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Behaviours in a dynamical model of traffic assignment with elastic demand*

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This paper investigates the dynamical behaviour of network traffic flow. Assume that trip rates may be influenced by the level of service on the network and travellers are willing to take a faster route. A discrete dynamical model for the day-to-day adjustment process of route choice is presented. The model is then applied to a simple network for analysing the day-to-day behaviours of network flow. It finds that equilibrium is arrived if network flow consists of travellers not very sensitive to the differences of travel cost. Oscillations and chaos of network traffic flow are also found when travellers are sensitive to the travel cost and travel demand in a simple network.

Keywords: discrete dynamical system, network traffic flow, traffic assignment problem, chaos
PACC: 0545, 0547, 0250

1. Introduction

The notion of equilibrium in the analysis of urban transportation networks stems from the dependence of the link travel costs on the link flows. Network equilibrium models have a long history in the traffic flow research literature.[1−3] Particularly, the classes of static equilibrium models have been seen to be versatile,[4,5] with extensions able to represent perceptual variations across the driver population, mixed vehicle types, and some mixed mode operations.

In the 1980s, with a growing international interest in real-time information and control systems, a couple of fundamental issues of network equilibrium models began to be questioned:[6,7] for example, what the required conditions for equilibrium are, if equilibrium is really reached, what the relationship between travellers' learning processes and network behaviour is, and also, their inability to deal with many of the dynamic aspects of traffic systems. Initially, the predominant focus was on efforts to generalize the equilibrium model to represent within-day dynamical models, that is to say, the dependence of route choice on departure time, while accounting for the proper temporal and spatial evolution of congestion across the network.[8] Subsequently, more fundamental questions began to be asked about the behaviour basis of the equilibrium paradigm, particularly in the sense of its usefulness for representing travellers' response to exogenous information. This led to a growing interest in day-to-day dynamical models, whereby the evolution over days of travel choices and traffic congestion is linked through a learning model based on travellers' past experiences.[9−11]

Many general contrasts may readily be made between the day-to-day dynamics and traditional equilibrium approaches. For example, the estimation of equilibrium has been typically achieved through the solution of some optimization, complementarity or variational inequality problem, yet this makes the approach rather restrictive in terms of the feasible generalizations. On the other hand, given a suitably constructed equilibrium solution algorithm, the interpretation of model outputs is typically straightforward, whereas in simulations of the day-to-day approach the interpretation stage is often the most challenging task.[12,13] However, perhaps the greatest contrast to be made is the fundamental basis of each approach. In traditional equilibrium models, the underlying hypothesis is the very notion of a market in equilibrium: a typically isolated, 'self consistent' state of the network which, if attained, would persist under certain rational rules of behaviour. In the day-to-day

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approach, the underlying belief is in the behavioural dynamics, namely how the behaviour on day n is affected by behaviour and the state of the network on days $n-1$ and earlier. It is therefore tempted to regard the two approaches as competing philosophies. Yet the classical preliminary analyses of dynamical approaches involve examining the 'equilibrium' behaviour of the system. While the concept of equilibrium of a dynamical system does not relate directly to the traditional economic equilibrium of traffic network models, there are many important links between the two approaches.[14−17]

Indeed, dynamical models can be seen as a generalization of equilibrium models since they simulate the convergence of the supply-demand system towards possibly different equilibrium states and transient states visited due to modifications of supply and/or demand.[18−20] Furthermore, under some rather mild assumptions, equilibrium configurations of the system can be modelled as attractors of the system, for example, states in which the evolution of the system stops. Finally, the dynamical approaches allow analysis of the stability of equilibrium configurations and provide a complete statistical description of the system's evolution.[21]

Theoretical day-to-day dynamical models can be classified into two groups according to whether they model the system in continuous time or discrete time. The former is mainly differential equation systems, which assumes revisions on a route, may be in continuous time.[22,23] The latter is formulated as difference equations or simulation models. In reality, repeated trips are subjected to active constraints which mean that they are not continuously adjustable in time (i.e. over days), therefore, a more appropriate representation would seem to be a discrete time system, critically, this allows an interpretation to be given to the rate of adjustment.[24,25]

In this study, we consider a general network equilibrium model with elastic demands. It is assumed that one has associated each origin–destination (O-D) pair in the network with a travel disutility function. A model for the process by which a route choice is adjusted day-to-day is formulated as a discrete dynamical system. Then this model is applied to a simple network for a behaviour analysis of network flow.

