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Exact solutions of the Klein–Gordon equation with Makarov potential and a recurrence relation

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In this paper, the Klein–Gordon equation with equal scalar and vector Makarov potentials is studied by the factorization method. The energy equation and the normalized bound state solutions are obtained, a recurrence relation between the different principal quantum number n corresponding to a certain angular quantum number ℓ is established and some special cases of Makarov potential are discussed.

Keywords: Makarov potential, Klein–Gordon equation, bound state, factorization method
PACC: 0365G, 1110Q, 1240Q

1. Introduction

It is well known that when a particle is in a strong potential field, the relativistic effect yields the correction for non-relativistic quantum mechanics.[1] Taking the relativistic effect into account, one could apply the Klein–Gordon equation to the treatment of a zero-spin particle and apply the Dirac equation to that of a 1/2-spin particle. Fishbane *et al*[2] have showed that confining potential in the Dirac equation involving the interaction of fermions leads to no Klein paradoxes if the strength of the vector potential is appropriately limited in comparison with the scalar potential. Su and Zhang[3] have demonstrated that if we want to get a confining solution from the Dirac equation, a scalarlike potential must be introduced, which is equivalent to a dependence of the rest mass upon position. Some authors studied the bound states of the Klein–Gordon equation and Dirac equation with mixed typical potentials on the condition that each scalar potential of them is equal to or stronger than its vector potential,[4−17] etc. Originally, these potentials were proposed for describing the properties of diatomic molecules.

In recent years, considerable attention has been drawn to the non-central potentials. Due to the possible applications in quantum chemistry and nuclear physics to describe ring-shaped molecules like benzene and interactions between deformed pairs of nuclei, these potentials are studied from both the non-relativistic and relativistic quantum mechanical viewpoints with various approaches and include the Hartmann potential,[18,19] ring-shaped oscillator potential,[20,21] new anharmonic oscillator potential,[22,23] double ring-shaped oscillator potential,[24] and Coulomb potential plus a new ring-shaped potential[25] etc. In this domain, another non-central potential is proposed by Makarov *et al* and has the form[26]

$$V(r,\theta)=\frac{\alpha}{r}+\frac{\beta}{r^2\sin^2\theta}+\gamma\frac{\cos\theta}{r^2\sin^2\theta}, \qquad (1)$$

where α,β,γ are three dimensionless real parameters. The general solutions of the Schrödinger equation with Makarov potential are obtained in different ways.[27,28] Recently, Yasuk and co-workers solved the Klein–Gordon equation with equal scalar and vector Makarov potentials by the Nikiforov–Uvarov method,[29] in which the Klein–Gordon equation with Makarov potentials is divided into angular and radial parts, and each of them is transformed into a generalized equation of hypergeometric type, the radial and angular wavefunctions are given in terms of the Laguerre and Jacobi polynomials respectively.

In this study, we apply a different approach, the factorization method to solve the Klein–Gordon equation for this system.[30,31] In principle, the factorization method is an operator process, which could degenerate a second-order differential equation into a first-order differential equation and enables us to find immediately the eigenvalues and manufacturing process for the normalized eigenfunctions; this method is usually used in solving the bound state problem

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analytically and plays an important role in various domains.[22] Within the framework of this treatment, this paper is organized as follows. In Section 2 the angular and radial parts of Klein–Gordon equation with Makarov potentials are solved by the factorization method, and the bound state solutions of the angular and radial parts are presented in terms of the elementary functions respectively, which make the normalization of the wavefunctions much easier than that of the Laguerre and Jacobi polynomials or the other special functions. Furthermore, utilizing the radial solution we obtained the bound state condition, $\alpha < 0$, this demands that the first term of Makarov potential must be Coulomb potential. In Section 3 we established a recurrence formula between the different principal quantum number n corresponding to a certain angular quantum number ℓ by factorization method. This formula is the basis for the calculation of a large class of matrix elements involving the Hartmann potential, Coulomb potential, etc., and is useful in the calculation of transition probabilities in quantum mechanics. Finally, some discussions on Makarov potential and concluding remarks are given in Section 4.

