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The projective Riccati equation expansion method and variable separation solutions for the nonlinear physical differential equation in physics*

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Using the projective Riccati equation expansion (PREE) method, new families of variable separation solutions (including solitary wave solutions, periodic wave solutions and rational function solutions) with arbitrary functions for two nonlinear physical models are obtained. Based on one of the variable separation solutions and by choosing appropriate functions, new types of interactions between the multi-valued and single-valued solitons, such as a peakon-like semi-foldon and a peakon, a compacton-like semi-foldon and a compacton, are investigated.

Keywords: projective Riccati equation, nonlinear physical equation, variable separation solution, soliton

PACC: 0365, 0340, 0545

1. Introduction

Modern soliton theory is applied in almost all fields of physics, such as fluid mechanics, plasma physics, nonlinear optics, condensed matter physics, etc.[1] Particularly, some localized coherent theories of (2+1)-dimensional soliton systems have been proved to be of considerable interest. For instance, the discovery, via the Bäcklund transformation by Boiti *et al*[2] of dromion type of coherent solutions of the Daver–Stewartson I system has renewed the interest in (2+1)-dimensional soliton systems. In the recent decade, the main effort to study higher-dimensional physical models has focused on single valued localized excitations, such as solitoffs, dromions, rings, lumps, breathers, instantons, peakons, compactons, localized chaotic, and fractal patterns, etc.[3−8] However, in reality there are various complicated phenomena like the ordinary bubbles on a fluid surface, which are folded in a simple way but they cannot be described by the single valued functions obviously, neither can the various types of ocean waves. In Refs.[9,10], some multi-valued localized excitations have been introduced to describe the folded solitary wave and define a new type of soliton, i.e. foldon, if the interaction between the folded solitary waves (FSWs) is fully elastic (that is, the amplitudes, velocities and wave shapes of two solitons do not change after their interaction). Actually, the simple foldons in lower dimensions can be equivalently called the loop solitons which can be found in many (1+1)-dimensional integrable models[11,12] and have been applied to some physical branches like the quantum theory, string theory and particle physics.[13−16] However, comparing with these known lower-dimensional foldons, we know little about foldons in higher dimensions.

In this paper, we try to reconsider this case in higher dimensions. A transformation is first obtained from the well-known projective Riccati equation[17,18] and this transformation is taken as an intermediate to solve the (1+1)-dimensional nonlinear Schrödinger equations

$$u(x,t)_{xx} + u(x,t)v(x,t) - ku(x,t) = 0, \quad (1)$$

$$u(x,t)_x - v(x,t)_t = 0, \quad (2)$$

where k is an arbitrary constant, and the (2+1)-dimensional modified dispersive water-wave (MDWW) system

$$u_{yt} + u_{xxy} - 2v_{xx} - (u^2)_{xy} = 0, \quad (3)$$

$$v_t - v_{xx} - 2(uv)_x = 0, \quad (4)$$

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where $u \equiv u(x, y, t)$ and $v \equiv v(x, y, t)$ are two functions of the independent variables $\{x, y, t\}$. As a result, three types of new variable separation solutions including the traditional travelling wave solutions of these celebrated equations are derived. And then, from a particular variable separation solution and by choosing appropriate functions, novel types of interactions between the multi-valued and single-valued solitons are discussed.

2. Solutions of the projective Riccati equation and its expansion method

The well-known projective Riccati equations read

$$f'(\xi) = pf(\xi)g(\xi), \tag{5}$$

$$g'(\xi) = q + pg^2(\xi) - rf(\xi), \tag{6}$$

where $p \neq 0$ is a real constant, q and r are two real constants.[17,18] Through a direct verification, we have the following lemma.

Lemma When $p = \pm 1$, the functions $f \equiv f(\xi)$ and $g \equiv g(\xi)$ of the projective Riccati equations (5) and (6) have the following solutions:[19]

(a) When $pq < 0$, Eqs.(5) and (6) have a solitary solution

$$f = \frac{q}{r + s\cosh(\sqrt{-pq}\xi) + h\sinh(\sqrt{-pq}\xi)}, \quad \text{and} \quad g = -\frac{\sqrt{-pq}}{p}\frac{s\sinh(\sqrt{-pq}\xi) + h\cosh(\sqrt{-pq}\xi)}{r + s\cosh(\sqrt{-pq}\xi) + h\sinh(\sqrt{-pq}\xi)}, \tag{7}$$

where s and h are arbitrary constants. Here, the relation between f and g is

$$g^2 = -\frac{1}{p}\left[q - 2rf + \frac{r^2 + h^2 - s^2}{q}f^2\right]. \tag{8}$$