2. Model

Consider in a urban transportation $G = (N, A)$, where N is the set of nodes, A is the links set. Assume that trip rates may be influenced by the level of service on the network and travellers' willingness of taking a faster route. For example, as congestion increases, travellers may decide to use a different mode of travel (e.g. subway), shift the time of travel (outside the design period), or forgo some trips altogether. The trip rate, q_{rs}, between every O-D pair $r-s$ in the network can be assumed to be a function of travel time between origin r and destination s. In other words,

$$q_{rs} = D_{rs}(u_{rs}), \quad \forall_{r,s}. \tag{1}$$

Where u_{rs} is the minimum travel cost between r and s, and $D_{rs}(\cdot)$ is the demand function (for vehicular trips) between r and s. It can include that O-D specific parameters reflect population size, income distribution, and vehicle ownership for origin nodes, as well as employment intensity or retail activity variables for destination nodes.

The demand function can be expected to be monotonically decreasing (or at least non-increasing) in the O-D travel time. In other words, as u_{rs} grows, q_{rs} decreases, and vice versa. This function is also bounded from above.[26] For example, the maximum number of vehicular trips generated between a given origin and a given destination in a certain design period maybe bounded by the population size at the origin.

The concept behind the traffic flow dynamics (or the adjustment process) is that path flow along paths increases if path cost is less than the O-D travel cost, while it decreases if the path cost exceeds O-D travel cost; similarly, the O-D travel cost increases if total path flow on the same O-D pair is less than the O-D demand flow, while it decreases if the path flow exceeds the O-D demand flow.[27] This may be seen as natural, and is similar to the projected dynamical system. These dynamics can be expressed by the following difference equation:

$$f_p(i+1) - f_p(i) = \Psi_p\lfloor f_p(i)\rfloor, \tag{2a}$$

$$\Psi_p[f_p(i)] = \begin{cases} \max\left\{\eta\left[u_{rs}(i) - c_p(i)\right]^{\lambda}, -f_p(i)\right\}, & \text{if } \ c_p(i) - u_{rs}(i) > 0, \\ \eta\left[u_{rs}(i) - c_p(i)\right]^{\lambda}, & \text{otherwise.} \end{cases} \tag{2b}$$

$$u_{rs}(i+1) - u_{rs}(i) = \Phi_{rs}[u_{rs}(i)], \tag{3a}$$

$$\Phi_{rs}[u_{rs}(i)] = \begin{cases} \max\left\{\eta\left[q_{rs}[u_{rs}(i)] - \sum\limits_{p\in P_{rs}} f_p(i)\right]^{\lambda}, -u_{rs}(i)\right\}, & \text{if } \sum\limits_{p\in P_{rs}} f_p(i) - q_{rs}[u_{rs}(i)] > 0, \\ \eta\left[q_{rs}[u_{rs}(i)] - \sum\limits_{p\in P_{rs}} f_p(i)\right]^{\lambda}, & \text{otherwise.} \end{cases} \tag{3b}$$

Define the function $F_p(f_p)$, $U_{rs}(u_{rs})$ as follows:

$$F_p(f_p) = f_p + \Psi_p(f_p) = \begin{cases} \max\left\{f_p(i) + \eta[u_{rs}(i) - c_p(i)]^{\lambda}, 0\right\}, & \text{if } \ c_p(i) - u_{rs}(i) > 0, \\ f_p(i) + \eta[u_{rs}(i) - c_p(i)]^{\lambda}, & \text{otherwise.} \end{cases} \tag{4a}$$

$$\begin{aligned} U_{rs}(u_{rs}) &= u_{rs} + \Phi_{rs}(u_{rs}) \\ &= \begin{cases} \max\left\{u_{rs}(i) + \eta\left[q_{rs}[u_{rs}(i)] - \sum\limits_{p\in P_{rs}} f_p(i)\right]^{\lambda}, 0\right\} & \text{if } \sum\limits_{p\in P_{rs}} f_p(i) - q_{rs}[u_{rs}(i)] > 0, \\ u_{rs}(i) + \eta\left[q_{rs}[u_{rs}(i)] - \sum\limits_{p\in P_{rs}} f_p(i)\right]^{\lambda}, & \text{otherwise.} \end{cases} \end{aligned} \tag{4b}$$

