Vol 16 No 7, July 2007
1009-1963/2007/16(07)/1858-05

Chinese Physics

© 2007 Chin. Phys. Soc.
and IOP Publishing Ltd

An important property of entanglement: pairwise entanglement that can only be transferred by an entangled pair*

Xi Xiao-Qiang(惠小强)[a)b)†] and Liu Wu-Ming(刘伍明)[b)]

[a)] *Department of Applied Mathematics and Physics, Xi'an Institute of Posts and Telecommunications, Xi'an* 710061, *China*
[b)] *Beijing National Laboratory for Condensed Matter Physics, Institute of Physics, Chinese Academy of Sciences, Beijing* 100080, *China*

(Received 11 September 2006; revised manuscript received 25 January 2007)

Based on the calculation of all the pairwise entanglements in the n ($n \leq 6$)-qubit Heisenberg XX open chain with system impurity, we find an important result: pairwise entanglement can only be transferred by an entangled pair. The non-nearest pairwise entanglements will have the possibility to exist as long as there has been even number of qubits in their middle. This point indicates that we can obtain longer distance entanglement in a solid system.

Keywords: pairwise entanglement, transfer of entanglement, Heisenberg XX chain
PACC: 0365, 7510J

1. Introduction

It is well known that entanglement has some prominent applications. Firstly, it can be used to test some fundamental questions of the quantum mechanics. Secondly, it is the key ingredient of the quantum information processing such as quantum teleportation, superdense coding, quantum computation, quantum communication, quantum computational speed-ups and quantum cryptographic protocols. Thirdly, the entangled states can be used to the sensitivity of interferometric measurements such as quantum lithography,[1] quantum optical gyroscope,[2] quantum clock synchronization and positioning[3] and frequency metrology.[4] Fourthly, entanglement plays a central role in the study of strongly correlated quantum systems;[5] ground-state entanglement can help us to understand the quantum phase transition,[6] Mott insulator-superfluid transition and quantum magnet–paramagnet transition.

In all the implications of entanglement, the most important thing is to find the entangled pair. Photons,[7] the energy level of different trapped ions[8] (or different atoms[9]), qubit in crystal lattices,[10] qubit in Josephson junctions[11] and Bose–Einstein condensates[12−14] are often used to produce entanglement, but their complicated equipment will prevent them from using cosmically. While using spin chains[15] to produce entanglement seems more convenient, because it can be scaled easily and its equipment will be very simple, many interesting works are developed around entanglement in a spin chain (see Ref.[16] and references therein), while most of them[17,18] are concerned with the nearest pairwise entanglement, which is important but far from perfect. The best entanglement should be the non-nearest pairwise, for example, in a three-qubit Heisenberg XX open chain, the entanglement between the first and the third qubits (if exists) is more practical than that between the first and the second because the former has a farther entanglement distance.

If we can find the law of the non-nearest pairwise entanglement in a spin chain, then we have the possibility to construct a more distant entangled pair; this is the aim of this paper. We choose the Heisenberg XX open chain as the object for study, because it is simple and very useful in quantum information processing: e.g., it can be used for quantum computation[19−22] and quantum communication.[23−26]

Recently, non-nearest pairwise entanglement in the Heisenberg chain has been also discussed;[27−29]

*Project supported in part by the National Natural Science Foundation of China (Grant No 10547008), and the Foundation of Xi'an Institute of Posts and Telecommunications, China (Grant No 105-0414).
†E-mail: xxq@xiyou.edu.cn

the non-nearest entanglement comes from the introduction of magnetic field, which leads to the direct interactions between the non-nearest spin qubits. They have two disadvantages: the first is that the entanglement cannot obtain the maximal value 1 (the numerical results[27,28] show that the non-nearest entanglement does not exceed 0.5), because the magnetic field only causes the interaction between the z components of the non-nearest spin qubits; the second is that the magnetic field will cause complexity of the system and its application.

