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The Holevo capacity of a generalized amplitude-damping channel*

Hou Li-Zhen(侯丽珍) and Fang Mao-Fa(方卯发)†

Department of Physics, Hunan Normal University, Changsha 410081, *China*

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The Holevo capacity of a generalized amplitude-damping channel is investigated by using a numerical method. It is shown that the Holevo capacity depends on the channel parameters representing the ambient temperature and fidelity. In particular, under a special condition, the Holevo capacity of the generalized amplitude-damping channel can be written as an analytical expression.

Keywords: the Holevo capacity, generalized amplitude-damping channel, channel parameters
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In quantum information theory, a central problem is how to calculate the capacities of quantum channels.[1−12] Physically, a quantum channel may be described by a completely positive and trace-preserving (CPTP) map from input-state density matrices to output-state density matrices.[13,14] Unlike classical channels, which are adequately characterized by a single capacity, quantum channels have several distinct capacities, such as the Holevo classical channel capacity,[15] entanglement-assisted classical capacity,[16] adaptive classical capacity,[17] and so on. Here, we shall focus on the Holevo classical channel capacity. The Holevo capacity is the asymptotic classical communication that may be achieved by using joint measurements on output states, but the input states are unentangled.[15] The Holevo capacity of a quantum channel Φ is determined by

$$C(\Phi) = \sup_{p_i,\rho_i} S[\Phi(\bar{\rho})] - \Sigma_i p_i S[\Phi(\rho_i)], \qquad (1)$$

where $\bar{\rho} = \Sigma p_i \rho_i$, and $S(\rho) = -\mathrm{Tr}\rho \log_2 \rho$ is the von Neumann entropy of density operator ρ. The p_i is the probability of the input state ρ_i, and therefore must be non-negative, and the total probabilities of all the input states must sum to 1. In general, it is difficult to determine the Holevo capacity $C(\Phi)$ by expression (1). Up to now, many researchers have developed only some special methods and calculated the Holevo capacity of some quantum channels.[18−21] However, to the best of our knowledge, the Holevo capacity of the generalized amplitude-damping channel (GADC) has not been calculated yet. The GADC is one of the most important noisy quantum channels, and it can describe the effect of dissipation to an environment at a finite temperature.[22] In this paper, we will calculate the Holevo capacity of the GADC and discuss its dependence of channel parameters. In particular, we will consider a special case where the Holevo capacity of the GADC may be exactly expressed by an analytic formula.

In the Bloch sphere representation, a general qubit density operator may be expressed as

$$\rho = \frac{1}{2}(1 + \boldsymbol{r} \cdot \boldsymbol{\sigma}), \qquad (2)$$

where $\boldsymbol{\sigma}$ is the vector of Pauli operators $(\sigma_x, \sigma_y, \sigma_z)^{\mathrm{T}}$. The Bloch vector $\boldsymbol{r}$ is a real three-dimensional vector $(x, y, z)^{\mathrm{T}}$, which is equal to 1 in magnitude when representing a pure state density matrix, but less than 1 for a mixed (non-pure) density matrix. The components of vector $\boldsymbol{r}$ determine a position of state in the Bloch Sphere.

A qubit channel Φ maps the sphere of possible input states into an ellipsoid, and may be expressed as[20]

$$\Phi(\rho) = \frac{1}{2}[1 + (\Lambda \boldsymbol{r} + \boldsymbol{t}) \cdot \boldsymbol{\sigma}], \qquad (3)$$

where

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \text{ and } \boldsymbol{t} = \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix}. \qquad (4)$$

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†Corresponding author. E-mail: mffang@hunnu.edu.cn

The λ_k and t_k $(k = 1, 2, 3)$ are real parameters which must satisfy certain constrains in order to ensure that the matrix above represents a CPTP qubit map. The unital maps correspond to $\boldsymbol{t} = 0$.

For the GADC, the qubit channel can be determined by[22]

$$\Phi(\rho) = \sum_i A_i \rho A_i^\dagger, \tag{5}$$

where

$$\begin{aligned} A_1 &= \sqrt{p}\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\eta} \end{pmatrix}, \\ A_2 &= \sqrt{p}\begin{pmatrix} 0 & \sqrt{1-\eta} \\ 0 & 0 \end{pmatrix}, \\ A_3 &= \sqrt{1-p}\begin{pmatrix} \sqrt{\eta} & 0 \\ 0 & 1 \end{pmatrix}, \\ A_4 &= \sqrt{1-p}\begin{pmatrix} 0 & 0 \\ \sqrt{1-\eta} & 0 \end{pmatrix}. \end{aligned} \tag{6}$$

