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Complete synchronization between two bi-directionally coupled chaotic systems via an adaptive feedback controller*

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In this paper, we apply a simple adaptive feedback control scheme to synchronize two bi-directionally coupled chaotic systems. Based on the invariance principle of differential equations, sufficient conditions for the global asymptotic synchronization between two bi-directionally coupled chaotic systems via an adaptive feedback controller are given. Unlike other control schemes for bi-directionally coupled systems, this scheme is very simple to implement in practice and need not consider coupling terms. As examples, the autonomous hyperchaotic Chen systems and the new non-autonomous 4D systems are illustrated. Numerical simulations show that the proposed method is effective and robust against the effect of weak noise.

Keywords: complete synchronization, bi-directionally coupled systems, adaptive control
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1. Introduction

The synchronization of oscillations has been well known to scientists since the historical observation of the phenomenon by Huygens in a pendulum clock,[1] which is an omnipresent phenomenon in natural systems and artificial oscillators. In particular, since the idea of synchronizing two identical chaotic systems under different initial conditions was first introduced by Pecora and Carroll in 1990,[2] chaotic synchronization, as an important topic in nonlinear science, has been realized to have great potential applications in a large number of areas, such as physics,[3] biology,[4] signal engineering,[5] chemistry[6] and ecology science,[7] etc. Because of its importance, chaotic synchronization has been studied extensively by many scientists in recent years, and various modern control methods have been proposed to synchronize chaotic systems with feedback or non-feedback, such as backstepping design,[8] nonlinear controller,[9,10] adaptive controller,[11−13] active controller,[14] or adaptive active controller,[15] etc.

In fact, many natural and synthetic systems can be described by coupled systems correspondingly. Synchronization is also a common phenomenon in coupled systems, and has been extensively studied in laser dynamics, electronic circuits, chemical and biological systems. It is well known that, for coupled systems, the influence of the coupling terms on the synchronization is very important. In this paper, based on the invariance principle of differential equation, we apply an adaptive feedback controller to synchronizing two kinds of bi-directionally coupled systems globally without considering coupling terms.

The rest of this paper is organized as follows. In Section 2, a chaotic system considered in this paper is presented, the criterion for chaotic synchronization is established. In Section 3, in order to validate the proposed synchronization approach, two examples are considered, i.e. the synchronizations between two bi-directionally coupled autonomous hyperchaotic Chen systems and between two bi-directionally coupled new non-autonomous 4D systems, respectively, via an adaptive feedback controller. Finally, some conclusions are drawn in Section 4.

2. Adaptive synchronization scheme

Consider the bi-directionally coupled ordinary differential equations described by

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$$\dot{x} = f(x,t) + C(y-x),$$
$$\dot{y} = f(y,t) + C(x-y), \quad (1)$$

where $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ and $y = (y_1, y_2, \ldots, y_n)^{\mathrm{T}} \in R^n$ are the state vectors of two oscillators and $f = (f_1, f_2, \ldots, f_n)^{\mathrm{T}} : R^n \to R^n$ is the non-linear vector field of a single oscillator in the absence of coupling, and $C = (c_{ij})_{n\times n}$ is the coupling link matrix, which is a matrix of each node's variable used in the coupling. For the vector function $f(x,t)$, we give the following assumption.[11,12]

Assumption Let $\Omega \in R_n$ be a chaotic bounded set of Eq.(1) which is globally attractive, for any $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ and $y = (y_1, y_2, \ldots, y_n)^{\mathrm{T}} \in \Omega$, there exists a positive constant l satisfying

$$|f_i(x,t) - f_i(y,t)| \le l \max|x_i - y_i|, \quad j = 1,2,\ldots,n$$
$$i = 1,2,\ldots,n, \quad (2)$$

In fact, a lot of chaotic systems satisfy the above condition. If $f(x,t)$ degenerates into $f(x)$, i.e. $f(x,t)$ is independent of time t, the above condition becomes very general. For example, condition (2) holds as long as $\dfrac{\partial f_i}{\partial x_j}(i,j = 1,2,\ldots,n)$ are bounded. If $f(x,t)$ depends on time t, condition (2) is also very general. For example, it is not difficult to find a kind of chaotic system with $f(x,t) = A(t)g(x) + b(t)$ satisfying condition (2), where $A(t) \in R^n \times R^n$ is a bounded matrix function, $b(t) \in R^n$ is a bounded vector function, $g(x) \in R^n$ is a vector function satisfying condition (2).