We want to seek the relation between the pairwise entanglement and the system impurity, since the system impurity is obviously simpler than magnetic impurity, and maybe we can find the maximal non-nearest pairwise entanglement in this system. This idea comes from our previous paper,[30] in which we have introduced the system impurity to three-qubit Heisenberg XX spin ring and find that the system impurity can control the nearest pairwise entanglement, so we suppose that the system impurity will help us to understand the non-nearest pairwise entanglement.

The main tool for discussing pairwise entanglement is the concept of entanglement formation and concurrence.[31−33] Entanglement formation can be used to measure the pairwise entanglement; and concurrence C ranges from zero to one and is linearly related to the former, so concurrence is often used to measure the pairwise entanglement. We adopt this measurement and use C_{ij} to express the pairwise entanglement between the ith and jth qubits. $C_{ij} = \max\{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0\}$, where λ_k $(k = 1, 2, 3, 4)$ are the square roots of the eigenvalues of the operator $\hat{\rho}_{ij} = \rho_{ij}(\sigma_i^y \otimes \sigma_j^y)\rho_{ij}^*(\sigma_i^y \otimes \sigma_j^y)$ in descending order, $\rho_{ij} = \mathrm{Tr}_{\mathrm{non}(ij)}\rho$ is the reduced density matrix of the system.

In order to introduce the impurity, we write the Hamiltonian of the N-qubit Heisenberg XX open chain as follows:

$$H = \sum_{i=1}^{N-1} J_i J_{i+1}(\sigma_i^+ \sigma_{i+1}^- + \sigma_{i+1}^+ \sigma_i^-), \qquad (1)$$

where $\sigma^{\pm} = \frac{1}{2}(\sigma^x \pm \mathrm{i}\sigma^y)$, and $\sigma^x, \sigma^y, \sigma^z$ are the Pauli matrices, J_i is the contribution of the ith lattice to the exchange hopping and $J_i J_{i+1}$ is the exchange hopping between the ith and the $(i+1)$th lattices. If the ith lattice is impurity, then we let J_i be the impurity parameter, $J_k = 1$ $(k \neq i)$ are the normal parameters.

2. Results

Firstly we begin with the simplest case of the above Hamiltonian, i.e. $N = 3$, different impurities can be seen from Fig.1. The results of calculation show that using J_1 or J_3 can make $C_{12} = 1$, but using J_2 can only make $(C_{12})_{\max} = 0.457$. This can be understood easily: increasing J_1 or decreasing J_3 will make the first and the second qubits couple stronger, i.e. increasing their interaction and entanglement, while J_2 is an overall factor, in this sense this impurity system is equivalent to a normal system. Next we discuss the relation between C_{13} and impurity parameter J_i. Unfortunately, no matter how we change the impurity parameter, we have $C_{13} = 0$, which seems to give no confusion because there is no direct interaction between them.

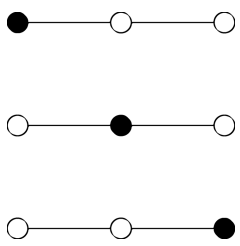


Fig.1. The diagrams of the impurity position in the 3-qubit open chain, where the solid circle is the impurity qubit, the hollow circle is the normal qubit, similarly as in Fig.2. and Fig.3. The sites are labelled 1, 2, 3 from left to right.

Then let us see the next simplest case, i.e. $N = 4$. As the impurity changing from the first to the fourth site, we need to discuss the value of C_{12}, C_{23}, C_{13}, and C_{14}. C_{12} and C_{23} are the nearest pairwise entanglement, C_{13} and C_{24} are the non-nearest pairwise entanglement. Different impurities can be seen from Fig.2.

Tedious calculations show the following results.