(b) When $pq > 0$, we have the trigonometric function solution

$$f = \frac{q}{r + s\cos(\sqrt{pq}\xi) + h\sin(\sqrt{pq}\xi)}, \quad \text{and} \quad g = \frac{\sqrt{pq}}{p}\frac{s\sin(\sqrt{pq}\xi) - h\cos(\sqrt{pq}\xi)}{r + s\cos(\sqrt{pq}\xi) + h\sin(\sqrt{pq}\xi)}, \tag{9}$$

where s and h are arbitrary constants. Here, the relation between f and g is

$$g^2 = -\frac{1}{p}\left[q - 2rf + \frac{r^2 - h^2 - s^2}{q}f^2\right]. \tag{10}$$

(c) When $q = 0$, Eqs.(5) and (6) admit a rational solution

$$f = \frac{2}{pr\xi^2 + C_1\xi - C_2},$$

and

$$g = -\frac{2pr\xi + C_1}{(pr\xi^2 + C_1\xi - C_2)p}, \tag{11}$$

where C_1, C_2 and r are arbitrary constants. The relation between f and g then satisfies

$$g^2 = \frac{2r}{p}f + \frac{C_1^2 + 4C_2pr}{4}f^2. \tag{12}$$

The basic idea of our algorithm is as follows. For a given nonlinear partial differential equation (NPDE) with independent variables $X \equiv (x_0 = t, x_1, x_2, \cdots, x_m)$, and dependent variable $u \equiv u(X)$,

$$P(u, u_t, u_{x_i}, u_{x_ix_j}, \cdots) = 0, \tag{13}$$

where P is a polynomial function of its argument, and the subscripts denote the partial derivatives, we assume that its solution is written in the standard truncated Painlevé expansion, namely

$$u = A_0(X) + \sum_{i=1}^{n}(A_i(X)f(\xi(X)) + B_i(X)g(\xi(X)))f^{i-1}(\xi(X)), \tag{14}$$

where $A_0(X), A_i(X)$ and $B_i(X)$ $(i = 1, \cdots, n)$ are arbitrary functions of the variable X to be determined, and $f \equiv f(\xi(X))$ and $g \equiv g(\xi(X)))$ satisfy the projective Riccati equations (5) and (6).

To determine u explicitly, one can take the following procedure. First, in a way similar to the usual function transformation approach, we can de-

termine n by balancing the highest-order nonlinear term with the highest-order partial derivative term in Eq.(13). Second, substituting Eq.(14) along with Eqs.(5) and (6) into the given NPDE, collecting the coefficients of the polynomials of $f^i g^j$ $(i = 0, 1, \cdots, j = 0, 1)$ and eliminating each of them, we can derive a set of partial differential equations of $A_0(X), A_i(X), B_i(X)$ $(i = 1, \cdots, n)$ and $\xi(X)$. Third, to obtain $A_0(X), A_i(X), B_i(X)$ $(i = 1, \cdots, n)$ and $\xi(X)$, we solve these partial differential equations. Finally, substituting $A_0(X), A_i(X), B_i(X)$ $(i = 1, \cdots, n), \xi(X)$ and the solutions (7), (9) and (11) into Eq.(14), one can derive exact solutions of the given NPDE.

3. Applications to the given equations

3.1. The (1+1)-dimensional nonlinear Schrödinger equations

The (1+1)-dimensional nonlinear Schrödinger equations (1) and (2) have been proven to be related to the sine-Gordon equation by Miura transformation[20] and are also called negative Korteweg-de Vries system (NKdV).[21]

