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The classification of travelling wave solutions and superposition of multi-solutions to Camassa–Holm equation with dispersion

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Under the travelling wave transformation, the Camassa–Holm equation with dispersion is reduced to an integrable ordinary differential equation (ODE), whose general solution can be obtained using the trick of one-parameter group. Furthermore, by using a complete discrimination system for polynomial, the classification of all single travelling wave solutions to the Camassa–Holm equation with dispersion is obtained. In particular, an affine subspace structure in the set of the solutions of the reduced ODE is obtained. More generally, an implicit linear structure in the Camassa–Holm equation with dispersion is found. According to the linear structure, we obtain the superposition of multi-solutions to Camassa–Holm equation with dispersion.

Keywords: classification of travelling wave solution, symmetry group, Camassa–Holm equation with dispersion, superposition of solutions

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1. Introduction

The Camassa–Holm (CH) equation[1−4] reads as

$$u_t + 2ku_x - u_{xxt} + 3uu_x = 2u_xu_{xx} + uu_{xxx}, \quad (1)$$

which describes shallow water waves. The CH equation has been studied from several aspects.[1−15] Camassa and Holm have found that this equation exhibits 'peakons' when $k = 0$, that is, solitary wave solutions with discontinuous first derivative at their crests. This peakons have the following form:

$$u = c\exp(-|x - ct|). \quad (2)$$

For an arbitrary value of k, Liu *et al*[13,14] have shown that the CH equation has peakons as

$$u = (c + k)\exp(-|x - ct|) - k. \quad (3)$$

Liu and Qian[15] have also obtained a kind of generalized kink and anti-kink wave solutions. In 2001, based on the CH equation, Dullin[16] derived the Camassa–Holm equation with dispersion, expressed as

$$u_t + 2ku_x + 3uu_x - \varepsilon(u_{xxt} + 2u_xu_{xx} + uu_{xxx}) + \gamma u_{xxx} = 0, \quad (4)$$

where k, ε and γ are constants. If $k = \epsilon = 0$ and $\gamma = 1$, then the Camassa–Holm equation with dispersion becomes the Korteweg de Vries (KdV) equation. If $\varepsilon = 1$ and $\gamma = 0$, the Camassa–Holm equation with dispersion becomes the CH equation. Dullin[16] has obtained peakons in the case of $k = \gamma = 0$. The peakons in some special cases are given in Refs.[17–21]. These peakons are the so-called weak solutions or generalized solutions. Here we consider classical solutions.

In the present paper, we give the complete classification of classical travelling wave solutions of Camassa–Holm equation with dispersion. Applying the invariance property of the reduced Camassa–Holm equation with dispersion under a one-parameter group, we obtain a general solution of the reduced equation. Furthermore using a direct integral method and complete discrimination system for polynomial,[22−24] we give the classifications of travelling wave solutions to Camassa–Holm equation with dispersion. This complete result is valuable for studying the physical properties of Camassa–Holm equation with dispersion. In addition, we find a two-dimensional affine subspace structure in the set of the solutions of the reduced ODE. More generally, an implicit linear structure in Camassa–Holm equation with dispersion is found. According to the linear structure, we give the superposition of multi-solutions to Camassa–Holm equation with dispersion.

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2. Classification of travelling wave solutions to Camassa–Holm equation with dispersion

Under the travelling wave transformation $u = u(\xi)$, and $\xi = x - ct$, the Camassa–Holm equation with dispersion is reduced to the following ODE:

$$(2k-c)u' + 3uu' - \varepsilon(uu''' + 2u'u'' - cu''') + \gamma u''' = 0. \tag{5}$$

Integrating it once yields the following equation:

$$u'' + \frac{1}{2(u-c-\gamma/\varepsilon)}(u')^2 - \frac{3u^2 - 2(c-2k)u + c_0}{2\varepsilon(u-c-\gamma/\varepsilon)} = 0, \tag{6}$$

where c_0 is an arbitrary constant.

So we only need to solve Eq.(6).