(1) Changing impurity sites, increasing J_1 (or J_4) and decreasing J_2 (or J_3) can make $C_{12} = 1$. The physical meaning is clear: increasing J_1 (or J_4) and decreasing J_2 (or J_3) mean that we isolate the first and the second qubits from the whole chain, and their interaction is equivalent to the two-qubit case, of course it can lead to the maximal entanglement. (2) The maximal value of C_{23} is 0.457, because increasing J_1 (or J_4) or decreasing J_2 (or J_3) will make the middle qubits couple more weakly, so their entanglement will be decreased. Decreasing J_1 (or J_4) or increasing J_2 (or J_3) means that a qubit will be taken away from

the whole chain, the four-qubit chain then becomes a three-qubit chain with a uniform coupling; its maximal value of pairwise entanglement is just 0.457. (3) No matter how we change the impurity parameters, $C_{13} = 0$, i.e. there is no entanglement between the next nearest qubits. We believe that the reason is that there is no direct interaction between them. (4) There is entanglement between the boundary qubits, its maximal value $(C_{14})_{\max} = 0.457$, C_{14} increases with decreasing J_1 (or J_4) or increasing J_2 (or J_3). Obviously, there is no direct interaction between the boundary qubits, i.e. 'no direct interaction no entanglement' is invalid for explaining the existence of C_{14}, hence there must be other reason, which is just what we are interested in.

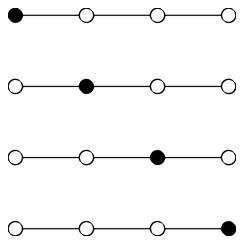


Fig.2. The diagrams of the impurity position in the four-qubit open chain. The sites are labelled 1, 2, 3, 4 from left to right.

Since 'no direct interaction no entanglement' cannot present a harmonious explanation between $C_{13} = 0$ and $C_{14} \neq 0$, so we concentrate on the difference between the qubits in the middle of the chain. Let us see this difference from Fig.3.

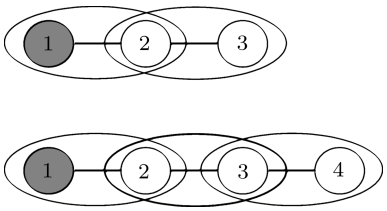


Fig.3. The comparison diagrams of the three-qubit and four-qubit open chain. The ellipse denotes that there is entanglement between the qubits in it.

The difference is very clear: there is one qubit in the middle of the first and the third qubits, while two qubits (an entangled pair) between the first and the fourth qubits. Obviously, it is this difference that makes the different entanglement of C_{13} and C_{14}.

An explanation is given as follows: for C_{13}, there is only one qubit in the middle, one qubit cannot entangle with itself, although $C_{12} \neq 0$ and $C_{23} \neq 0$, C_{13} is still equal to zero, as if the entanglement is broken in the middle. For the four-qubit case, there is an entangled pair in the middle, $C_{12} \neq 0, C_{23} \neq 0$ and $C_{34} \neq 0$, so $C_{14} \neq 0$, i.e. C_{23} acts as a bridge to transfer the pairwise entanglement; that is to say, pairwise entanglement can be transferred by entangled pair.

For further understanding of C_{14}, we plot the comparison diagrams for C_{12}, C_{23}, C_{34} and C_{14} at fixed $J_1 = J$ (i.e. the first site is impurity) and fixed temperature (see Fig.4).

From Fig.4 we can see the transfer law of pairwise entanglement: C_{14} depends on the smallest one of C_{12}, C_{34} and C_{23} ($\min(C_{12}, C_{34}, C_{23})$), when temperature is lower, C_{14} is equal to $\min(C_{12}, C_{34}, C_{23})$ approximately; when temperature is higher, C_{14} is smaller than $\min(C_{12}, C_{34}, C_{23})$ but with similar trend. Changing the impurity sites, similar results can be obtained. If we want to find the maximal C_{14}, we must increase $\min(C_{12}, C_{34}, C_{23})$. For the specificity of C_{23}, we call it 'entangled kernel'.

We wonder how C_{14} changes if we make the entangled kernel $C_{23} = 1$. Considering the nearest interaction theory (for nearest pair, the stronger the interaction the bigger the pairwise entanglement), we need to make the interaction between the second and the third qubits strong; such a model is constructed in Eq.(1). Let $N = 4$, $J_1 = J_4 = J$, $J_2 = J_3 = 1$, i.e. the middle qubits are normal sites and the boundary qubits are impurity sites; as long as J is smaller than 1, the above condition is satisfied, and obviously we have $C_{12} = C_{34}$ under this condition. Numerical results under certain conditions are shown in Fig.5.