2. Solutions of the Klein–Gordon equation with Makarov potential

The Klein–Gordon equation with scalar potential $S(\boldsymbol{r})$ and vector potential $V(\boldsymbol{r})$ is ($\hbar = c = 1$)

$$[\hat{p}^2 + (M + S(\boldsymbol{r}))^2 - (E - V(\boldsymbol{r}))^2]\psi(\boldsymbol{r}) = 0, \quad (2)$$

where $\hat{p}$ is the momentum operator, E is the energy and M is the rest mass of the particle respectively. Under the condition of equal scalar and vector Makarov potential, Eq.(2) becomes

$$\Big[\hat{p}^2 + (M^2 - E^2) + 2(M + E)\Big(\frac{\alpha}{r} + \frac{\beta}{r^2 \sin^2\theta} + \gamma\frac{\cos\theta}{r^2 \sin^2\theta}\Big)\Big]\psi(\boldsymbol{r}) = 0. \quad (3)$$

In spherical coordinates, separate Eq.(3) into variables and select

$$\psi(\boldsymbol{r}) = r^{-1}u(r)H(\theta)\mathrm{e}^{\mathrm{i}m\varphi}\ (m = 0, \pm1, \pm2\ldots). \quad (4)$$

Substituting Eq.(4) into Eq.(3), we obtain the differential equations for $H(\theta)$ and $u(r)$ as

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\Big(\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\Big)H(\theta) - \Big[\frac{m^2 + 2(M + E)(\beta + \gamma\cos\theta)}{\sin^2\theta}\Big]H(\theta) + \lambda H(\theta) = 0, \quad (5)$$

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}u(r) - \Big[\frac{\lambda}{r^2} + \frac{2(M + E)\alpha}{r}\Big]u(r) + (E^2 - M^2)u(r) = 0. \quad (6)$$

Where λ is the separation constant. Equations (5) and (6) are angular and radial equations respectively, and they will be solved by factorization method briefly in the following.

Now we study the angular equation (5). Let

$$Y(\theta) = \sin^{1/2}\theta H(\theta), \quad (7)$$

Eq.(5) could be arranged as

$$\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}Y(\theta) - \Big[\frac{m^2 - 1/4 + 2(M + E)(\beta + \gamma\cos\theta)}{\sin^2\theta}\Big]Y(\theta) + \Big(\lambda + \frac{1}{4}\Big)Y(\theta) = 0. \quad (8)$$

The potential function in Eq.(8) is exactly the type A factorization. Compared with the general expression of type A factorization, one only lets

$$a = 1, \quad c = -\frac{1}{2}, \quad p = 0, \quad d = \sqrt{2(M + E)\beta} = g, \quad \gamma = \sqrt{\frac{2\beta m^2}{E + M}}, \quad x = \theta, \quad (9)$$

and replaces λ by $\lambda+1/4$, where c is another constant. Therefore, the factorization is given by

$$k(\theta,m)=\Big(m-\frac{1}{2}\Big)\cot\theta+\frac{g}{\sin\theta}, \tag{10}$$

$$L(m)=\Big(m-\frac{1}{2}\Big)^2. \tag{11}$$

Since $L(m)$ is an increasing function of m, hence Eq.(8) belongs to the Class I problem of type A factorization and the eigenvalues are

$$\lambda+\frac{1}{4}=L(\ell+1), \tag{12}$$

so that

$$\lambda=\ell(\ell+1), \tag{13}$$

where $\ell=0,1,2\cdots$ and $\ell\geq m$. Correspondingly, the normalized eigenfunction $Y_\ell^\ell(\theta)$ is obtained by

$$\begin{aligned}Y_\ell^\ell(\theta)&=C\mathrm{e}^{\int k(\theta,\ell+1)\mathrm{d}\theta}=C\mathrm{e}^{\int[(\ell+1/2)\cot\theta+g/\sin\theta]\mathrm{d}\theta}\\&=C'\sin^{(\ell+1/2+g)}\frac{\theta}{2}\cos^{(\ell+1/2-g)}\frac{\theta}{2}.\end{aligned} \tag{14}$$