Lemma The general solution of Eq.(6) is as follows:

$$\pm(\xi - \xi_0) = \int \sqrt{\frac{\varepsilon(u-c-\gamma/\varepsilon)}{(u-c-\gamma/\varepsilon)^3 + d_2(u-c-\gamma/\varepsilon)^2 + d_1(u-c-\gamma/\varepsilon) + d_0}}\,\mathrm{d}u, \tag{7}$$

where ξ_0 and d_0 are two arbitrary constants, $d_2 = 2c + 2k + 3\gamma/\varepsilon$, and $d_1 = 3(c+\gamma/\varepsilon)^2 + 2(2k-c)(c+\gamma/\varepsilon) + c_0$.

Proof Since Eq.(6) is invariant under the following one-parameter group

$$\xi* = \xi + \epsilon, \tag{8}$$

$$u* = u, \tag{9}$$

we can reduce Eq.(6) to a first-order ODE. Therefore, we take a new variable $z = u'$, and then we have $u'' = z(\mathrm{d}z/\mathrm{d}u)$. Substituting those terms into Eq.(6) and denoting $z^2 = W$ yield

$$\frac{\mathrm{d}W}{\mathrm{d}u} + \frac{1}{u-c-\gamma/\varepsilon}W - \frac{3u^2 - 2(c-2k)u + c_0}{\varepsilon(u-c-\gamma/\varepsilon)} = 0. \tag{10}$$

Let

$$p(u) = \frac{1}{u-c-\gamma/\varepsilon}$$

and

$$q(u) = -\frac{3u^2 - 2(c-2k)u + c_0}{\varepsilon(u-c-\gamma/\varepsilon)}$$

and using the method of variation of constants, then we will have the general solution of Eq.(10) as follows:

$$W(u) = \exp\Big(-\int p(u)\mathrm{d}u\Big) \times \Big[c_1 - \int q(u)\exp\Big(\int p(u)\mathrm{d}u\Big)\mathrm{d}u\Big], \tag{11}$$

where c_1 is an integral constant. Since $W = z^2 = (\mathrm{d}u/\mathrm{d}\xi)^2$, substituting the expressions of $p(u)$ and $q(u)$ into Eq.(11), rescaling arbitrary constant c_1 and arranging the result obtained yield Eq.(7), therefore the proof is completed.

Now we give the classification of all single travelling wave solutions to Camassa–Holm equation with dispersion. We have the following results.

Theorem 1 All travelling wave solutions to the Camassa–Holm equation with dispersion can be classified as follows:

Case 1. $\epsilon > 0$.

Case 1.1. $d_0 = 0$. Denote $\Delta = d_2^2 - 4d_1$. There exist two cases to be discussed:

Case 1.1.1. If $\Delta = 0$, then the corresponding solutions to Eq.(5) are

$$u = c_1\exp\Big(\pm\frac{\xi}{\sqrt{\varepsilon}}\Big) - k - \frac{\gamma}{2\varepsilon}, \tag{12}$$

where c_1 is an arbitrary constant.

Case 1.1.2. If $\Delta > 0$ or $\Delta < 0$, then the corresponding solution of Eq.(5) is

$$u = c_1\exp\Big(\frac{\xi}{\sqrt{\varepsilon}}\Big) + c_2\exp\Big(-\frac{\xi}{\sqrt{\varepsilon}}\Big) - k - \frac{\gamma}{2\varepsilon}, \tag{13}$$

where c_1 and c_2 are two arbitrary constants.

Case 1.2. $d_0 \neq 0$. Denote

$$\Delta = -27\Big(\frac{2d_2^3}{27} + d_0 - \frac{d_1d_2}{3}\Big)^2 - 4\Big(d_1 - \frac{d_2^2}{3}\Big)^3, \tag{14}$$

$$D_1 = d_1 - \frac{d_2^2}{3}, \tag{15}$$

where Δ and D_1 constitute a complete discrimination system for

$$F(u) = \Big(u + \frac{\omega}{k}\Big)^3 + d_2\Big(u + \frac{\omega}{k}\Big)^2 + d_1\Big(u + \frac{\omega}{k}\Big) + d_0.$$

There are the following four cases to be discussed:

