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Chaos control and reduced-order generalized synchronization for the Chen–Liao system*

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This paper deals with the problem of chaos control and synchronization of the Chen–Liao system. From rigorous mathematic justification, the chaotic trajectories of the Chen–Liao system are led to a type of points whose four-dimensional coordinates have a particular functional relation among them. Meanwhile, a new synchronization manner, reduced-order generalized synchronization (RGS), is proposed which has the characteristic of having a functional relation between the slave and the partial master systems. It is shown that this new synchronization phenomenon can be realized by a novel technique. Numerical simulations have verified the effectiveness of the proposed scheme.

Keywords: Chen–Liao chaotic system, chaos control, RGS
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1. Introduction

Since the beginning of the 1990s a new direction has appeared in the theory of dynamical system, which is called 'controlling chaos'.[1] Generally speaking, chaos suppression and synchronization are two leading application domains of chaos control. Nowadays, after the pioneering work of Ott *et al*,[2] different techniques have been proposed to realize chaos control, such as linear and nonlinear feedback control,[3,4] adaptive control,[5,6] and back-stepping design,[7] etc. As to chaos suppression, most control methods aim to lead chaotic trajectories to their equilibrium point[8] or periodic orbit.[9] As far as we know, to date, driving all states of a chaotic system to an arbitrary point in the phase space has not been fully reported yet.

Methods for synchronizing chaotic systems have been developed simultaneously with the studies of chaos suppression. Since the research made by Pecora and Carroll,[10] synchronization phenomena have been extensively studied for various potential applications.[11] Recently, chaos synchronization of nonlinear dynamical systems with different orders, namely reduced-order synchronization,[12] has received increasing attention due to its ubiquity in real life. Its appearance stems from the research on the movement of neurons.[13,14] One study shows that it is possible that synchronous activity can be yielded in the thalamic and hippocampal neurons network.[15] As another example, we can observe that circulatory and respiratory systems behave in a synchronous way. Here, the master system is strictly different from the slave system; the difference may involve different orders. The research results in Ref.[12] have shown evidence that this problem has been solved. The main feature of the reduced-order synchronization is that the order of the slave system is less than the master one. Many authors[16−18] have analysed the reduced-order synchronization of several representative dynamical systems and the synchronization type which they discussed is the identical synchronization, so one might just as well call it reduced-order identical synchronization (RIS). In consideration of the difficulty for realizing RIS in real life due to various uncertain factors, we will take the generalized synchronization along with the reduced-order synchronization into consideration in this paper. That is, a functional relation exists between the slave and the partial master systems. Thus, we could name it reduced-order generalized synchronization (RGS) which is different from the generalized reduced-order synchronization (GRS).[19]

In view of the above considerations, the object of this paper is to undertake an investigation on the

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chaos controlling and the RGS for nonlinear dynamical system. The Chen–Liao system,[20] which was derived from a generalized model with distributed delays, is used as an illustration. The rest of this paper is organized as follows. The chaotic system is briefly introduced in Section 2. In Section 3, by the aid of Lyapunov analysis, the chaotic Chen–Liao system is controlled to the type of points, whose coordinates have a specific functional relation. In Section 4, the RGS is achieved between the Chen–Liao system and a class of systems using the novel control approach. Section 5 displays the corresponding numerical simulations, and conclusions are finally drawn in Section 6.

2. Description of the system

The chaotic dynamical system considered in this paper is described by[20]

$$\begin{aligned}
\dot{x}_1 &= (c-a)x_4 - x_2x_4 + cx_1,\\
\dot{x}_2 &= x_1x_4 - bx_2,\\
\dot{x}_3 &= a(x_1 - x_3),\\
\dot{x}_4 &= a(x_3 - x_4),
\end{aligned} \tag{1}$$

where $X = (x_1, x_2, x_3, x_4)$ is the variable and $\Theta = (a, b, c)$ is the positive system parameter.

When b and c are fixed($b = 4,\ c = 2.45$) and a varies, a stable limit cycle ($a = 0.85$) arises through a supercritical Hopf bifurcation. As a continues to increase, a new periodic solution with period-doubling bifurcates from the unstable cycle ($a = 1.2$). For $a > 2.6$, chaos-like solutions appear, and typical chaotic solutions are found in the case of $a = 3.1$.[20] Projections of the chaotic attractor of the Chen–Liao system under the condition $(0.5, 1.0, 2.0, -3.0)^{\mathrm{T}}$ on space $((x_1 - x_2 - x_3)$ and $(x_1 - x_3 - x_4))$ are shown in Figs.1(a) and 1(b).