According to the balancing procedure, we assume that the solutions of Eqs.(1) and (2) are

$$u = A + Bf(\xi) + Cf^2(\xi) + Dg(\xi) + Ef(\xi)g(\xi), \quad (15)$$

$$v = a + bf(\xi) + cf^2(\xi) + dg(\xi) + ef(\xi)g(\xi), \quad (16)$$

where $A, B, C, D, E, a, b, c, d, e$ and ξ are arbitrary functions of $\{x, t\}$ to be determined, $f \equiv f(\xi)$ and $g \equiv g(\xi)$ satisfy the projective Riccati equations (5) and (6). Substituting Eqs.(15) and (16) along with Eqs.(5) and (6) into Eqs.(1) and (2), collecting coefficients of polynomials of $f^i g^j$ $(i = 0, 1, 2, 3, 4, j = 0, 1)$ and setting each of the coefficients to zero, we can derive a set of partial differential equations of $A, B, C, D, E, a, b, c, d, e$ and ξ. It is difficult to obtain the general solution of these algebraic equations based on the solutions of Eqs.(5) and (6). Fortunately, in the case of $\xi = \chi(x) + \varphi(t)$, where $\chi \equiv \chi(x)$ and $\varphi \equiv \varphi(t)$ are two arbitrary variable separated functions of x and t, respectively, we can obtain solutions of Eqs.(1) and (2), that is

(a) For $p = \pm 1$, q, r, s and h are arbitrary constants $(pq < 0, h^2 + r^2 < s^2)$, we have the solitary solutions

$$\begin{aligned} u = {} & \frac{1}{q\chi_x}(-q\chi_{xxx} + qk\chi_x + pq^2\chi_x^3 + 3p(\pm\sqrt{(s^2 - h^2 - r^2)pq}\chi_x\chi_{xx} - qr\chi_x^3)f(\xi) \\ & -3p(s^2 - h^2 - r^2)\chi_x^3 f^2(\xi) - 3pq\chi_x\chi_{xx}g(\xi) \pm 3p\sqrt{(s^2 - h^2 - r^2)pq}\chi_x^3 f(\xi)g(\xi)), \\ v = {} & \frac{3}{q}p\chi_x\varphi_t f(\xi)(-qr - (s^2 - h^2 - r^2)f(\xi) \pm \sqrt{(s^2 - h^2 - r^2)pq}g(\xi)), \end{aligned} \quad (17)$$

where the coupled functions $f(\xi)$ and $g(\xi)$ satisfy Eq.(7).

(b) For $p = \pm 1$, q, r, s and h are arbitrary constants $(pq > 0, s^2 + h^2 > r^2)$, we have the trigonometric function solutions

$$\begin{aligned} u = {} & \frac{1}{q\chi_x}(-q\chi_{xxx} + qk\chi_x + pq^2\chi_x^3 + 3p(\pm\sqrt{(s^2 + h^2 - r^2)pq}\chi_x\chi_{xx} - qr\chi_x^3)f(\xi) \\ & -3p(s^2 + h^2 - r^2)\chi_x^3 f^2(\xi) - 3pq\chi_x\chi_{xx}g(\xi) \pm 3p\sqrt{(s^2 + h^2 - r^2)pq}\chi_x^3 f(\xi)g(\xi)), \\ v = {} & \frac{3}{q}p\chi_x\varphi_t f(\xi)(-qr - (s^2 - h^2 - r^2)f(\xi) \pm \sqrt{(s^2 - h^2 - r^2)pq}g(\xi)), \end{aligned} \quad (18)$$

where the coupled functions $f(\xi)$ and $g(\xi)$ satisfy Eq.(9).

(c) For $p = \pm 1$, $q = 0$, C_1, C_2 and r are arbitrary constants $(C_1^2 + 4C_2pr > 0)$, we obtain the rational solutions

$$\begin{aligned} u = {} & \frac{1}{4\chi_x}(-4\chi_{xxx} + 4k\chi_x + 6p(\pm\sqrt{C_1^2 + 4C_2pr}\chi_x\chi_{xx} - 2r\chi_x^3)f(\xi) \\ & -3p^2(C_1^2 + 4C_2pr)\chi_x^3 f^2(\xi) - 12p\chi_x\chi_{xx}g(\xi) \pm 6p^2\sqrt{C_1^2 + 4C_2pr}\chi_x^3 f(\xi)g(\xi)), \\ v = {} & \frac{3}{4}p\chi_x\varphi_t f(\xi)(-4r - p(C_1^2 + 4C_2pr)f(\xi) \pm 2p\sqrt{C_1^2 + 4C_2pr}g(\xi)), \end{aligned} \quad (19)$$

where the coupled functions $f(\xi)$ and $g(\xi)$ satisfy Eq.(11).