In which $C'=2C$ is the normalization constant and is determined by the condition

$$\int_0^\pi\big[Y_\ell^\ell(\theta)\big]^2\mathrm{d}\theta=1. \tag{15}$$

From Eqs.(14) and (15) we get

$$\begin{aligned}Y_\ell^\ell(\theta)=&\Big[\frac{\Gamma(2\ell+2)}{\Gamma(\ell+g+1)\Gamma(\ell-g+1)}\Big]^{1/2}\\&\times\sin^{(\ell+1/2+g)}\frac{\theta}{2}\cos^{(\ell+1/2-g)}\frac{\theta}{2}.\end{aligned} \tag{16}$$

It is obvious that $Y_\ell^\ell(\theta)$ satisfies the bound state conditions of Eq.(8)

$$Y_\ell^\ell(\theta)|_{\theta=0}=0,\quad Y_\ell^\ell(\theta)|_{\theta=\pi}=0. \tag{17}$$

The whole normalized ladder eigenfunctions belong to $\lambda=\ell(\ell+1)$ are

$$Y_\ell^0,Y_\ell^1,Y_\ell^2\cdots Y_\ell^\ell, \tag{18}$$

in which

$$\begin{aligned}Y_\ell^{m-1}(\theta)=&\Big[(\ell+m)(\ell+1-m)\Big]^{-1/2}\\&\times\Big[\Big(m-\frac{1}{2}\Big)\cot\theta+\frac{g}{\sin\theta}+\frac{\mathrm{d}}{\mathrm{d}\theta}\Big]Y_\ell^m(\theta).\end{aligned} \tag{19}$$

When m is a negative integer, we have the following relation:

$$Y_\ell^{-m}(\theta)=(-1)^mY_\ell^m(\theta). \tag{20}$$

Finally, the normalized angular eigenfunctions are obtained as

$$H_{\ell m}(\theta)=(\sin\theta)^{-\frac{1}{2}}Y_\ell^m(\theta). \tag{21}$$

Substituting Eq.(13) into Eq.(6) we obtain

$$\begin{aligned}&\frac{\mathrm{d}^2}{\mathrm{d}r^2}u(r)-\Big[\frac{\ell(\ell+1)}{r^2}+\frac{2(M+E)\alpha}{r}\Big]u(r)\\&+(E^2-M^2)u(r)=0.\end{aligned} \tag{22}$$

It is easy to see that potential function in Eq.(22) is the type F factorization. Compared with the general expression of type F factorization, we let

$$q=(M+E)\alpha,\quad x=r, \tag{23}$$

and replace m by ℓ and

$$\lambda'=(E^2-M^2). \tag{24}$$

Therefore, the factorization is given by

$$k(x,m)=k(r,\ell)=\frac{\ell}{r}+\frac{(M+E)\alpha}{\ell}, \tag{25}$$

$$L(m)=L(\ell)=-\frac{(M+E)^2\alpha^2}{\ell^2}. \tag{26}$$

Since $L(\ell)$ is an increasing function of ℓ, hence Eq.(21) belongs to the Class I problem of type F factorization and the eigenvalues are

$$\lambda'=L(\ell+1)=-\frac{(M+E)^2\alpha^2}{(\ell+1)^2}. \tag{27}$$

With a new notation

$$n=\ell+1,\quad \ell+1=1,2,3\cdots\leq n, \tag{28}$$

where n is the principal quantum number and ℓ is the angular quantum number, we obtain the energy equation from Eqs.(24), (26) and (27)

$$E=M\Big(\frac{n^2-\alpha^2}{n^2+\alpha^2}\Big). \tag{29}$$

This result is consistent with that of the function analysis method given in Ref.[32]. The solutions of Eq.(22) belonging to eigenvalue $\lambda'=L(\ell+1)=L(n)$ are

$$\begin{aligned}u_n^\ell(r)=u_n^{n-1}(r)&=C\mathrm{e}^{\int k(r,n)\mathrm{d}r}\\&=Cr^n\mathrm{e}^{(M+E)\alpha r/n}.\end{aligned} \tag{30}$$