The parameter $p \in [0, 1]$ represents the temperature of the environment, and $\eta \in [0, 1]$ is a parameter representing fidelity. This channel induces the following affine maps into the Stokes parameter space

$$\begin{aligned} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} &= \begin{pmatrix} \sqrt{\eta} & 0 & 0 \\ 0 & \sqrt{\eta} & 0 \\ 0 & 0 & \eta \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ &\quad + \begin{pmatrix} 0 \\ 0 \\ (2p-1)(1-\eta) \end{pmatrix}, \end{aligned} \tag{7}$$

that is, $\boldsymbol{r} \to \Lambda \boldsymbol{r} + \boldsymbol{t}$, where

$$\begin{aligned} \Lambda &= \begin{pmatrix} \sqrt{\eta} & 0 & 0 \\ 0 & \sqrt{\eta} & 0 \\ 0 & 0 & \eta \end{pmatrix}, \\ \boldsymbol{t} &= \begin{pmatrix} 0 \\ 0 \\ (2p-1)(1-\eta) \end{pmatrix}. \end{aligned} \tag{8}$$

The stationary state of the GADC is

$$\rho_\infty = \begin{pmatrix} p & 0 \\ 0 & 1-p \end{pmatrix}, \tag{9}$$

which satisfies $\Phi_{\mathrm{GADC}}(\rho_\infty) = \rho_\infty$. When $p = 0$ or 1, the channel reduces to an amplitude-damping channel.[22]

Besides expression (1), the Holevo capacity can also be calculated by using relative entropy[19] as follows:

$$C = C(\Phi) = \min_{\psi_0} \max_{\rho_0} D[\Phi(\rho_0) \| \Phi(\psi_0)], \tag{10}$$

where D is a relative entropy

$$D(\rho \| \psi) = \mathrm{Tr}(\rho \log \rho - \rho \log \psi). \tag{11}$$

Note that we usually take our logarithms to the base 2. The relative entropy can be evaluated using the following useful formula from Ref.[18]:

$$D(\rho \| \psi) = \frac{1}{2}[f(r) - \log(1 - q^2) - r\cos(\theta) f'(q)], \tag{12}$$

where

$$f(x) = (1 + x)\log(1 + x) + (1 - x)\log(1 - x), \tag{13}$$

$$f'(x) = \log\Big(\frac{1 + x}{1 - x}\Big). \tag{14}$$

The Bloch vectors for ρ and ψ are $\boldsymbol{r}$ and $\boldsymbol{q}$, respectively, and we also define $r = |\boldsymbol{r}|$, $q = |\boldsymbol{q}|$, and $\cos(\theta) = \boldsymbol{r} \cdot \boldsymbol{q}/rq$.

We call an ensemble of channel output states an optimal ensemble if this ensemble achieves the Holevo capacity.[19] In the following, we shall take the states $\rho = \Phi(\rho_0)$ and $\psi = \Phi(\psi_0)$ as output states from the channel. If $\psi = \Sigma_k p_k \rho_k$ is an average output density operator for an optimal ensemble, the operators ρ_k that maximize $D(\rho_k \| \psi)$ are possible output states for this ensemble. It is shown in Ref.[18] that the optimal ensemble is not necessarily unique because there may be different ways of choosing the probabilities such that $\Sigma_k p_k \rho_k = \psi$. However, the optimal average output state is unique.

In our particular case, in order to calculate the capacity of the GADC, we must know the optimal ensemble. So, it is important to find the optimal ensemble of the GADC. If we consider only the restricted case of channels Φ such that the x and y components of $\boldsymbol{t}$ both are zero, and use the notation $t = t_3$, then $\boldsymbol{t}$ in expression (4) is given by

$$\boldsymbol{t} = \begin{pmatrix} 0 \\ 0 \\ t \end{pmatrix}. \tag{15}$$

For this kind of special channels, Ref.[21] has given the following theorems.

Theorem 1 For a CPTP map $\Phi_{t,\Lambda}$, with Λ and $\boldsymbol{t}$ given by expressions (4) and (15) respectively, if $\lambda_{\mathrm{m}} = |\lambda_3|$ or $A \notin (0, 1/2)$, where

$$A = \frac{t^2 \lambda_3^2}{\lambda_{\mathrm{m}}^2 - \lambda_3^2} - 1 + \lambda_{\mathrm{m}}^2 + t^2 \tag{16}$$

and $\lambda_{\mathrm{m}} = \max(|\lambda_1|, |\lambda_2|)$, then there exists an ensemble that gives the maximum output Holevo information and has two states.