Figure 5 shows that at lower temperature, C_{14} is only dependent on C_{23} even $C_{12} = C_{34} = 0$ (but their interaction cannot be too small). Choosing proper J and temperature, we can make $C_{14} = 1$. This is an important support for the importance of entangled kernel.

As a further support for our theory, we calculate the pairwise entanglement in the five-qubit open chain with boundary impurities and find that $C_{13} = 0, C_{15} = 0$ and $C_{14} \neq 0$. For more than five-qubit case, the calculation will be very difficult and tedious, here we only construct a simple six-qubit case with $J_1 = J_6 = 0.1, J_2 = J_5 = 1, J_3 = J_4 = 10$, and obtain $C_{16} = 0.96098$ in the ground state (its entanglement is maximal), of course we can construct a longer distant spin chain with maximal pairwise entanglement as long as the interaction strength decreases monotonically from the middle to the boundary.

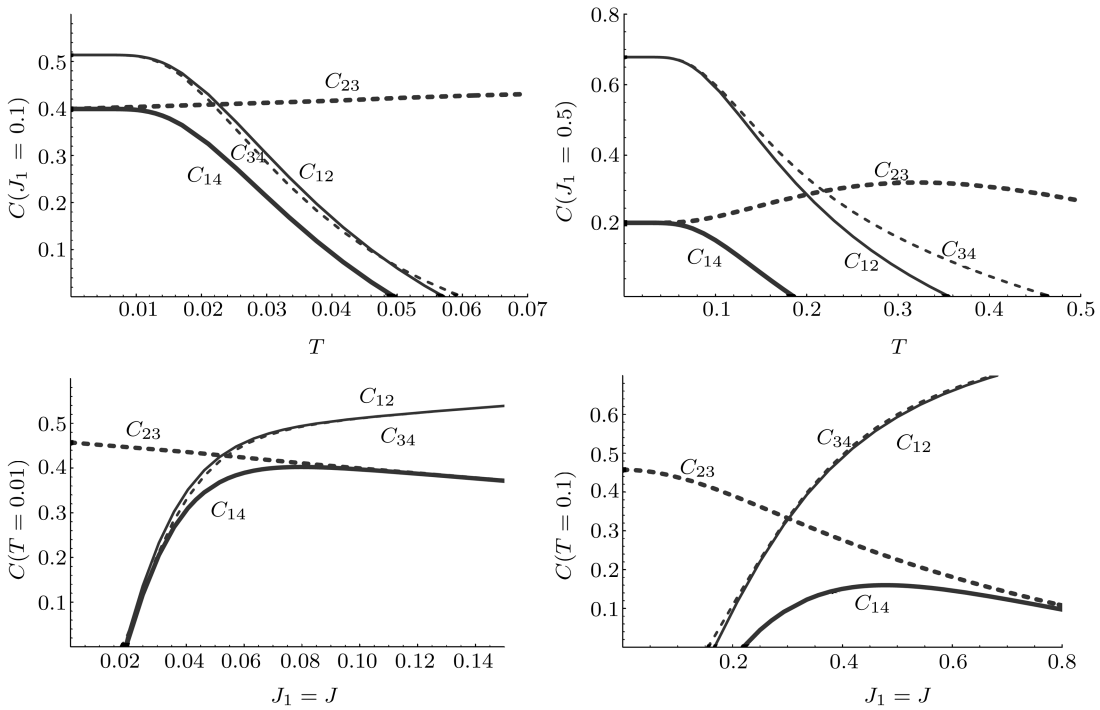


Fig.4. The comparison diagrams for C_{12}, C_{23}, C_{34} and C_{14} at fixed $J_1 = J$ and temperatures.

