathrm{i}\theta t}[|0\rangle_1|0\rangle_2|1\rangle_3 + |1\rangle_1|0\rangle_2|0\rangle_3\right. \\ &\quad \left. + |0\rangle_1|1\rangle_2|0\rangle_3 + |1\rangle_1|0\rangle_2|1\rangle_3 - |0\rangle_1|1\rangle_2|1\rangle_3 - |1\rangle_1|1\rangle_2|0\rangle_3 \otimes |0\rangle_{\mathrm{E}}]\right\}. \end{aligned} \tag{19}$$

When the qubits without interaction interact with the common environment, it is easy to see from the above analysis that the environment induces the indirect interaction between qubits and causes the collapse and revival of the entanglement witness of the qubits. In the case of weak coupling ($\omega \gg g$), although the system and the environment are in the product state, the system is still in entangled states and shows the exactly periodical collapse and revival of the entanglement witness for the three-qubit system. In fact, if the initial state of the environment is in coherent state $|\alpha\rangle$, we can still obtain the result of Eq.(18).

Let us consider the case in which initial state of the environment is in a coherent state $|\alpha\rangle$, i.e., the initial state of the system and environment is

$$|\Psi(0)\rangle = |\psi(0)\rangle \otimes |\alpha\rangle_{\rm E} = \frac{1}{\sqrt{2}}\Big(|+\rangle_1|0\rangle_2|+\rangle_3 + |-\rangle_1|1\rangle_2|-\rangle_3\Big) \otimes |\alpha\rangle_{\rm E}.$$

Then from Eq.(4), we obtain the total state $|\Psi(t)\rangle$ as follows:

$$\begin{aligned}|\Psi(t)\rangle = \frac{1}{2\sqrt{2}}\Big\{ & e^{9i\theta t}\Big[|000\rangle \otimes e^{-24i\alpha^2\sin(\omega t)}|\alpha_1(t)\rangle_{\rm E} + |111\rangle \otimes e^{-12i\alpha^2\sin(\omega t)}|\alpha_2(t)\rangle_{\rm E}\Big] \\ & + e^{i\theta t}\Big[(|0\rangle_1|0\rangle_2|1\rangle_3 + |1\rangle_1|0\rangle_2|0\rangle_3 + |0\rangle_1|1\rangle_2|0\rangle_3) \otimes e^{-4i\alpha^2\sin(\omega t)}|\alpha_3(t)\rangle_{\rm E} \\ & + (|1\rangle_1|0\rangle_2|1\rangle_3 - |0\rangle_1|1\rangle_2|1\rangle_3 - |1\rangle_1|1\rangle_2|0\rangle_3 \otimes |\alpha\rangle_{\rm E})\Big]\Big\}, \end{aligned} \tag{20}$$

where

$$\begin{aligned}|\alpha_1(t)\rangle_{\rm E} &= \left|\alpha\left(4e^{-i\omega t} - 3\right)\right\rangle_{\rm E}, \\ |\alpha_2(t)\rangle_{\rm E} &= \left|-\alpha\left(2e^{-i\omega t} - 3\right)\right\rangle_{\rm E}, \\ |\alpha_3(t)\rangle_{\rm E} &= \left|\alpha\left(2e^{-i\omega t} - 1\right)\right\rangle_{\rm E}.\end{aligned}$$

Using Eq.(14), we obtain the entanglement witness of the 3-qubit system in Eq.(20)

$$\begin{aligned}\langle\Psi(t)|W_1^{(C_3)}|\Psi(t)\rangle = 1 & - e^{-4\alpha^2(1-\cos(\omega t))}\cos\gamma_3(t) \\ & - \frac{1}{2}e^{-4\alpha^2(1-\cos(\omega t))}[\cos\gamma_1(t) \\ & + \cos\gamma_2(t)],\end{aligned} \tag{21}$$

where

$$\begin{aligned}\gamma_1(t) &= 8\theta t - 22\alpha^2\sin(\omega t), \\ \gamma_2(t) &= 8\theta t - 10\alpha^2\sin(\omega t), \\ \gamma_3(t) &= 6\alpha^2\sin(\omega t).\end{aligned}$$