Theorem 2 For a CPTP map $\Phi_{t,\Lambda}$, with Λ and $\boldsymbol{t}$ given by expressions (4) and (15) respectively, the condition that $\lambda_{\rm m} = |\lambda_3|$ or $A \notin (0, 1/2)$ may be expressed as two alternative mutually exclusive conditions:

condition 1. $\lambda_{\rm m} \leq |\lambda_3|$ or $A \geq 1/2$,

condition 2. $\lambda_{\rm m} > |\lambda_3|$ and $A \leq 0$.

If condition 1 is satisfied, the optimal ensemble consists of two states on the z-axis. If condition 2 is satisfied, there is an optimal ensemble consisting of two states that are equidistant from the z-axis and lie on a line perpendicular to and intersecting the z-axis.

For the GADC, from expression (8), we know $\lambda_1 = \lambda_2 = \sqrt{\eta}$, $\lambda_3 = \eta$ and $t = (2p-1)(1-\eta)$. So the value of A in Theorem 1 can be easily calculated. We obtain $A = 4p(1-\eta)(p-1)$. $\eta \in [0,1]$ and $p \in [0,1]$, hence $A \leq 0$, that is, the condition $A \notin (0,1/2)$ in Theorem 1 is satisfied. Thus the optimal ensemble of the GADC requires two states. At the same time, $\sqrt{\eta} > \eta$ ($\eta \neq 0, 1$) and $A \leq 0$, hence condition 2 in Theorem 2 holds. Consequently these two states are equidistant from the z-axis and lie on a line perpendicular to and intersecting the z-axis.

By using the method similar to that provided in Ref.[21], we take it that the input state has $\boldsymbol{r} = (\sin\phi, 0, \cos\phi)^{\rm T}$, then the output state has $\boldsymbol{r} = (\lambda_1 \sin\phi, 0, t+\lambda_3\cos\phi)^{\rm T}$, and the average output state ψ has $\boldsymbol{q} = (0, 0, q_z)^{\rm T}$. From the above analyses, we know that the optimal ensemble of the GADC consists of two states, which are equidistant from the z-axis and lie on a line perpendicular to and intersecting the z-axis. Therefore, we can assume that, when $\phi = \pm\phi_0$, the $D(\rho\|\psi)$ is maximum. Then, according to expressions (10)–(12), the Holevo capacity of the GADC becomes

$$C = D(\rho_\pm\|\psi) = \frac{1}{2}[f(r_\pm) - \log(1-q_z^2) - (t+\lambda_3\cos\phi_0)f'(q_z)], \quad (17)$$

where ϕ_0 satisfies the following equation:[21]

$$[(\lambda_{\rm m}^2 - \lambda_3^2)\cos\phi_0 - t\lambda_3]f'(r_\pm)/r_\pm = -f'(q_z)\lambda_3, \quad (18)$$

and $\lambda_{\rm m} = \sqrt{\eta}$, $\lambda_3 = \eta$ and $t = (2p-1)(1-\eta)$.

Because ψ must be the average of the two $\rho_\pm$ and the z-components of the two $\boldsymbol{r}_\pm$ are equal in magnitude, the z component of $\boldsymbol{q}$ must be also equal to the z-component of $\boldsymbol{r}_+$ or $\boldsymbol{r}_-$ in magnitude. In other words, if ψ is optimal, the z-component of $\boldsymbol{r}$ should be equal to the z-component of $\boldsymbol{q}$ in magnitude. Therefore, the optimal ensemble may be found by finding the solution of expression (18) with $q_z = r_z$. That is to say,

$$q_z = t + \lambda_3\cos\phi_0 = (2p-1)(1-\eta) + \eta\cos\phi_0. \quad (19)$$

So

$$\cos\phi_0 = \frac{q_z - (2p-1)(1-\eta)}{\eta}, \quad (20)$$

$$r_\pm = \sqrt{\eta\sin^2\phi_0 + q_z^2} = \sqrt{\eta(1-\cos^2\phi_0) + q_z^2}. \quad (21)$$

Substituting expressions (20) and (21) into Eq.(18) and solving this equation, we can obtain the value of q_z numerically. Then, the capacity of the GADC may be determined numerically by substituting q_z into expression (17). It is evident that the Holevo capacity of the GADC is a function of the channel parameters p and η, i.e. $C = C(p, \eta)$.