Case 1.2.1. If $\Delta = 0$, and $D_1 < 0$, then we have $F(u) = (u-\alpha)^2(u-\beta)$, $\alpha \neq \beta$. Take the change of the variable

$$u = \frac{\beta v^2 - c - \gamma/\varepsilon}{v^2 - 1} \tag{16}$$

and its inverse transformation

$$\frac{u - c - \gamma/\varepsilon}{u - \beta} = v^2, \tag{17}$$

then we will have

$$\pm\frac{1}{\sqrt{\varepsilon}}(\xi - \xi_0) = \ln\Big|\frac{v+1}{v-1}\Big| + \sqrt{\frac{\alpha - c - \gamma/\varepsilon}{\alpha - \beta}}\ln\left|\frac{v - \sqrt{\dfrac{\alpha - c - \gamma/\varepsilon}{\alpha - \beta}}}{v + \sqrt{\dfrac{\alpha - c - \gamma/\varepsilon}{\alpha - \beta}}}\right|, \quad \Big(\frac{\alpha - c - \gamma/\varepsilon}{\alpha - \beta} > 0\Big) \tag{18}$$

and

$$\pm\frac{1}{\sqrt{\varepsilon}}(\xi - \xi_0) = \ln\Big|\frac{v+1}{v-1}\Big| - 2\sqrt{\frac{\alpha - c - \gamma/\varepsilon}{\beta - \alpha}}\arctan\Big(v\sqrt{\frac{\beta - \alpha}{\alpha - c - \gamma/\varepsilon}}\Big), \quad \Big(\frac{\alpha - c - \gamma/\varepsilon}{\beta - \alpha} > 0\Big). \tag{19}$$

Case 1.2.2. If $\Delta = 0$, and $D_1 = 0$, then we have $F(u) = (u-\alpha)^3$, the solution is as follows:

$$\pm\frac{1}{2\sqrt{\varepsilon}}(\xi - \xi_0) = \pm\sqrt{\frac{u - c - \gamma/\varepsilon}{u - \alpha}} + \frac{1}{2}\ln\left|\frac{\sqrt{\dfrac{u - c - \gamma/\varepsilon}{u - \alpha}} \mp 1}{\sqrt{\dfrac{u - c - \gamma/\varepsilon}{u - \alpha}} \pm 1}\right|. \tag{20}$$

Case 1.2.3. If $\Delta > 0$, and $D_1 < 0$, then $F(u) = (u-\alpha)(u-\beta)(u-\rho)$. Assume $\alpha > \beta > \rho$, and take the change of variable

$$u = \frac{\alpha v^2 - c - \gamma/\varepsilon}{v^2 - 1}, \tag{21}$$

then the corresponding integral will become

$$\begin{aligned}&\pm\frac{\sqrt{(\alpha-\beta)(\alpha-\rho)}}{2\sqrt{\varepsilon}(\alpha - c - \gamma/\varepsilon)}(\xi - \xi_0)\\ &= \int\Big\{1 + \frac{1}{2}\Big(\frac{1}{v-1} - \frac{1}{v+1}\Big)\Big\}\\ &\quad\times\frac{1}{\sqrt{(v^2+A)(v^2+B)}}\mathrm{d}v,\end{aligned} \tag{22}$$

where

$$A = \frac{\beta - c - \gamma/\varepsilon}{\alpha - \beta}$$

and

$$B = \frac{\rho - c - \gamma/\varepsilon}{\alpha - \rho}.$$

It is easy to see that the integral in Eq.(22) can be expressed by the elliptic integrals of the first kind and the third kind.

Case 1.2.4. If $\Delta < 0$, then we have $F(u) = (u-\alpha)(u^2 + pu + q)$, $p^2 - 4q < 0$. Take the change of variable

$$v^2 = \frac{u - c - \gamma/\varepsilon}{u - \alpha},$$

then the corresponding integral will become

$$\begin{aligned}&\pm\frac{\xi - \xi_0}{\sqrt{\varepsilon}(\alpha - c - \gamma/\varepsilon)}\\ &= \int\Big(1 + \frac{1}{v-1}\Big)\frac{1}{\sqrt{v(Av^2 + Bv + C)}}\mathrm{d}v,\end{aligned} \tag{23}$$

where

$$A = p\alpha + q + \alpha^2,$$

$$B = -(2\alpha + p)\Big(c + \frac{\gamma}{\varepsilon}\Big) - p\alpha - 2q,$$

and

$$C = \Big(c + \frac{\gamma}{\varepsilon}\Big)^2 + p\Big(c + \frac{\gamma}{\varepsilon}\Big) + q,$$

with $B^2 - 4AC < 0$. It is easy to see that the corresponding integral in Eq.(23) can be expressed by the elliptic integrals of the first kind and the third kind.