3.2. The (2+1)-dimensional MDWW system

The (2+1)-dimensional MDWW system was used to model nonlinear and dispersive long gravity waves travelling in two horizontal directions on shallow waters of uniform depth, and can also be derived from the well-known Kadomtsev–Petviashvili (KP) equation using the symmetry constraint.[22] Abundant propagating localized excitations were derived by Tang *et al*[3] with the help of Painlevé–Bäcklund transformation and a multilinear variable separation approach.

According to the balancing procedure, we can also assume that the solutions of Eqs.(3) and (4) are

$$u = A + Bf(\xi) + Cg(\xi), \tag{20}$$

$$v = a + bf(\xi) + cf^2(\xi) + dg(\xi) + ef(\xi)g(\xi), \tag{21}$$

where A, B, C, a, b, c, d, e and ξ are arbitrary functions of $\{x, y, t\}$ to be determined, $f \equiv f(\xi)$ and $g \equiv g(\xi)$ satisfy the projective Riccati equations (5) and (6). Substituting Eqs.(20) and (21) along with Eqs.(5) and (6) into Eqs.(3) and (4), collecting coefficients of polynomials of $f^i g^j$ $(i = 0, 1, 2, 3, 4, j = 0, 1)$ and setting each of the coefficients to zero, we can derive a set of partial differential equations of A, B, C, a, b, c, d, e and ξ. In the case of $\xi = \chi(x, t) + \varphi(y)$, where $\chi \equiv \chi(x, t)$ and $\varphi \equiv \varphi(y)$ are two arbitrary variable separated functions of $\{x, t\}$ and y, respectively, we can obtain solutions of Eqs.(3) and (4), that is

(a) For $p = \pm 1$, q, r, s and h are arbitrary constants $(pq < 0, h^2 + r^2 < s^2)$, we have the solitary solutions

$$\begin{aligned} u &= \frac{1}{2\chi_x}(\chi_t - \chi_{xx}) \pm \frac{1}{2q}\sqrt{(s^2 - h^2 - r^2)pq}\chi_x f(\xi) - \frac{1}{2}p\chi_x g(\xi), \\ v &= \frac{1}{2q}p\chi_x\varphi_y f(\xi)(-qr - (s^2 - h^2 - r^2)f(\xi) \pm \sqrt{(s^2 - h^2 - r^2)pq}g(\xi)), \end{aligned} \tag{22}$$

where the coupled functions $f(\xi)$ and $g(\xi)$ satisfy Eq.(7).

(b) For $p = \pm 1$, q, r, s and h are arbitrary constants $(pq > 0, s^2 + h^2 > r^2)$, we have the trigonometric function solutions

$$\begin{aligned} u &= \frac{1}{2\chi_x}(\chi_t - \chi_{xx}) \pm \frac{1}{2q}\sqrt{(s^2 + h^2 - r^2)pq}\chi_x f(\xi) - \frac{1}{2}p\chi_x g(\xi), \\ v &= \frac{1}{2q}p\chi_x\varphi_y f(\xi)(-qr - (s^2 + h^2 - r^2)f(\xi) \pm \sqrt{(s^2 + h^2 - r^2)pq}g(\xi)), \end{aligned} \tag{23}$$

where the coupled functions $f(\xi)$ and $g(\xi)$ satisfy Eq.(9).

(c) For $p = \pm 1$, $q = 0$, C_1, C_2 and r are arbitrary constants $(C_1^2 + 4C_2pr > 0)$, we obtain the rational solutions

$$\begin{aligned} u &= \frac{1}{2\chi_x}(\chi_t - \chi_{xx}) \pm \frac{1}{4}\sqrt{C_1^2 + 4C_2pr}\chi_x f(\xi) - \frac{1}{2}p\chi_x g(\xi), \\ v &= \frac{1}{8}p\chi_x\varphi_y f(\xi)(-4r - p(C_1^2 + 4C_2pr)f(\xi) \pm 2p\sqrt{C_1^2 + 4C_2pr}g(\xi)), \end{aligned} \tag{24}$$

where the coupled functions $f(\xi)$ and $g(\xi)$ satisfy Eq.(11).