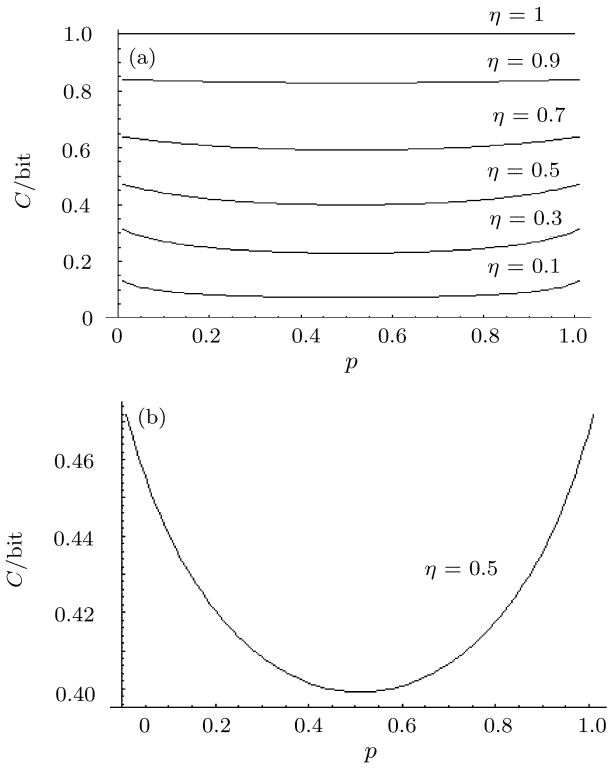


Fig.1. The Holevo capacity C versus channel parameter p for (a) different values of η and (b) a fixed value $\eta = 0.5$.

Through the numerical computation, we obtain the relationship between the Holevo capacity and the channel parameters as shown in Figs.1 and 2. Examining these diagrams, we have the following conclusions. (1) For an arbitrary fixed value of η ($\eta \neq 0, 1$), the values of C are symmetric about the channel parameter $p = 0.5$. That is to say, $C(0.5+x, \eta) = C(0.5-x, \eta)$ with $x \in [-0.5, 0.5]$. These can be seen from Fig.1. Furthermore, when $p = 0.5$, the corresponding value

of C is minimum for any fixed value of η. It is shown in this situation that the corresponding temperature of the environment is the worst for transmitting the information for a GADC, because p is a parameter that represents the temperature of the environment. (2) For any fixed value of p, the larger η is, the larger C will be. In other words, the Holevo capacity C is a monotonically ascending function of η for any fixed value of p. This is reasonable, since η is a parameter that describes fidelity. Moreover, $C \to 0$ as $\eta \to 0$, and $C \to 1$ as $\eta \to 1$. These can be easily seen from Fig.2.

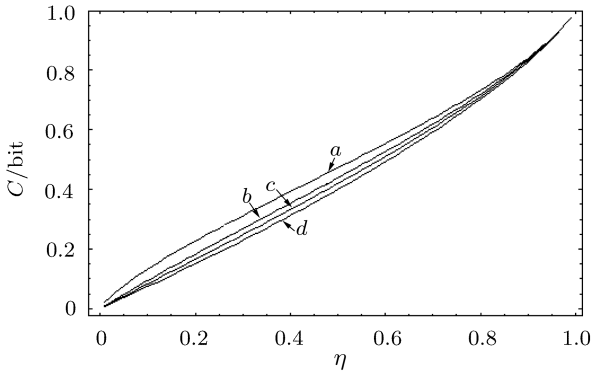


Fig.2. The Holevo capacity C versus channel parameter η for different values of p : 0 (curve a), 0.1 (curve b), 0.2 (curve c) and 0.5 (curve d).

Now, we consider a special case. When $p = 0.5$, $\boldsymbol{t}$ in expression (8) reduces to $\boldsymbol{t} = 0$, i.e., the GADC reduces to a unital channel. Using the method provided in Ref.[18], we obtain the Holevo capacity of the GADC as follows:

$$\begin{aligned} C &= 1 - S(|\lambda^{\text{major axis}}|) = 1 - S(\sqrt{\eta}) \\ &= 1 + \Big(\frac{1+\sqrt{\eta}}{2}\Big)\log_2\Big(\frac{1+\sqrt{\eta}}{2}\Big) \\ &\quad + \Big(\frac{1-\sqrt{\eta}}{2}\Big)\log_2\Big(\frac{1-\sqrt{\eta}}{2}\Big). \end{aligned} \tag{22}$$

This shows in this case the Holevo capacity of the GADC can be exactly written as an analytical expression. From expression (22), we can easily obtain the plot of the Holevo capacity C versus the channel parameter η in Fig.3. It is well identical with curve d in Fig.2 obtained by the numerical method.