Case 2. $\varepsilon < 0$.

Case 2.1. $d_0 = 0$. Let $\Delta = d_2^2 - 4d_1$. If $\Delta \leq 0$, then there exist no solutions at all. So we only consider the case of $\Delta > 0$. When $-2k - \gamma/(2\varepsilon) - \sqrt{\Delta}/2 < u < -2k - \gamma/(2\varepsilon) + \sqrt{\Delta}/2$, we have

$$u = -2k - \frac{\gamma}{\varepsilon} \pm \frac{\sqrt{\Delta}}{2}\sin\Big(\frac{1}{\sqrt{-\varepsilon}}(\xi - \xi_0)\Big), \tag{24}$$

which can be rewritten as follows

$$u = -2k - \frac{\gamma}{\varepsilon} + c_1\sin\Big(\frac{1}{\sqrt{-\varepsilon}}\xi\Big) + c_2\cos\Big(\frac{1}{\sqrt{-\varepsilon}}\xi\Big), \tag{25}$$

where c_1 and c_2 are two arbitrary constants. These are periodic solutions.

Case 2.2. $d_0 \neq 0$. Δ, D_1 and $F(u)$ are the same as those above. There appear also the following four cases that are to be discussed:

Case 2.2.1. If $\Delta = 0$, and $D_1 < 0$, then we have $F(u) = (u-\alpha)^2(u-\beta)$, $\alpha \neq \beta$. Take the change of the variable

$$u = \frac{\beta v^2 + c + \gamma/\varepsilon}{1+v^2}, \tag{26}$$

its inverse transformation

$$\frac{u - c - \gamma/\varepsilon}{u-\beta} = -v^2, \tag{27}$$

then we will have

$$\pm\frac{1}{\sqrt{-\varepsilon}}(\xi-\xi_0) = 2\arctan v - \sqrt{\frac{\alpha-c-\gamma/\varepsilon}{\beta-\alpha}}\ln\left|\frac{v-\sqrt{\dfrac{\alpha-c-\gamma/\varepsilon}{\beta-\alpha}}}{v+\sqrt{\dfrac{\alpha-c-\gamma/\varepsilon}{\beta-\alpha}}}\right|, \quad \left(\frac{\alpha-c-\gamma/\varepsilon}{\beta-\alpha} > 0\right) \tag{28}$$

and

$$\pm\frac{1}{\sqrt{-\varepsilon}}(\xi-\xi_0) = 2\arctan v - 2\sqrt{\frac{\alpha-c-\gamma/\varepsilon}{\alpha-\beta}}\arctan\left(v\sqrt{\frac{\alpha-\beta}{\alpha-c-\gamma/\varepsilon}}\right), \quad \left(\frac{\alpha-c-\gamma/\varepsilon}{\alpha-\beta} > 0\right). \tag{29}$$

Case 2.2.2. If $\Delta = 0$, and $D_1 = 0$, then we have $F(u) = (u-\alpha)^3$, its solutions are as follows:

$$\pm\frac{1}{2\sqrt{-\varepsilon}}(\xi-\xi_0) = 2\sqrt{\frac{u-c-\gamma/\varepsilon}{\alpha-u}} - 2\arctan\left(\sqrt{\frac{u-c-\gamma/\varepsilon}{\alpha-u}}\right). \tag{30}$$

Case 2.2.3. If $\Delta > 0$, and $D_1 < 0$, then $F(u) = (u-\alpha)(u-\beta)(u-\rho)$. Assume $\alpha > \beta > \rho$, and take the change of variable

$$u = \frac{\alpha v^2 - c - \gamma/\varepsilon}{v^2-1}, \tag{31}$$

then the corresponding integral will become

$$\begin{aligned}&\pm\frac{\sqrt{(\alpha-\beta)(\alpha-\rho)}}{2\sqrt{-\varepsilon}(\alpha-c-\gamma/\varepsilon)}(\xi-\xi_0)\\ &= \int\left\{1+\frac{1}{2}\left(\frac{1}{v-1}-\frac{1}{v+1}\right)\right\}\\ &\quad\times\frac{1}{\sqrt{-(v^2+A)(v^2+B)}}\mathrm{d}v,\end{aligned} \tag{32}$$

where

$$A = \frac{\beta-c-\gamma/\varepsilon}{\alpha-\beta}$$

and

$$B = \frac{\rho-c-\gamma/\varepsilon}{\alpha-\rho}.$$

If $A > 0$ and $B > 0$, it is easy to see that no solution can be given. In other cases, it is easy to see that the integral in Eq.(32) can be expressed by the elliptic integrals of the first kind and the third kind.