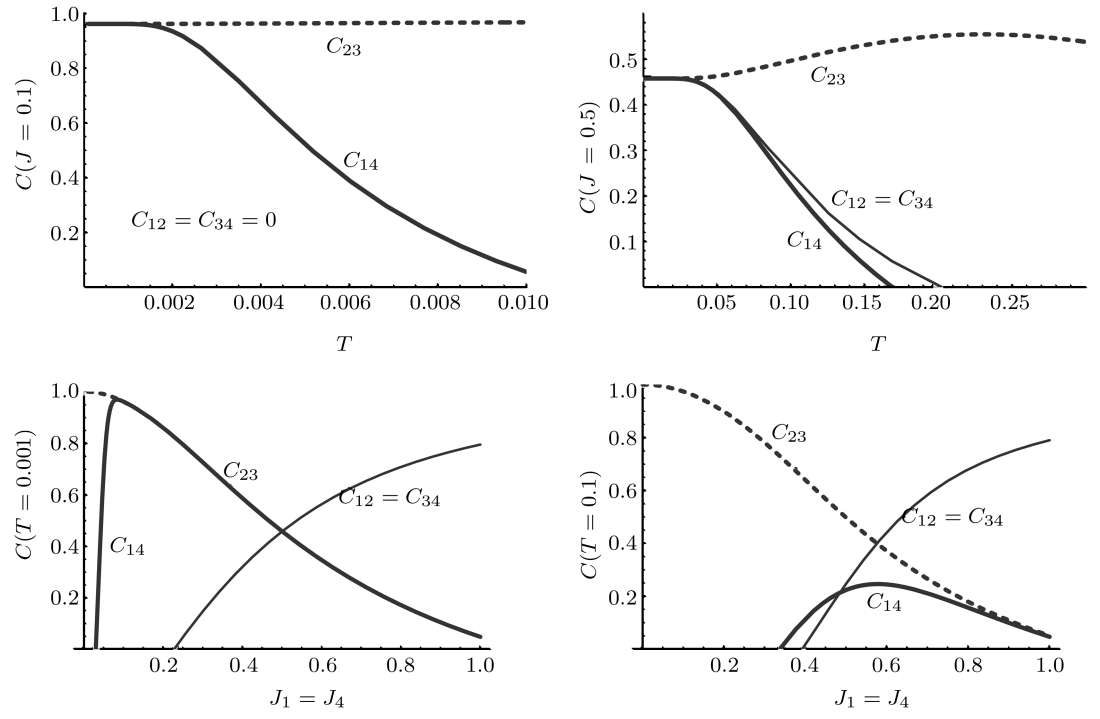


Fig.5. The comparison diagrams for C_{12}, C_{23}, C_{34} and C_{14} at fixed $J_1 = J_4 = J$ and temperatures.

3. Discussions

In this paper, we calculate all the pairwise entanglements in the three- and four-qubit Heisenberg open chains with system impurity, and find some interesting results. For the nearest pairwise entanglement, the stronger the interaction, the bigger the entanglement. For the non-nearest pairwise entanglement, the condition of the existence of entanglement is determined by the number of qubits in the middle, in which 'odd' means no entanglement and 'even' means the possibility of existence of entanglement. This conclusion is

supported by $C_{13} = 0$, $C_{15} = 0$ and $C_{14} \neq 0$. The attenuation of non-nearest pairwise entanglement (if exists) with temperature or impurity is quicker than that of the nearest pairwise entanglement, i.e. the transferred entanglement is more sensitive to the temperature and impurity.

The above conclusions are based on the open chain with qubit number no more than six, but they may be general to other qubit cases. Firstly, pairwise entanglement is always connected with double qubits, so its transfer must depends on the entangled kernel, at this point they are coincident. Secondly, from the viewpoint of physics, the transfer of pairwise entanglement is the transfer of pairwise interaction, the best example is the four-qubit chain with boundary impurities.

These conclusions are very important for a solid system. Using the transfer of entanglement we can construct pairwise entanglement with a longer distance, we can increase this distance to a practical scale for solid state quantum information. As we know, there are at least two possible ways to realize such a system: electrons floating on liquid helium and electron spins in coupled semiconductor quantum dots.