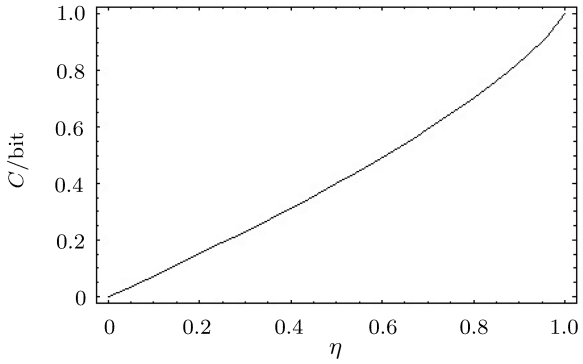


Fig.3. The channel capacity C versus the channel parameter η when $p = 0.5$.

When $p = 0$ or 1, the GADC reduces to an amplitude-damping channel. In expressions (17)–(21), letting $p = 0,\ \eta = 0.36$, or $p = 1,\ \eta = 0.36$, we can obtain $C = 0.360022$ through numerical calculation. This is well consistent with the result in Ref.[18].

In summary, we have calculated the Holevo channel capacity of the GADC. The numerical calculation illustrates clearly that the Holevo capacity C of the GADC varies with the channel parameters p and η. When the value of p is fixed, the Holevo capacity C is a monotonically ascending function of the channel parameter η. When the value of η is fixed, the capacity is symmetric about $p = 0.5$. Moreover, when $p = 0.5$, the corresponding capacity is minimum for a fixed value of η ($\eta \neq 0, 1$). It is shown that the capacity of the GADC to transmit information is worst in this environment, because p is a parameter that represents the temperature of the environment. In particular, when $p = 0.5$, we obtain an analytic relationship between the Holevo capacity C and the channel parameter η. At the same time, when $p = 0$ or 1, the GADC reduces to an amplitude-damping channel. The Holevo capacity of an amplitude-damping channel is a special case of our results about the GADC. Calculating the capacities of quantum noisy channels is an important task in quantum information theory. We expect our results to be useful for the transmission of information through the quantum noisy channels and to arouse more interest in this subject.

References

[1] Cortese J C 2004 *Phys. Rev.* A **69** 022302

[2] Chen J L and Kuang L M 2004 *Chin. Phys. Lett.* **21** 12

[3] Sohma M and Hirota O 2003 *Phys. Rev.* A **68** 022303

[4] Wang J S, Feng J, Liu T K and Zhan M S 2002 *J. Phys.*

B **35** 2411
[5] Giovannetti V, Lloyd S, Maccone L and Shor P W 2003 *Phys. Rev. Lett.* **91** 047901
[6] Wu Y and Yang X 2004 *Phys. Rev.* A **70** 063812
[7] Ball J, Dragan A and Banaszek K 2004 *Phys. Rev.* A **69** 042324
[8] Zheng Y Z, Gu Y J, Wu G C and Guo G C 2003 *Chin. Phys.* **12** 1070
[9] Jun J W 2006 *Phys. Rev.* A **73** 064301
[10] Gao T, Yan F L and Wang Z X 2005 *Chin. Phys.* **14** 893
[11] Arshed N and Toor A H 2006 *Phys. Rev.* A **73** 014304
[12] Liu T K, Wang J S, Feng J and Zhan M S 2005 *Chin. Phys.* **14** 0536
[13] Kraus K 1983 *Effects and Operations: Fundamental Notions of Quantum Theory, Lecture Notes in Physics* (Berlin: Springer) p190
[14] Schumacher B 1996 *Phys. Rev.* A **54** 2614
[15] Schumacher B and Westmoreland M D 1997 *Phys. Rev.* A **56** 131
[16] Bennett C H, Shor P W, Smolin J A and Thapliyal A V 1999 *Phys. Rev. Lett.* **83** 3081
[17] Shor P W 2002 quant-ph/0206058
[18] Cortese J 2002 quant-ph/0207128
[19] Schumacher B and Westmoreland M D 2001 *Phys. Rev.* A **63** 022308
[20] King C and Ruskai M B 2001 *IEEE Trans. Info. Theory* A **47** 192
[21] Berry D W 2005 *Phys. Rev.* A **71** 032334
[22] Nielsen M A and Chuang I L 2000 *Quantum Computation and Quantum Information* (Cambridge: Cambridge University Press) p380