In order to realize the synchronization, the controller $u = (u_1, u_2, \ldots, u_n)^{\mathrm{T}} \in R^n$ is added to the second system

$$\dot{y} = f(y,t) + C(x-y) + u. \quad (3)$$

The synchronization problem actually is how to design a controller u which can synchronize the states of the coupled system. If we define the error vector as $e = y - x$, then the equation of synchronization errors can be expressed as

$$\dot{e} = f(y,t) - f(x,t) - 2C(y-x) + u, \quad (4)$$

where $e = (e_1, e_2, \ldots, e_n)^{\mathrm{T}} \in R^n$. The aim of synchronization is to achieve $\lim\limits_{t\to\infty} ||e(t)|| = 0$.

The problem of synchronization between the coupled systems can be converted into a problem of how to realize the asymptotical stabilization of error system (4). So our aim is to design a controller u to make error system (4) asymptotically stable *ab origine*.

We introduce the following simple adaptive-feedback controller u:

$$u_i = \varepsilon_i(y_i - x_i), \quad i = 1,2,\ldots,n, \quad (5)$$

where $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)^{\mathrm{T}} \in R^n$ is the variable feedback strength. Instead of linear feedback, the feedback strength ε will be adapted according to the following law:

$$\dot{\varepsilon}_i = -\gamma_i e_i^2, \quad i = 1,2,\ldots,n, \quad (6)$$

where $\gamma_i > 0 (i = 1,2,\ldots,n)$, i.e. they are arbitrary positive constants.

Then error system (4) can be rewritten as

$$\dot{e}_i = f_i(y,t) - f_i(x,t) - 2\sum_{j=1}^{n} c_{ij}e_j + \varepsilon_i e_i,$$
$$i = 1,2,\ldots,n. \quad (7)$$

The following theorem will give the sufficient condition for coupled system (1) with a controller to be globally asymptotically synchronized.

Theorem 1 If system (1) satisfies the above assumption and controller u follows update law (5), then error system (4) is globally asymptotically stable *ab origine*.

Proof We introduce the following non-negative function:

$$V = \frac{1}{2}\sum_{i=1}^{n} e_i^2 + \frac{1}{2}\sum_{i=1}^{n}\frac{1}{\gamma_i}(\varepsilon_i + L)^2, \quad (8)$$

where L is a constant. By differentiating the function V along with system (4), we obtain

$$\begin{aligned}\dot{V} = & \sum_{i=1}^{n} e_i(\dot{y}_i - \dot{x}_i) + \sum_{i=1}^{n}\frac{1}{\gamma_i}(\varepsilon_i + L)\dot{\varepsilon}_i \\ = & \ e^{\mathrm{T}}[f(y) - f(x) - 2Ce] + \sum_{i=1}^{n}\varepsilon_i e_i^2 - \sum_{i=1}^{n}(\varepsilon_i + L)e_i^2\end{aligned}$$

$$\begin{aligned} &= e^{\mathrm{T}}[f(y) - f(x) - (C^{\mathrm{T}} + C)e] + \sum_{i=1}^{n} \varepsilon_i e_i^2 - \sum_{i=1}^{n} (\varepsilon_i + L) e_i^2 \\ &\leq e^{\mathrm{T}}[nlI - C^{\mathrm{T}} - C - LI]e, \end{aligned} \tag{9}$$

where I is an identity matrix. It is obvious that $nlI - C^{\mathrm{T}} - C$ is a real symmetric matrix. Therefore, according to the theory of matrix,[16] if L is big enough, matrix $nlI - C^{\mathrm{T}} - C - LI$ is definitely negative, i.e. $\dot{V} \leq 0$.

In the above proof, we have used condition (2) in the above assumption. Obviously, $\dot{V} = 0$ if and only if $e_i = 0$ $(i = 1, 2, \ldots, n)$. Namely, the set $E = \{(e, \varepsilon) \in R^{2n} | e = 0, \varepsilon = \varepsilon_0 \in R^n\}$ is the largest set contained in $\dot{V} = 0$ for system (4). Then according to the well-known La Salle invariance principle of differential equation,[17] starting with arbitrary initial values of system (4), the orbit will converge asymptotically to the set E, i.e. $y \to x$ and $\varepsilon \to \varepsilon_0$ as $t \to \infty$.

3. Application of the above-mentioned method

In this section, two examples are given to illustrate the effectiveness of the proposed scheme, and the numerical simulations are performed to validate the proposed method. As is well known, mixing of the useful signal and noise takes place during the transfer of information through a communication channel. So noise is considered in the following two examples, and we find that the adaptive synchronization scheme is robust against the effect of weak noise.