The dynamics can then be rewritten as the following equations:

$$f_p(i+1) = F_p\lfloor f_p(i)\rfloor, \tag{5a}$$

$$u_{rs}(i+1) = U_{rs}[u_{rs}(i)]. \tag{5b}$$

Or,

$$f_p(i+1) = \max\{0, \eta[u_{rs}(i) - c_p(i)]^{\lambda} + f_p(i)\}, \ \ \forall p \in P^{rs}, \ \ \forall_{r,s}, \tag{6a}$$

$$u_{rs}(i+1) = \max\left\{0, \eta\left[q_{rs}(u_{rs}(i)) - \sum_{p\in P_{rs}} f_p(i)\right]^{\lambda} + u_{rs}(i)\right\}, \ \ \forall_{r,s}, \tag{6b}$$

where $f_p(i)$ is path flow on day i and on path p, $c_p(i)$ is path cost on day i and on path p, $u_{rs}(i)$ is the cost on day i between origin r to destination s, η and λ are positive parameters. $q_{rs}(u_{rs}(i))$ is the trip demand of origin r to destination s on day i. The former parameter, η, represents the sensitivity of the adjustment. If the parameter, λ, is more than 1.0, travellers react more sensitively when the difference is large. Otherwise, if λ is less than 1.0, the adjustment function is depressive about the difference of the travel times. From the above model, if path flow $f_p(i)$ equals to 0, which means that the path travel cost is larger than O-D cost; if O-D cost $u_{rs}(i)$ equals to 0, which means that the path flow is larger than O-D demand.

Let $F'_p(f_p)$, $U'_{rs}(u_{rs})$ denote the derivative of $F_p(f_p)$ and $U_{rs}(u_{rs})$ respectively. The derivative of $F_p(f_p)$ and $U_{rs}(u_{rs})$ at equilibrium, $F'_p(f^*_p)$, $U'_{rs}(u^*_{rs})$,

can be classified into three cases for the stability analysis:

$$F'_p(f^*_p) = \begin{cases} 1, & \lambda \geq 1, \\ \infty, & 0 < \lambda < 1, \end{cases} \quad \forall p \in P^{rs}, \tag{7}$$

$$U'_{rs}(u^*_{rs}) = \begin{cases} 1, & \lambda > 1 \\ 1 + \eta q'_{rs}(u^*_{rs}), & \lambda = 1 \\ \infty, & 0 < \lambda < 1 \end{cases} \quad \forall r, s. \tag{8}$$

From Eqs.(7) and (8), when $0 < \lambda < 1$, both $|F'_p(f^*_p)|$ and $|U'_{rs}(u^*_{rs})|$ are greater than 1 and the equilibrium is not stable. When $\lambda > 1$, the equilibrium is locally stable. When $\lambda = 1$, in the case that η is sufficiently large, it is clear that the equilibrium is not (locally) stable because $|U'_{rs}(u^*_{rs})| > 1$, at this time, the step size, $u_{rs}(i+1) - u_{rs}(i)$, is very large. In the case that η is sufficiently small but not 0, the equilibrium is locally stable because $\eta > 0, \lambda > 0$ and $q'_{rs}(u^*_{rs}) < 0$, in this case, the equilibrium might also be globally stable because the step size, $u_{rs}(i+1) - u_{rs}(i)$, is so small that u_{rs} approaches the equilibrium gradually. Furthermore, when $0 < \lambda < 1$, the equilibrium is not stable and the flow does not converge even if η is small. This could imply that the network flow does not converge to the equilibrium and is not stable when travellers react more sensitively to small cost differences than to large.

3. Numerical analysis

The numerical test was implemented on a computer conducted with double precision arithmetic on an Intel® Pentium® 4 CPU 1.75 GHz, 512 MB RAM, by using the Microsoft Window XP (SP2) operating system. All of the coding was carried out in Matlab 6.0. We study the discrete dynamical model on a simple network, which is shown in Fig.1. For this network, there are three nodes: 1, 2, 3; three links: a, b, c; and a single O-D pair (1, 3). Let path $p_1 = (a, c)$ and path $p_2 = (b, c)$.

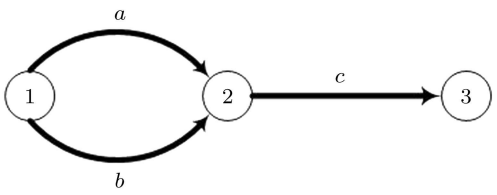


Fig.1. An elastic demand example.