4. New types of interactions between solitons in the (2+1)-dimensional MDWW system

In this section, we discuss a novel localized structure for the physical quantity v expressed by Eq.(23). For simplicity, we take the constants $p = -1, q = -1, r = 3, s = 2$ and $h = 4$, then formula (23) becomes the following trigonometric function solution:

$$V \equiv v = \frac{(6 + \sqrt{11})\sin(\chi + \varphi) + (3 - 2\sqrt{11})\cos(\chi + \varphi) + 10}{(4\sin(\chi + \varphi) + 2\cos(\chi + \varphi) + 3)^2}\chi_x\varphi_y, \tag{25}$$

where $\chi \equiv \chi(x,t)$ is an arbitrary function of $\{x,t\}$ and $\varphi \equiv \varphi(y)$ is an arbitrary function of y. Owing to the arbitrariness of the functions χ and φ included in the expression (25), the physical quantity V may obviously extend to several different types of structures, such as the multi-soliton solutions, multi-dromion and dromion lattice solutions, multiple ring soliton solutions, breathers, instantons, oscillating solitons, peakons, compactons, chaotic and fractal localized solutions. For instance, a dromion solution which is localized in all directions can be driven by multiple straight-line and straight-line ghost solitons. When the functions χ and φ are, for simplicity chosen as

$$\chi = \sum_{i=1}^{M} \mathrm{e}^{K_i x + L_i t + x_{0i}},$$

$$\varphi = 1 + \sum_{i=1}^{N} \mathrm{e}^{k_i y + y_{0i}},$$

where K_i, L_i, k_i, x_{0i} and y_{0i} are arbitrary constants and M and N are positive integers, we can obtain a particular multi-dromion solution for the physical quantity V. Meanwhile, when functions χ and φ are considered to be some piecewise smooth functions, we can derive some multi-peakon excitations for the physical quantity V. For example, when

$$\chi = \sum_{i=1}^{M} \begin{cases} \chi_i(x+kt), & x+kt \le 0, \\ -\chi_i(-x-kt) + 2\chi_i(0), & x+kt > 0, \end{cases}$$

$$\varphi = \sum_{i=1}^{N} \begin{cases} \varphi_i(y), & y \le 0, \\ -\varphi_i(-y) + 2\varphi_i(0), & y > 0, \end{cases}$$

where the functions $\chi_i(x+kt)$ and $\varphi_i(y)$ are the differentiable functions of the indicated arguments and possess the boundary conditions

$$\chi_i(\pm\infty) = A_{\pm i}, \ (i = 1, 2, \cdots, M),$$

$$\varphi_i(\pm\infty) = B_{\pm i}, \ (i = 1, 2, \cdots, N),$$

with $A_{\pm i}$ and $B_{\pm i}$ being constants or even infinity, we can derive a particular multi-peakon solution for the physical quantity V. Since these similar situations have been widely discussed in some previous literature,[23−27] the related plots are neglected in our present paper. Here we are interested in revealing some new types of interactions between solitons in the (2+1)-dimensional MDWW equation.

Based on the physical quantity V expressed in Eq.(25), semi-folded localized structures can be discussed, when the function φ is a single valued function and χ is chosen via the relations

$$\chi_x = \sum_{i=1}^{N_1} \alpha_i(\zeta - d_i t), \quad x = \zeta + \sum_{j=1}^{N_2} \beta_j(\zeta - e_j t),$$

and

$$\chi = \int^{\zeta} \chi_x x_\zeta \mathrm{d}\zeta, \tag{26}$$

where d_i and e_j $(i = 1, 2, \cdots, N_1, j = 1, 2, \cdots, N_2)$ are arbitrary constants, α_i and β_j are localized excitations with the properties $\alpha_i\ (\pm\infty) = 0$, and $\beta_j\ (\pm\infty)$ =const. From Eq.(26), one can know that ζ may be a multi-valued function in some suitable regions of x by choosing the functions β_j appropriately. Therefore, the function χ_x, which is obviously an interaction solution of N_1 localized excitations due to the property $\zeta|_{x\to\infty} \to \infty$, may be a multi-valued function of x in these areas, though it is a single valued function of ζ. Actually, most of the known multi-loop solutions are the particular cases of Eq.(26).