Obviously, in the special case of $\omega \gg g$, Eq.(21) becomes Eq.(18). In this case, $|\alpha_i(t)\rangle_{\rm E}$ $(i = 1, 2, 3)$ are very close to $|\alpha\rangle_{\rm E}$, i.e., the overlap $\langle\alpha \mid \alpha_i(t)\rangle \approx 1$. This means that the environment state is almost always in the coherent state.

In conclusion, we have studied the dynamical behaviour of the entanglement witness for a 3-qubit cluster state in the common environment. The test of entanglement for the 3-qubit cluster state can be realized by stabilizer operators in Ref.[14]. We have demonstrated the fundamental role of the environment in such an entanglement dynamics. The result shows that the environment can induce an interesting feature in the time evolution of the entanglement witness: i.e., the periodical collapse and revival of the entanglement dynamics. When the frequency ω of the harmonic oscillator for the environment and the coupling constant g satisfy the relation $\omega \gg g$, the system and the environment are in the product state, and dynamics of the entanglement witness for the system takes the cosine function. Using our method, one can study the entanglement dynamics of cluster states for an N-qubit system in the common environment by using the corresponding witness operators. And similar entanglement dynamics can be expected. In such an N-qubit system, the situation corresponds to the case where a one-way quantum computer interacts with a common environment.

References

[1] Einstein A, Podolsky B and Rosen N 1935 *Phys. Rev.* **47** 777

[2] Oh S and Kim J 2006 quant-ph/0605033

[3] Malinovsky V S and Sola I R 2006 *Phys. Rev. Lett.* **96** 050502

[4] Cubitt T S, Verstraete F, Dür W and Cirac J I 2003 *Phys. Rev. Lett.* **91** 037902

[5] Braun D 2002 *Phys. Rev. Lett.* **89** 277901

[6] Benatti F, Floreanini R and Piani M 2003 *Phys. Rev. Lett.* **91** 070402

[7] Walls D F and Milburn G J 1994 *Quantum Optics* (Berlin: Springer)

[8] Pan J W, Bouwmeester D, Daniell M, Weinfurter H and Zeilinger A 2000 *Nature* (London) **403** 515

[9] Briegel H J and Raussendorf R 2001 *Phys. Rev. Lett.* **86** 910

[10] Raussendorf R, Browne D E and Briegel H J 2003 *Phys. Rev.* A **68** 022312

[11] Yoran N and Reznik B 2003 *Phys. Rev. Lett.* **91** 037903

[12] Nielsen M A and Dawson C M 2004 *Phys. Rev.* A **71** 042323

[13] Browne D E and Rudolph T 2005 *Phys. Rev. Lett.* **95** 010501
[14] Toth G and Gühne O 2005 *Phys. Rev. Lett.* **94** 060501
Toth G and Gühne O 2005 *Phys. Rev.* A **72** 022340
[15] Gottesman D 1996 *Phys. Rev.* A **54** 1862
[16] Gottesman D 1997 Ph.D. Thesis *Stabilizer Codes and Quantum Error Correction* (California Institute of Technology, Pasadena, CA)
[17] Cubit T S, Verstraete F, Dür W and Cirac J I 2003 *Phys. Rev. Lett.* **91** 037902
[18] Quiroga L and Johnson N F 1999 *Phys. Rev. Lett.* **83** 2270
[19] Lu H X, Pan J W, Liu K J and Zhang Y D 2000 *Chin. Phys.* **9** 5
[20] Maham G D 1990 *Many-Particle Physics* (New York: Plenum)
[21] Lee T D 2006 *Chin. Phys.* **15** 1125
[22] Horodecki M, Horodecki P and Horodecki R 1996 *Phys. Lett.* A **223** 1