Assume that the user link cost functions are

$$c_a(f) = 5f_a + 2f_b + 15,$$
$$c_b(f) = 7f_b + f_a + 15,$$
$$c_c(f) = 3f_c + f_a + f_b + 12,$$

and the disutility (or inverse demand) function is given by

$$u_{13}(d_{13}) = -2d_{13} + 114.$$

And the dynamical system of the network is

$$f_{p_1}(i+1) = \max\left\{0, \eta[u_{13}(i) - c_{p_1}(i)]^\lambda + f_{p_1}(i)\right\},$$
$$f_{p_2}(i+1) = \max\left\{0, \eta[u_{13}(i) - c_{p_2}(i)]^\lambda + f_{p_2}(i)\right\},$$
$$u_{13}(i+1) = \max\left\{0, \eta\left[q_{13}(u_{13}(i)) - \sum_{p \in P_{13}} f_p(i)\right]^\lambda + u_{13}(i)\right\}.$$

From the discrete dynamic model (6a),(6b), under some parameters, we find complicate behaviours happening in this test network. Oscillations and some apparently irregular behaviours were found in certain intervals of different parameters. Under the irregular behaviours, it is also found that the model is sensitive to initial conditions, when the initial condition is changed slightly, the O-D trip flow changes clearly.

Figures 2–4 demonstrate the different characters of two path flows under different parameters. The total iteration number is 1000 and the bifurcation parameter $\eta \in [0, 1.2]$ with step size 0.003, for better

show the characters of traffic flow, the figures visualized the values of iteration number from 1 to 150. Figure 2 shows the two path flows of the test network, with the same initial conditions for $\eta = 0.1051$. It can be found that the two path flows arrive equilibria (5,4) with total O-D cost $u_{rs} = 96$. Figure 3 shows that the path flows have oscillation with $\eta = 0.3001$, the path flow behaviour is very complex in Fig.4 with $\eta = 0.9001$.

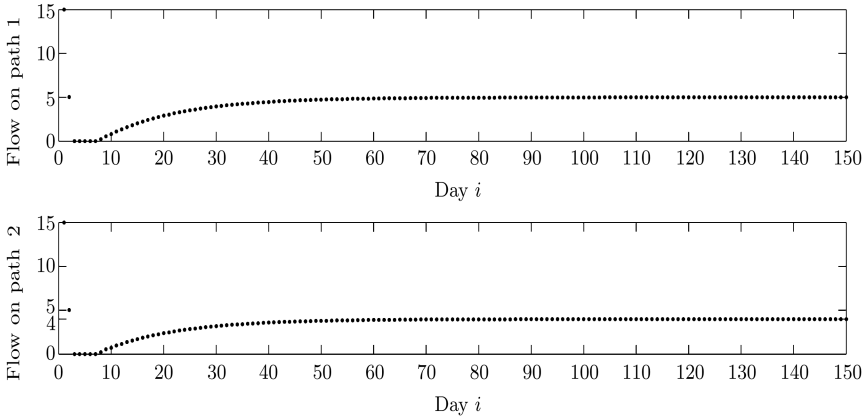


Fig.2. Equilibrium flow on two paths with $\eta = 0.1051$.

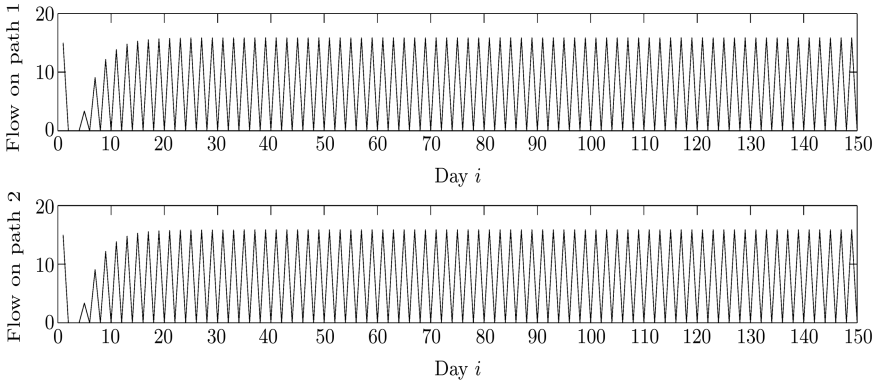


Fig.3. Oscillation flow on two paths with $\eta = 0.3001$.