4.1. Interaction between a peakon-like semi-foldon and a peakon

Owing to the arbitrariness of the functions in Eq.(25), we can discuss the interaction behaviour between a peakon-like semi-foldon and a peakon by choosing the function φ, a single valued function, and χ as Eq.(26) with $N_1 = N_2 = 1$, namely

$$\chi_x = 0.8\mathrm{sech}^2\zeta + 0.4\mathrm{sech}^2(\zeta - 0.3t),$$
$$x = \zeta - 1.5\tanh(\zeta - 0.3t), \quad \chi = \int^{\zeta} \chi_x x_\zeta \mathrm{d}\zeta, \tag{27}$$

$$\varphi = 8 + \begin{cases} \dfrac{1}{4}\mathrm{e}^{2y}, & y \le 0, \\ 1 - \dfrac{1}{4}\mathrm{e}^{-2y}, & y > 0. \end{cases} \tag{28}$$

The corresponding evolution profiles are depicted in Fig.1, from which one can find that the semi-folded coherent structure possesses a novel property, namely, it folds as a loop soliton in the x-direction but localizes as a peakon in the y-direction, so we call it a peakon-like semi-foldon. From Fig.1(d), we can see that the interaction between the multi-valued peakon-like semi-foldon and the single-valued peakon is non-fully elastic since the amplitudes of the static peakon and the moving semi-peakon are both changed. This property is different from the fully elastic interaction between either the single-valued and single-valued solitons,[26] or the multi-valued and multi-valued solitons (foldons).[27]

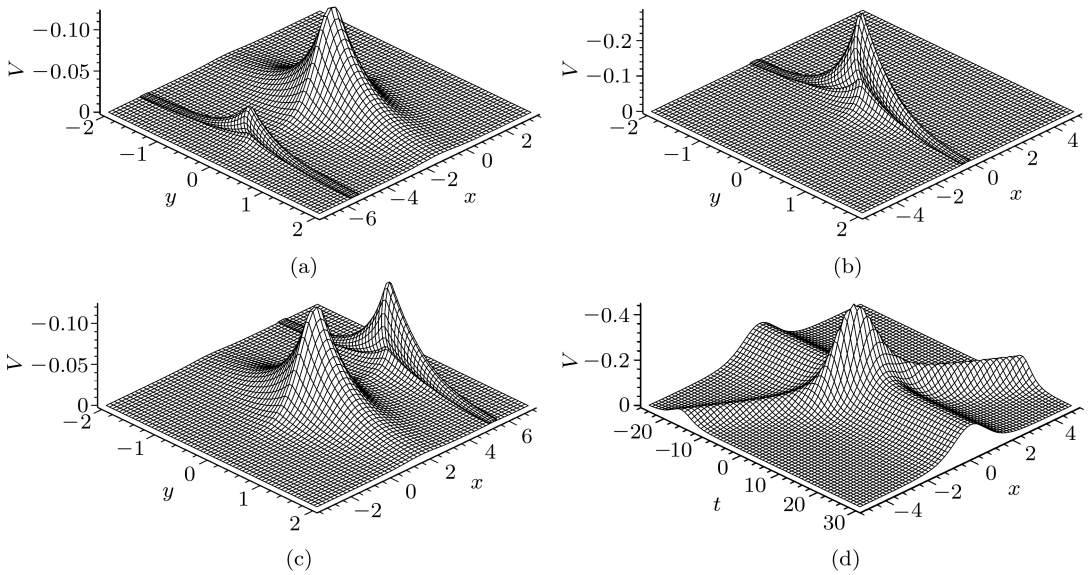


Fig.1. Evolution profiles of the interaction between a peakon-like semi-foldon and a peakon determined by Eq.(25) with conditions (27) and (28) at (a) $t=-18$, (b) $t=-1$, (c) $t=18$; (d) The evolution of V with the variables x and t at $y=0$.

4.2. Interaction between a 2-compacton-like semi-foldon and a 2-compacton

Similarly, one of the most simple choices for discussing the interaction between a compacton-like semi-foldon and a compacton is to take an arbitrary multi-valued function and an arbitrary single-valued function, i.e. χ is chosen as Eq.(27), and φ has the following form:

$$\varphi = 8 + \begin{cases} 0, & x \leq -\dfrac{\pi}{2}, \\ 0.005\sin^3 x, & -\dfrac{\pi}{2} < x \leq \dfrac{\pi}{2}, \\ 2, & x > \dfrac{\pi}{2}. \end{cases} \tag{29}$$

From Fig.2, we can also find that the interaction between the multi-valued 2-compacton-like semi-foldon and the single-valued 2-compacton is non-fully elastic since the amplitudes of the static compactons and the moving semi-foldons both increase with the time increasing.