For the principal quantum number $n>0$, then $M+E=M-|E|>0$, the bound state conditions of Eq.(22) confine the parameter α in Eq.(30) to be negative. Therefore, we can write Eq.(30) as

$$u_n^\ell(r)=u_n^{n-1}(r)=Cr^n\mathrm{e}^{-(M+E)|\alpha|r/n}, \tag{31}$$

and the normalization constant C is determined by the condition

$$\int_0^\infty [u_n^{n-1}(r)]^2 \mathrm{d}r = 1. \tag{32}$$

Then we obtain

$$u_n^{n-1}(r) = [(2n)!]^{-1/2}\Big[\frac{2(M+E)|\alpha|}{n}\Big]^{n+1/2} \times r^n \mathrm{e}^{-(M+E)|\alpha|r/n}. \tag{33}$$

When $r = 0, u_n^{n-1}(r) = 0$, and when $r \to \infty$, $u_n^{n-1}(r) \to 0$. Indeed, $u_n^{n-1}(r)$ is the top of bound state solutions of Eq.(22). The whole normalized ladder eigenfunctions belonging to $\lambda' = L(\ell+1) = L(n)$ are

$$u_n^0, u_n^1, u_n^2 \cdots u_n^{n-1}, \tag{34}$$

in which

$$u_n^{\ell-1}(r) = {}^+\mathcal{H}_n^\ell \, u_n^\ell(r). \tag{35}$$

Where the operator in this case has the form

$${}^\pm\mathcal{H}_n^\ell = \frac{n\ell}{(M+E)|\alpha|}[(n-\ell)(n+\ell)]^{-1/2} \times \Big[\frac{\ell}{r} - \frac{(M+E)|\alpha|}{\ell} \pm \frac{\mathrm{d}}{\mathrm{d}r}\Big]. \tag{36}$$

Finally, we write $u_n^\ell(r)$ as $u_{n\ell}(r)$ and get the exact solutions of the Klein–Gordon equation with Makarov potential

$$\psi(\boldsymbol{r}) = r^{-1}u_{n\ell}(r)H_{\ell m}(\theta)\mathrm{e}^{\mathrm{i}m\phi}. \tag{37}$$

3. A recurrence relation between the different principal quantum number n

For the radial Eq.(22) satisfies the type F factorization, from Eq.(33) we could establish the recurrence formula between the different principal quantum number n corresponding to a certain angular quantum number ℓ by factorization method (see Chapter 8 in Ref.[31]). Introducing a new variable

$$\rho = (M+E)|\alpha|r, \tag{38}$$

Eq.(33) could be rewritten as

$$u_n^{n-1}(\rho) = \Big(\frac{2}{n}\Big)^{n+1/2}[(2n)!]^{-1/2} \times [(M+E)|\alpha|]^{1/2}\rho^n \mathrm{e}^{-\rho/n}. \tag{39}$$

The $u_n^\ell(\rho)$ is defined by

$$u_n^{\ell-1}(\rho) = {}^+\mathcal{H}_n^\ell u_n^\ell(\rho), \tag{40}$$

and Eq.(36) becomes

$${}^\pm\mathcal{H}_n^\ell = n\ell\Big[(n-\ell)(n+\ell)\Big]^{-1/2}\Big[\frac{\ell}{\rho} - \frac{1}{\ell} \pm \frac{\mathrm{d}}{\mathrm{d}\rho}\Big]. \tag{41}$$