Case 2.2.4. If $\Delta < 0$, then we have $F(u) = (u-\alpha)(u^2+pu+q)$, $p^2-4q < 0$. Take the change of variable

$$v^2 = \frac{u-c-\gamma/\varepsilon}{\alpha-u},$$

then the corresponding integral will become

$$\begin{aligned}&\pm\frac{\xi-\xi_0}{\sqrt{-\varepsilon}(\alpha-c-\gamma/\varepsilon)}\\ &= \int\left(1-\frac{1}{v-1}\right)\frac{1}{\sqrt{v(Av^2+Bv+C)}}\mathrm{d}v,\end{aligned} \tag{33}$$

where $A = p\alpha+q+\alpha^2$, $B = -(2\alpha+p)(c+\gamma/\varepsilon)+p\alpha+2q$, and $C = (c+\gamma/\varepsilon)^2 - p(c+\gamma/\varepsilon)+q$, with $B^2-4AC < 0$. It is easy to see that the corresponding integral in Eq.(33) can be expressed by the first kind and the third kind of elliptic integrals.

Remark 1 By calculating the integral in Eq.(7), we can directly obtain the solutions of Eq.(6). Of course, those solutions are also the solutions of Eq.(5). If we rescale the arbitrary constants ξ_0 and c_0, then we can give the solutions of Eq.(5) in more simple forms such as solutions (12) and (13). On the other hand, we can verify easily solutions (13) to Eq.(5) by substituting expression (13) directly into Eq.(5) and using $u''' = u'/\varepsilon$, and $u'' = (1/\varepsilon)(u+k+\gamma/\varepsilon)$.

3. Superposition of multi-solutions to Camassa–Holm equation with dispersion

According to case 1.2 in Section 2, we find an interesting fact, that is, affine superposition of two solutions $u_1 = \exp(x - ct)$ and $u_2 = \exp(-(x - ct))$ is also a solution to Eq.(5). This is because Eq.(5) has a special structure. In fact, Eq.(5) includes two parts, one part is $-cu' + c\varepsilon u'''$, the other is $2ku' + 3uu' - \varepsilon uu''' - 2\varepsilon u'u'' + ru'''$. If we take

$$-cu' + c\varepsilon u''' = 0, \tag{34}$$

$$2ku' + 3uu' - \varepsilon uu''' - 2\varepsilon u'u'' + ru''' = 0, \tag{35}$$

then from Eq.(34), we have $u' = \varepsilon u'''$, therefore Eq.(35) becomes $u+k+r/(2\varepsilon) = \varepsilon u''$. It is easy to see that these two equations are compatible. Therefore, if we take $v = u + k + r/(2\varepsilon)$, these two equations become linear equations $v'' = v/\varepsilon$ essentially. In this case, the superposition property of the solutions v is found. Thus there is a two-dimensional linear subspace in the set of the solutions of v-equations. When $\varepsilon > 0$, the corresponding bases are $e_1 = \exp(x - ct)$ and $e_2 = \exp(-(x - ct))$. When $\varepsilon < 0$, the bases are $e_1 = \sin(\xi/\sqrt{-\varepsilon})$ and $e_2 = \cos(\xi\sqrt{-\varepsilon})$. This means that there is a two-dimensional affine subspace in the set of the solutions of u-equations.