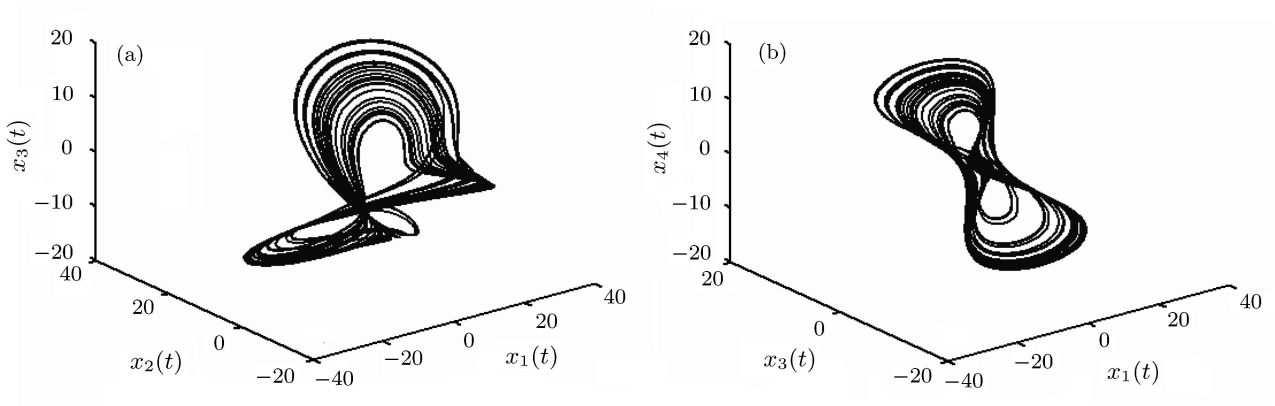


Fig.1. The chaotic attractor for Chen–Liao system with $a = 3.1$, $b = 4$, $c = 2.45$.

3. Controlling chaos to specific points

In this section, chaotic trajectories of the Chen–Liao system is controlled to any given point $P = (r, r^2/b, r, r)$. In the case of $r = 0$, P is the equilibrium point of system (1). For this purpose, assume that the controlled dynamical system is given by

$$\begin{aligned}
\dot{x}_1 &= (c-a)x_4 - x_2x_4 + cx_1 + u,\\
\dot{x}_2 &= x_1x_4 - bx_2,\\
\dot{x}_3 &= a(x_1 - x_3),\\
\dot{x}_4 &= a(x_3 - x_4),
\end{aligned} \tag{2}$$

where u is the undetermined controller.

Define the Lyapunov function as follows:

$$V(e_1, e_3) = \frac{1}{2}(e_1^2 + e_3^2), \tag{3}$$

where $e_1 = x_1 - r$ and $e_3 = x_3 - r$ are the state errors. Taking the time derivative of V along the trajectories of system (2) leads to

$$\begin{aligned}
\dot{V} &= e_1\dot{e}_1 + e_3\dot{e}_3\\
&= e_1[(c-a)x_4 - x_2x_4 + cx_1 + u]\\
&\quad + e_3[a(x_1 - x_3)]\\
&= -ae_3^2 + e_1[(c-a)x_4 - x_2x_4\\
&\quad + cx_1 + ae_3 + u].
\end{aligned} \tag{4}$$

Then, if we select the controller as

$$u = -[(c-a)x_4 - x_2x_4 + cx_1 + ae_3], \tag{5}$$

it is easily deduced according to the Lyapunov stability theorem that $e_3 \to 0$, which implies $x_3 \to r$ as $t \to \infty$. According to the Lagrange mean value theorem in differential calculus, one obtains $\lim\limits_{t\to\infty} \dot{x}_3(t) = 0$, thus we can easily find $\lim\limits_{t\to\infty} x_1(t) = r$ by combining the third equation of system (2) with Eq.(5). Furthermore, we may prove that $\lim\limits_{t\to\infty} x_2(t) = r^2/b$ and $\lim\limits_{t\to\infty} x_4(t) = r$ with the choice of proposed controller. For this reason, we introduce the following theorem.

Let a nonlinear dynamic system be given by

$$\dot{X}(t) = \Lambda X(t) + f(X(t)), \tag{6}$$

where

$$X = (x_1, x_2, \ldots, x_n)$$

is the n-dimensional state vector, Λ is a constant matrix, and $f(X) = (f_1(X), f_2(X), \ldots, f_n(X))$ is a nonlinear vector function. Suppose X is defined on a set $\Omega \subset R^n$, which is globally bounded.