Example 1 Consider two coupled identical autonomous hyperchaotic Chen systems[18] with a couple link matrix:

$$C = \begin{pmatrix} 0.1 & 0.1 & 0 & 0 \\ 0.2 & 0.3 & 0.2 & 0 \\ 0 & 0.4 & 0.1 & 0 \\ 0 & 0 & 0 & 0.2 \end{pmatrix}.$$

So the two bi-directionally coupled identical hyperchaotic systems can be described by the following differential equations:

$$\begin{aligned} \dot{x}_1 &= a(x_2 - x_1) + x_4 + 0.1 \cdot (y_1 - x_1) + 0.1 \cdot (y_2 - x_2), \\ \dot{x}_2 &= dx_1 - x_1 x_3 + cx_2 + 0.2 \cdot (y_1 - x_1) + 0.3 \cdot (y_2 - x_2) + 0.2 \cdot (y_3 - x_3), \\ \dot{x}_3 &= x_1 x_2 - bx_3 + 0.4 \cdot (y_2 - x_2) + 0.1 \cdot (y_3 - x_3), \\ \dot{x}_4 &= x_2 x_3 + rx_4 + 0.2 \cdot (y_4 - x_4), \\ \dot{y}_1 &= a(y_2 - y_1) + y_4 + 0.1 \cdot (x_1 - y_1) + 0.1 \cdot (x_2 - y_2) + u_1, \\ \dot{y}_2 &= dy_1 - y_1 y_3 + cy_2 + 0.2 \cdot (x_1 - y_1) + 0.3 \cdot (x_2 - y_2) + 0.2 \cdot (x_3 - y_3) + u_2, \\ \dot{y}_3 &= y_1 y_2 - by_3 + 0.4 \cdot (x_2 - y_2) + 0.1 \cdot (x_3 - y_3) + u_3, \\ \dot{y}_4 &= y_2 y_3 + ry_4 + 0.2 \cdot (x_4 - y_4) + u_4, \end{aligned} \tag{10}$$

where $u = (u_1, u_2, u_3)^{\mathrm{T}}$ is an adaptive-feedback controller with update laws (4) and (5); a, b, c, d, r are all positive real constant parameters. Here, let $a = 38$, $c = 12$, $d = 7$, and $r = 0.5$. Obviously, the single oscillator is hyperchaotic in the absence of coupling. It has been known that the hyperchaotic Chen system is similar to the Lorenz system, and the suitable control of the second component may synchronize the Lorenz system, so we let $\varepsilon_1 = \varepsilon_3 = \varepsilon_4 = 0$, and set $\gamma_2 = 0.1$.

For the numerical simulation, we use the fourth-order Runge–Kutta method to solve the system of differential equation (10) with a time step of 0.001, and the initial states of two coupled systems are set as $x = (2, 3, 7, 3)^{\mathrm{T}}$, $y = (3, 4, 1, 2)^{\mathrm{T}}$ and $\varepsilon^0 = (0, 0, 0, 0)^{\mathrm{T}}$. The simulation results are shown in Fig.1. In the figure, one can see that the synchronization errors con-

verge to zero finally and the feedback strength ε_2 converges to a constant. We consider the robustness of the scheme against the noise by adding a uniformly distributed random noise in the range [0,1], with amplitude of 0.25, to the time series x_2. The corresponding numerical simulation is given in Fig.2. The simulation results show that the synchronization errors eventually approach zero and in most of time slight fluctuations occur around zero.

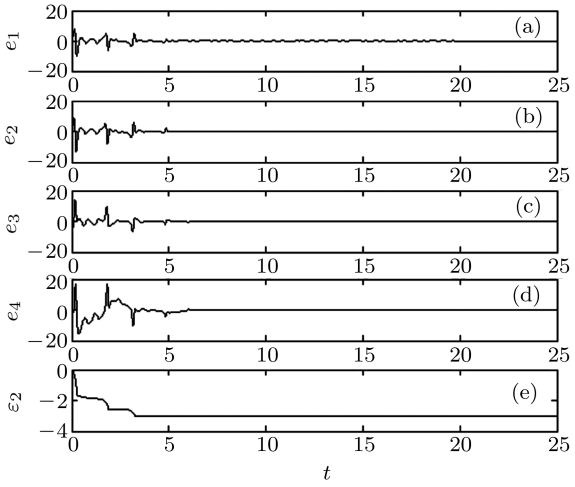


Fig.1. The chaotic synchronization error between the two bi-directionally coupled hyperchaotic Chen systems via an adaptive feedback controller, where (a)–(d) show temporal evolutions of the synchronization error, (e) shows the evolution of the feedback strength ε_2.