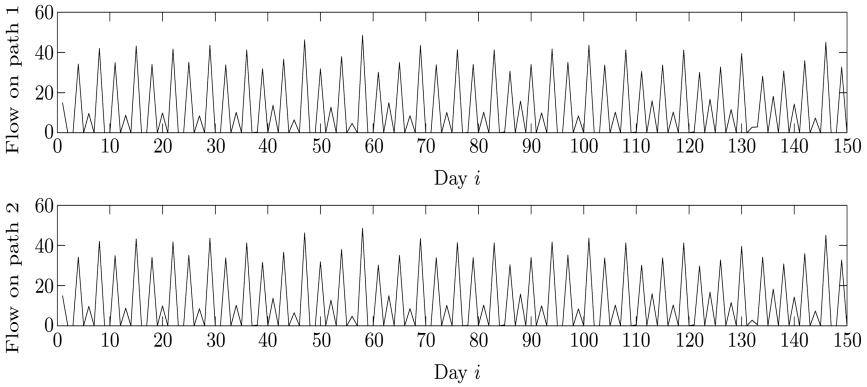


Fig.4. Chaos flow on two paths with $\eta = 0.9001$.

There are different choices for the initial O-D flow at every iteration. For convenience of comparison, we choose two ways: one uses the same initial conditions for all steps of the parameter, the other uses the final states of the system at the previous step of the parameter, given the initial condition at the first step. From Figs.5 and 6, by starting from different initial conditions, different path flows may be detected if there is more than one attractor for the same value of the parameter. Figures 7 and 8 show the O-D travel

cost behaviour about η under different initial conditions. From the figures, it is found that when the sensitivity parameter is in the interval $[0, 0.15]$, that is, travellers are not so sensitive to the travel cost, the network flow equilibrium state can be arrived at; when $\eta \in [0.15, 0.82]$, the path flow begin to oscillate, which causes the path flow arriving at non-equilibrium state and oscillating between path flow less than O-D flow and path flow larger than O-D flow; when $\eta \in [0.82, 0.94]$, more complex path flows appear, the path flow cannot be estimated; when $\eta \in [0.94, 1.2]$, the path flow will change alternatively under different states.

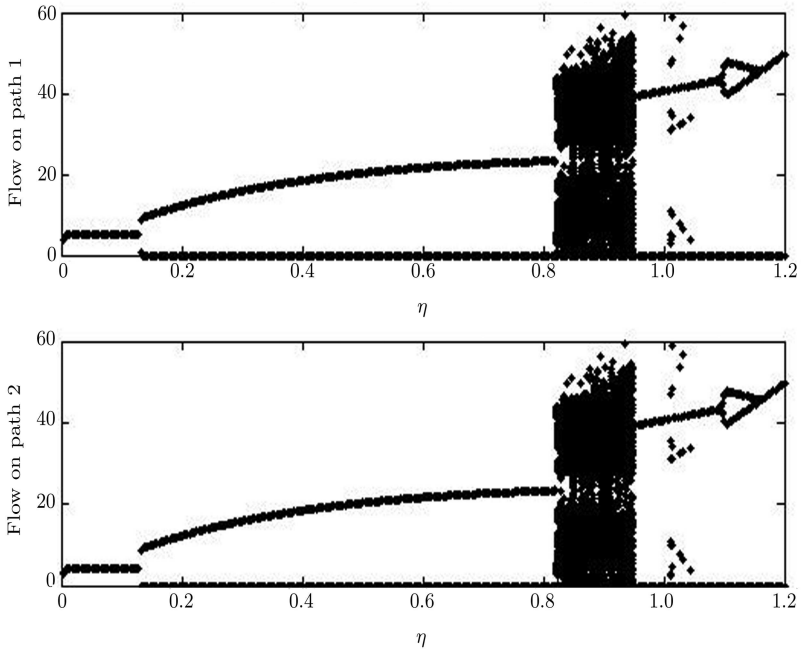


Fig.5. Bifurcation diagram of path flow about η, initial values are the same for all values of η.