More generally, we find an implicit linear structure in the Camassa–Holm equation with dispersion. In fact, we rewrite the Camassa–Holm equation with dispersion as

$$u_t - \varepsilon u_{xxt} = \varepsilon(2u_xu_{xx} + uu_{xxx}) - \gamma u_{xxx} - 2ku_x - 3uu_x. \tag{36}$$

Letting the left and the right sides both are zeros yields two equations

$$u_t - \varepsilon u_{xxt} = 0, \tag{37}$$

$$\varepsilon(2u_xu_{xx} + uu_{xxx}) - \gamma u_{xxx} - 2ku_x - 3uu_x = 0. \tag{38}$$

Integrating Eq.(37) with respect to t, we have

$$u = \varepsilon u_{xx} + f(x). \tag{39}$$

If we take $f(x) = -k - \gamma/(2\varepsilon)$, then $u_x = \varepsilon u_{xxx}$, and Eq.(38) is satisfied automatically. Therefore, we find a linear structure in the Camassa–Holm equation with dispersion, that is,

$$u + k + \frac{\gamma}{2\varepsilon} = \varepsilon u_{xx}. \tag{40}$$

Let $v = u + k + \gamma/(2\varepsilon)$, then the above equation becomes

$$v = \varepsilon v_{xx}. \tag{41}$$

When $\varepsilon > 0$, its general solution is as follows:

$$v(x,t) = \sum\Big[g_i(t)\exp\Big(\frac{x}{\sqrt{\varepsilon}} - s_i(t)\Big) + h_i(t)\exp\Big(-\frac{x}{\sqrt{\varepsilon}} - r_i(t)\Big)\Big], \tag{42}$$

where $g_i(t), h_i(t), s_i(t)$ and $r_i(t)$ all are arbitrary functions. The corresponding u-solution is

$$u(x,t) = \sum\Big[g_i(t)\exp\Big(\frac{x}{\sqrt{\varepsilon}} - s_i(t)\Big) + h_i(t)\exp\Big(-\frac{x}{\sqrt{\varepsilon}} - r_i(t)\Big)\Big] - k - \frac{\gamma}{2\varepsilon}. \tag{43}$$

When $\varepsilon < 0$, the general v-solution is

$$v(x,t) = \sum\Big[g_i(t)\sin\Big(\frac{x}{\sqrt{-\varepsilon}} - s_i(t)\Big) + h_i(t)\cos\Big(\frac{x}{\sqrt{-\varepsilon}} - r_i(t)\Big)\Big], \tag{44}$$

and the corresponding u-solution is

$$u(x,t) = \sum\Big[g_i(t)\sin\Big(\frac{x}{\sqrt{-\varepsilon}} - s_i(t)\Big) + h_i(t)\cos\Big(\frac{x}{\sqrt{-\varepsilon}} - r_i(t)\Big)\Big] - k - \frac{\gamma}{2\varepsilon}. \tag{45}$$

Through the above illustration, we show easily the affine superposition of multi-solutions to Camassa–Holm equation with dispersion using an implicit linear structure. Consequently, we have the following results:

Theorem 2 When $\varepsilon > 0$, the Camassa–Holm equation with dispersion has the following solution:

$$u(x,t) = \sum\Big[g_i(t)\exp\Big(\frac{x}{\sqrt{\varepsilon}} - s_i(t)\Big) + h_i(t)\exp\Big(-\frac{x}{\sqrt{\varepsilon}} - r_i(t)\Big)\Big] - k - \frac{\gamma}{2\varepsilon}; \tag{46}$$

when $\varepsilon < 0$, the corresponding solution becomes

$$u(x,t) = \sum\Big[g_i(t)\sin\Big(\frac{x}{\sqrt{-\varepsilon}} - s_i(t)\Big) + h_i(t)\cos\Big(\frac{x}{\sqrt{-\varepsilon}} - r_i(t)\Big)\Big] - k - \frac{\gamma}{2\varepsilon}, \tag{47}$$

where $g_i(t), h_i(t), s_i(t)$ and $r_i(t)$ all are arbitrary functions.

Remark 2 In order to compare with the travelling wave solutions, we have not written the expressions (46) and (47) in more simple forms.

4. Conclusions

In summary, according to the invariance property of Eq.(5) under a one-parameter group, we reduce it to the first-order ODE and give its general solutions. Furthermore applying a complete discrimination system for polynomial, we obtain the classifications of all single travelling wave solutions to the Camassa–Holm equation with dispersion. In addition, we find that a two-dimensional affine subspace structure in the set of solutions of the reduced ODE can be obtained. More generally, we find an implicit linear structure. Using the linear structure, we give easily the affine superposition of multi-solutions to Camassa–Holm equation with dispersion.

Remark 3 Because the Camassa–Holm equation with dispersion becomes a CH equation as $\varepsilon = 1$, and $r = 0$, we give the corresponding solution to CH equation in this parameters case.

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