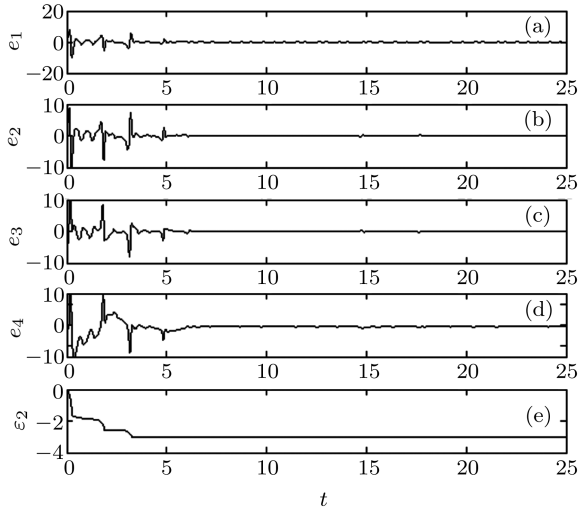


Fig.2. (a)–(e) show the effect on the synchronization and feedback strength in Fig.1 when a uniformly distributed noise in range [0, 1], of the strength 0.25, is present in the signal x_2.

Example 2 As the second example we consider the new non-autonomous 4D system described by[19]

$$\begin{aligned}
\dot{x}_1 &= a(x_2 - x_1) + x_2x_3x_4,\\
\dot{x}_2 &= b(x_1 + x_2) - x_1x_3x_4,\\
\dot{x}_3 &= -(c_1 + c_2\sin t)x_3 + x_1x_2x_4,\\
\dot{x}_4 &= -dx_4 + x_1x_2x_3,
\end{aligned} \tag{11}$$

where $(x_1, x_2, x_3, x_4)^{\mathrm{T}}$ are the state variables of the system and a, b, c_1, c_2, d are parameters of positive real constants. For the fixed parameters: $a = 30, b = 10, c_1 = 95, c_2 = 10, d = 10$, 4D-system (11) is chaotic. Considering coupling, we set coupling link matrix

$$C = \begin{pmatrix} 0.1 & 0 & 0.3 & 0\\ 0 & 0.6 & 0 & 0.4\\ 0.2 & 0 & 0.1 & 0\\ 0 & 0.3 & 0 & 0.4 \end{pmatrix}.$$

So the two bi-directionally coupled 4D-systems with a controller can be described by

$$\begin{aligned}
\dot{x}_1 &= a(x_2 - x_1) + x_2x_3x_4 + 0.1\cdot(y_1 - x_1) + 0.3\cdot(y_3 - x_3),\\
\dot{x}_2 &= b(x_1 + x_2) - x_1x_3x_4 + 0.6\cdot(y_2 - x_2) + 0.4\cdot(y_4 - x_4),\\
\dot{x}_3 &= -(c_1 + c_2\sin t)x_3 + x_1x_2x_4 + 0.2\cdot(y_1 - x_1) + 0.1\cdot(y_3 - x_3),\\
\dot{x}_4 &= -dx_4 + x_1x_2x_3 + 0.3\cdot(y_2 - x_2) + 0.4\cdot(y_4 - x_4),\\
\dot{y}_1 &= a(y_2 - y_1) + y_2y_3y_4 + 0.1\cdot(x_1 - y_1) + 0.3\cdot(x_3 - y_3) + u_1,\\
\dot{y}_2 &= b(y_1 + y_2) - y_1y_3y_4 + 0.6\cdot(x_2 - y_2) + 0.4\cdot(x_4 - y_4) + u_2,\\
\dot{y}_3 &= -(c_1 + c_2\sin t)y_3 + y_1y_2y_4 + 0.2\cdot(x_1 - y_1) + 0.1\cdot(x_3 - y_3) + u_3,\\
\dot{y}_4 &= -dy_4 + y_1y_2y_3 + 0.3\cdot(x_2 - y_2) + 0.4\cdot(x_4 - y_4) + u_4,
\end{aligned} \tag{12}$$

where $u = (u_1, u_2, u_3)^{\mathrm{T}}$ is the adaptive-feedback controller with the update laws (4) and (5). Here, we set $\varepsilon_3 = 0$, $\gamma_1 = 0.2$, $\gamma_2 = 0.3$ and $\gamma_4 = 0.4$.