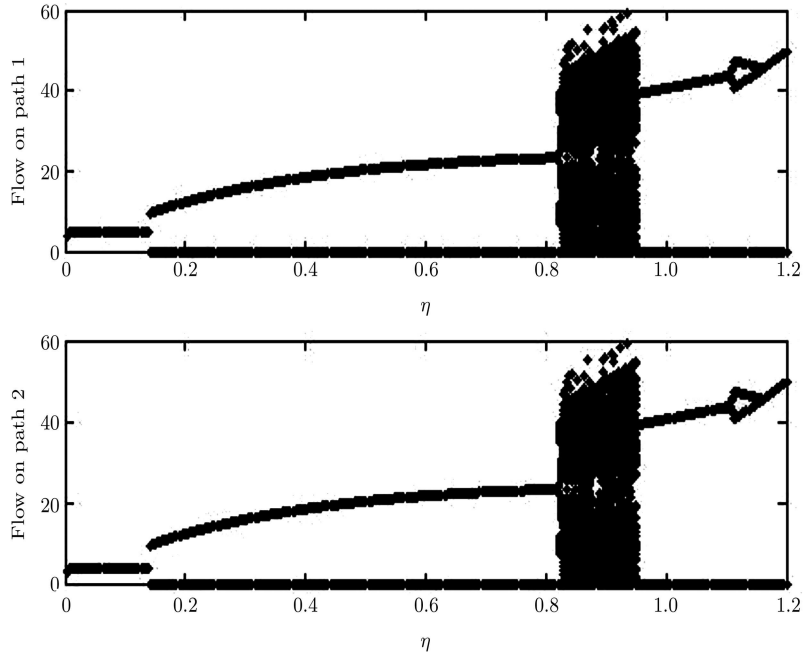


Fig.6. Bifurcation diagram of path flow about η, initial values are the final states of the previous step of η.

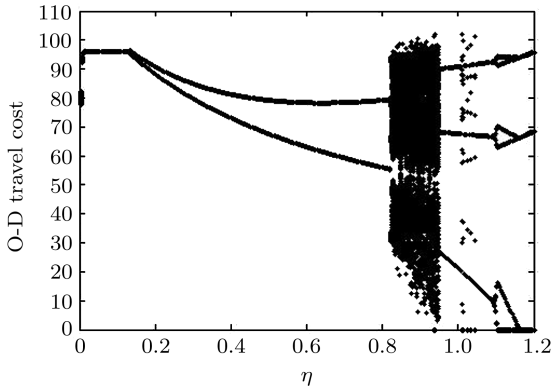


Fig.7. Bifurcation diagram of O-D travel cost about η, initial values are the same for all values of η.

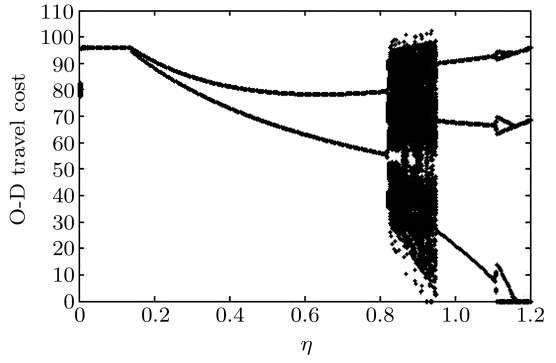


Fig.8. Bifurcation diagram of O-D travel cost about η, initial values are the final states of the previous step of η.

4. Conclusions

This paper assumes that travellers are willing to switch to a faster route, and a model of the day-to-day route adjustment process with elastic demand is formulated as a discrete dynamical system. The model is applied to a simple network, in which one origin-destination pair is connected by different paths. In order to examine the properties of the day-to-day dynamics of network flow, analysis is carried out to investigate the status of network flow.

The results of the study can be summarized as follows: the condition for stability of network flow is that the traveller is not very sensitive to differences of cost. At this time, the network equilibrium appears under the elastic demand. If travellers become sensitive to the path cost difference, the flow does not converge to equilibrium and begins to oscillate or has chaotic behaviours. When network flow becomes chaotic under certain conditions of travellers reaction in the test network, network flows cannot be observed with infinite accuracy. So, if this is the case long-term forecasts of flows cannot be made due to sensitivity to initial conditions, which is the nature of chaos.

As a future work, this model and its analysis should be expanded to a more general network. The question of whether or not there exists network flow in the real world for which chaotic behaviour occurs should be examined.

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