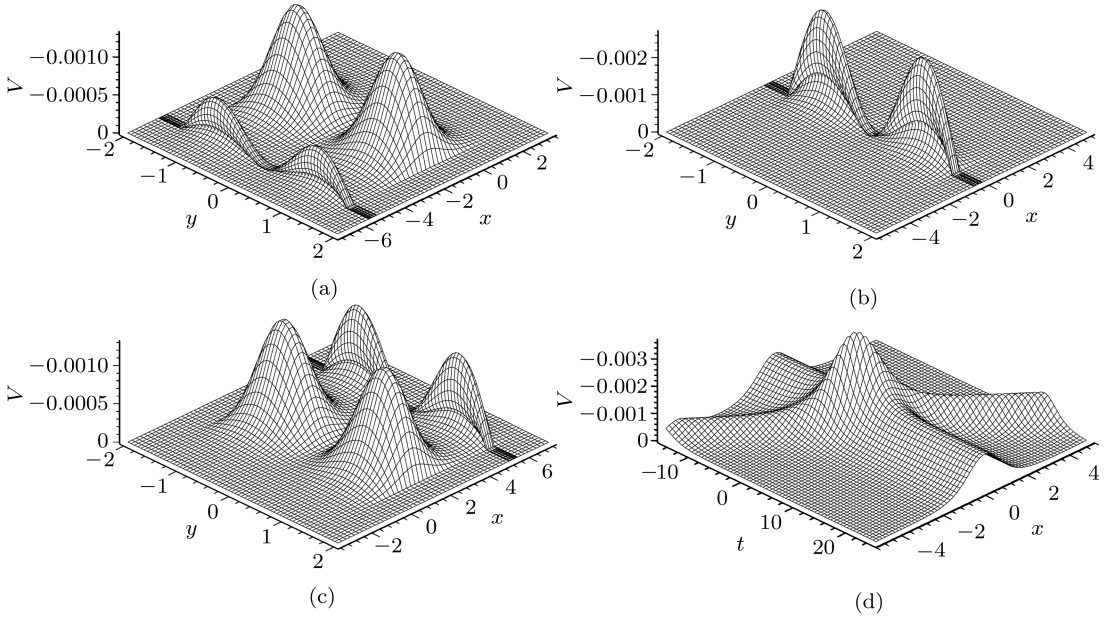


Fig.2. Evolution profiles of the interaction between a 2-compacton-like semi-foldon and a 2-compacton determined by Eq.(25) with conditions (27) and (29) at (a) $t=-18$, (b) $t=-1$, (c) $t=18$; (d) The evolution of V with the variables x and t at $y=1$.

5. Summary and discussion

We must point out that for the projective Riccati equations (5) and (6), when letting $f(\xi) = p(g(\xi) - r)$ and $r^2 = -\frac{q}{p}$, we can derive the traditional Riccati equation, namely $\phi^{'}(\xi) = \sigma + \phi^2(\xi)$, only if $\phi(\xi) \equiv p\left(g(\xi) - \frac{1}{2}r\right)$ and $\sigma \equiv \frac{1}{4}pq$. From this point, the approach we introduced here is a more generalized method than the mapping approach via the Riccati equation.[23−25]

With the help of the projective Riccati equation expansion method, we have obtained three types of variable separation solutions to the (1+1)-dimensional nonlinear Schrödinger equations and the (2+1)-dimensional modified dispersive water-wave system, respectively, which include solitary wave solutions, periodic wave solutions and rational function solutions.

In a usual case, based on the solitary wave solutions (22) and the variable separation solutions (24), one can find abundant localized excitations. However, in this paper, we have successfully revealed some localized patterns from the periodic wave solutions (23). From a special variable separation solution and by choosing appropriate functions, novel types of interactions between the multi-valued and single-valued solitons have been found. From the derived profiles, one find that their interactions are non-fully elastic, which is different from the fully elastic interaction between either the single-valued and single-valued solitons, or the multi-valued and multi-valued solitons (foldons).

Moreover, the work in this paper gives a worthwhile example about how to use direct and simple method to obtain variable separation solutions for other physical systems, since Zheng *et al* applied the mapping method to the BKK hierarchy. It is also implied that this simple and direct method is an alternative method to realize variable separation for the physical systems. The physical interpretation of these solutions and actual applications in reality will be investigated in future papers.

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