In numerical simulation, we use the fourth-order Runge–Kutta method to solve the system of differential equation (12) with a time step of 0.001. The initial states of two coupled systems are set as $x = (0, -0.8, 1.2, 3.0)^{\mathrm{T}}$, $y = (-1.0, 1.0, 2.5, 0.8)^{\mathrm{T}}$ and $\varepsilon^0 = (0, 0, 0, 0)^{\mathrm{T}}$. The simulation results are shown in Fig.3. In these figures, one can see that the synchronization error converges to zero finally and feedback strengths $\varepsilon_1, \varepsilon_2, \varepsilon_4$ converge to constants. We consider the robust of the scheme against the noise by adding a uniformly distributed random noise in the range [0,1], with amplitude 0.25, to the time series x_1, x_2, x_3 simultaneously. The corresponding numerical simulations are given in Fig.4. The simulation results show that the synchronization errors eventually approach zero and slight fluctuations appear around zero.

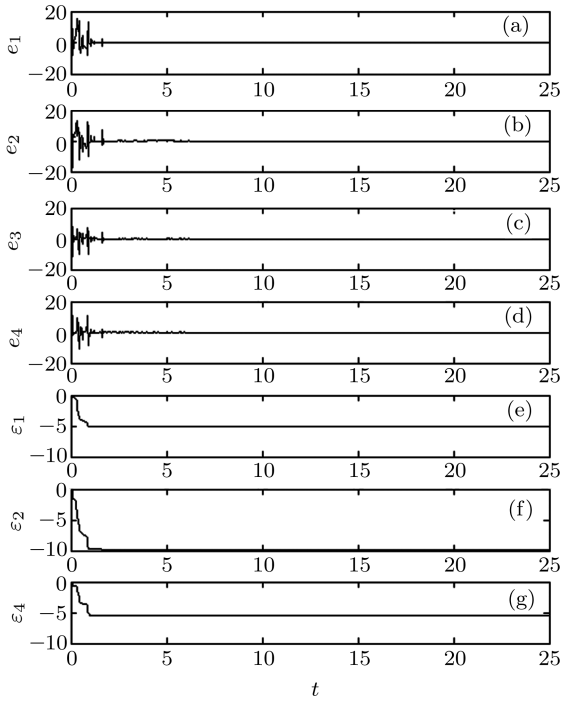


Fig.3. The chaotic synchronization error between the two bi-directionally coupled new 4D-systems via an adaptive feedback controller, where (a)–(d) show temporal evolutions of the synchronization error, (e)–(g) show the evolutions of the feed-back strengths $\varepsilon_1, \varepsilon_2, \varepsilon_4$.

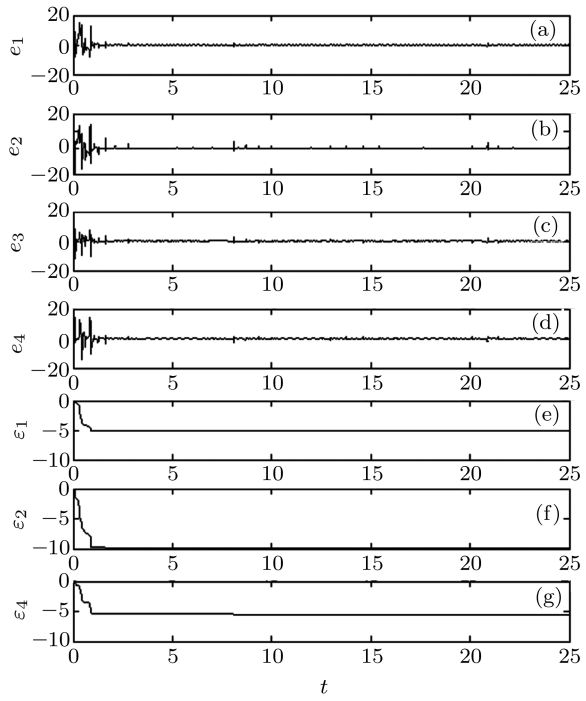


Fig.4. (a)–(g) show the effect on the synchronization and feedback strength in Fig.3 when a uniformly distributed noise in range [0, 1], of the strength 0.25, is present in the signals x_1, x_2, x_4.

4. Conclusion

In this paper, we apply a simple adaptive feedback control scheme to synchronizing two bi-directionally coupled chaotic systems. Based on the invariance principle of differential equations, the sufficient conditions for the global asymptotic synchronization of the two bi-directionally coupled chaotic systems are given. In order to validate the proposed synchronization scheme, two examples are illustrated, and numerical simulations are performed. The simulation results show that the proposed control method is very effective and robust against weak noise. Unlike other control methods for bi-directionally coupled systems, this method is very simple, and is available for a kind of chaotic system. We believe that this simple control method may have a practical application to bi-directionally coupled chaotic systems in the future.

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