We do not know if the transfer of entanglement in spin chain is also suitable for the entangled photons, because there is some difference in the essence of production. If it is possible, we will no longer worry about the attenuation of entangled photons. We also do not know if we can use the non-maximal pairwise entanglement; if we can, at what value can we obtain the satisfactory fidelity in quantum information processing?

References

[1] Boto A N, Kok P, Abrams D S, Braunstein S L, Williams C P and Dowling J P 2000 *Phys. Rev. Lett.* **85** 2733

[2] Dowling J P 1998 *Phys. Rev.* A **57** 4736

[3] Jozsa R, Abrams D S, Dowling J P and Williams C P 2000 *Phys. Rev. Lett.* **85** 2010

[4] Bollinger J J, Itano W M, Wineland D J and Heinzen D J 1996 *Phys. Rev.* A **54** R4649

[5] Preskill J 2000 *J. Mod. Opt.* **47** 127

[6] Osborne T J and Nielsen M A 2002 *Phys. Rev.* A **66** 032110

[7] Bouwmeester D, Pan J W, Mattle K, Eibl M, Weinfurter H and Zeilinger A 1997 *Nature* **390** 575

[8] Turchette Q A, Wood C S, Myatt C J, Turchette Q A, Leibfried D, Itano W M, Monroe C and Wineland D J 1998 *Phys. Rev. Lett.* **81** 1525

[9] Rauschenbeutel A, Nogues G, Osnaghi S, Bertet P, Brune M, Raimond J M and Haroche S 2000 *Science* **288** 2024

[10] Yamaguchi F and Yamamoto Y 1999 *Appl. Phys.* A **68** 1

[11] Makhlin Y, Schon G and Shnirman A 1999 *Nature* **398** 305

[12] Sørensen A, Duan L M, Cirac J I and Zoller P 2001 *Nature* (London) **409** 63

[13] Liu W M, Wu B and Niu Q 2000 *Phys. Rev. Lett.* **84** 2294

[14] Wei W 2005 *Chin. Phys.* **14** 2407

[15] Gershenfeld N A and Chuang I L 1997 *Science* **275** 350

[16] Arnesen M C, Bose S and Vedral V 2001 *Phys. Rev. Lett.* **87** 017901

[17] Wang X G 2002 *Phys. Rev.* A **66** 034302

[18] Xi X Q, Chen W X, Hao S R and Yue R H 2002 *Phys. Lett.* A **300** 567

[19] Lidar D A, Bacon D and Whaley K B 1999 *Phys. Rev. Lett.* **82** 4556

[20] Divincenzo D P, Bacon D, Kempe J, Burkard G and Whaley K B 2000 *Nature* **408** 339

[21] Santos L F 2003 *Phys. Rev.* A **67** 062306

[22] Imamoglu A, Awshalom D D, Burkard G, Divincenzo D P, Loss D, Sherwin M and Small A 1999 *Phys. Rev. Lett.* **83** 4204

[23] Bose S 2003 *Phys. Rev. Lett.* **91** 207901

[24] Subrahmanyam V 2004 *Phys. Rev.* A **69** 034304

[25] Osborne T J and Linden N 2004 *Phys. Rev.* A **69** 052315

[26] Liu D and Zhang J F 2006 *Chin. Phys.* **15** 272

[27] Cao M and Zhu S Q 2005 *Phys. Rev.* A **71** 034311

[28] Xi X Q, Chen W X, Liu Q and Yue R H 2006 *Acta Phys. Sin.* **55** 3026 (in Chinese)

[29] Venuti L C, Boschi C D E and Roncaglia M 2006 *Phys. Rev. Lett.* **96** 247206

[30] Xi X Q, Chen W X, Hao S R and Yue R H 2002 *Phys. Lett.* A **297** 291

[31] Wootters W 1998 *Phys. Rev. Lett.* **80** 2245

[32] Hill S and Wootters W K 1997 *Phys. Rev. Lett.* **78** 5022

[33] Coffman V, Kundu J and Wootters W K 2000 *Phys. Rev.* A **61** 052306