Since Eq.(41) is the same as that of the Kepler problem, we could establish the recurrence formula by using the same procedure. To do this, we introduce a new function $u_n^\ell(s)$. This function is defined by the same recurrence formula (40) as the corresponding function $u_n^\ell(\rho)$, the only difference is that function $u_n^{n-1}(s)$ are now taken to be

$$u_n^{n-1}(s) = \Big(\frac{2}{n}\Big)^{n+1/2} \times [(2n)!]^{-1/2}[(M+E)|\alpha|]^{1/2}\rho^n \mathrm{e}^{-s\rho}. \tag{42}$$

According to the discussion in Ref.[31], $u_n^\ell(s)$ are neither orthogonal or nor satisfying our differential equation, but they have the following important property:

$$[u_n^\ell(s)]_{s=1/n} = u_n^\ell(\rho). \tag{43}$$

We can now introduce an operator O_{n+1}^ℓ which enables us to change $u_n^\ell(s)$ into $u_{n+1}^\ell(s)$ from which our solutions can be got by Eqs.(43) and (38). In fact

$$u_{n+1}^\ell(s) = \mathrm{O}_{n+1}^\ell u_n^\ell(s), \tag{44}$$

where

$$\mathrm{O}_{n+1}^\ell = \frac{n^{\ell+2}}{(n+1)^{\ell+2}(2n+1)}\Big[\frac{n+\ell+1}{n-\ell}\Big]^{1/2} \times \Big[2n+1+\Big(s+\frac{1}{n}\Big)\frac{\mathrm{d}}{\mathrm{d}s}\Big]. \tag{45}$$

From Eqs.(41) and (45) it is easily seen that

$${}^+\mathcal{H}_{n+1}^\ell = \frac{n+1}{n}\Big[\frac{(n-\ell)(n+\ell)}{(n+1-\ell)(n+1+\ell)}\Big]^{1/2}{}^+\mathcal{H}_n^\ell \tag{46}$$

and

$$\mathrm{O}_{n+1}^{\ell-1} = \frac{n+1}{n}\Big[\frac{(n-\ell)(n+\ell)}{(n+1-\ell)(n+1+\ell)}\Big]^{1/2}\mathrm{O}_{n+1}^\ell. \tag{47}$$

Using Eqs.(40), (44), (45) and the above two equations we find

$$\begin{aligned} u_{n+1}^{\ell-1}(s) &= {}^+\mathcal{H}_{n+1}^\ell u_{n+1}^\ell(s) \\ &= \frac{n+1}{n}\Big[\frac{(n-\ell)(n+\ell)}{(n+1-\ell)(n+1+\ell)}\Big]^{1/2} \\ &\quad \times {}^+\mathcal{H}_n^\ell \mathrm{O}_{n+1}^\ell u_n^\ell(s) \\ &= \mathrm{O}_{n+1}^{\ell-1}u_n^{\ell-1}(s)\,. \end{aligned} \tag{48}$$

Since the operators $\mathcal{H}$ and O commute, Eq.(48) gives the recurrence relations between different principal quantum number n with a certain angular quantum number ℓ.

4. Conclusions

Makarov potential is a typical non-central potential. For $\gamma = 0$, $\alpha = -\eta\sigma^2 e^2$ and $\beta = \eta^2\sigma^2\hbar^2/2M$, Makarov potential is reduced to the Hartmann potential. For $\gamma = 0$, $\beta = 0$ and $\alpha = -ze^2$, Makarov potential is reduced to the Coulomb potential. Therefore, under the condition of equal scalar and vector potentials, the Klein–Gordon equation with Hartmann potential or Coulomb potential are the special cases of Makarov potential.

In this study, we have obtained the exact normalized solutions of the angular and radial parts of the Klein–Gordon equation with equal scalar and vector Makarov potentials by using the factorization method. The energy equation and the bound state condition are gained from the radial equation respectively. Furthermore, we established a recurrence formula between the different principal quantum number n corresponding to a certain quantum number ℓ by factorization method. The factorization method owes its existence primarily to a paper by Schrödinger,[33] and then his ideas have been considerably generalized. As shown above, factorization method is general and worth extending to the bound state problem with other non-central potentials.

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