Theorem 1 If the coefficient matrix Λ and nonlinear functions $f(X(t))$ are selected such that

(1) $\Lambda = \mathrm{diag}[a_i]$, with $-\infty < a_i < 0$, $i = 1, 2, \ldots, n$.

(2) $\lim\limits_{t\to\infty} f(X(t)) = A = [b_i]_{n\times 1}$, with b_i being bounded $i = 1, 2, \ldots, n, \forall X \in \Omega$.

Then the state vector $X(t)$ of system (6) tends to B asymptotically, where $B = [-b_i/a_i]_{n\times 1}$, $i = 1, 2, \ldots, n$. The detailed proof is given in the Appendix.

Now, we consider the last equation of system (2):

$$\dot{x}_4 = a(x_3 - x_4). \tag{7}$$

In contrast with Eq.(6), it is obvious that $\Lambda = -a$ and $f(X(t)) = ax_3$ for system (7), and $\lim\limits_{t\to\infty} f(X(t)) = \lim\limits_{t\to\infty} ax_3 = ar$ can be derived from the above analysis. By Theorem 1, we can deduce that $\lim\limits_{t\to\infty} x_4(t) = r$. Furthermore, $\lim\limits_{t\to\infty} x_2(t) = r^2/b$ can be obtained from the second equation of system (2) in a similar way. Therefore, the chaotic trajectories of controlled system (2) can be guided asymptotically to an arbitrarily given point $P = (r, r^2/b, r, r)$ by using the controller (5).

4. The RGS for Chen–Liao system

This section will address the generalized synchronization of chaotic systems with different orders and the slave system is given in the following form:

$$\begin{aligned} \dot{y}_1 &= y_2, \\ \dot{y}_2 &= y_3, \\ \dot{y}_3 &= f(y_1, y_2, y_3) + U, \end{aligned} \tag{8}$$

where

$$Y = (y_1, y_2, y_3)$$

is the state variable of system (8), U is the undetermined controller. In fact, the salve system is a general model, for many systems have the above form, such as Arneodo chaotic system[21] and Genesio chaotic system.[22] Then, we will give the systemic procedure to achieve the RGS for systems (1) and (8).

Define the first variable

$$z_1 = x_4 - y_1.$$

Its derivative along the solutions of systems (1) and (8) is

$$\dot{z}_1 = \dot{x}_4 - \dot{y}_1 = a(x_3 - x_4) - y_2.$$

Now define the second variable

$$z_2 = a(x_3 - x_4) - y_2.$$

Its time derivative along the solutions of systems (1) and (8) is

$$\dot{z}_2 = a(\dot{x}_3 - \dot{x}_4) - \dot{y}_2 = a^2(x_1 - 2x_3 + x_4) - y_3.$$

Define the third variable

$$z_3 = a^2(2x_3 - x_1 - x_4) + y_3.$$

Its time derivative along the solutions of systems (1) and (8) is

$$\begin{aligned} \dot{z}_3 &= a^2(2\dot{x}_3 - \dot{x}_1 - \dot{x}_4) + \dot{y}_3 \\ &= a^2[2a(x_1 - x_3) - (c-a)x_4 + x_2x_4 \\ &\quad - cx_1 - a(x_3 - x_4)] + f + U. \end{aligned}$$

Therefore, we have the following theorem.

Theorem 2 If the controller U is taken as

$$\begin{aligned} U = &-a^2[2a(x_1 - x_3) - (c-a)x_4 \\ &+ x_2x_4 - cx_1 - a(x_3 - x_4)] \end{aligned}$$

$$-f + k_1 z_1 + k_2 z_2 + k_3 z_3, \tag{9}$$

and the feedback coefficients satisfy the following conditions:

$$(1) \quad k_1 > 0,\ k_2 > 0,\ k_3 < 0,\ (2) \quad k_1 < -k_2 k_3, \tag{10}$$

then the RGS between systems (1) and (8) can be achieved.

Proof With the choice of controller (9), we can obtain the following error system:

$$\begin{aligned} \dot{z}_1 &= z_2, \\ \dot{z}_2 &= -z_3, \\ \dot{z}_3 &= k_1 z_1 + k_2 z_2 + k_3 z_3. \end{aligned} \tag{11}$$

It is obvious that the error system has the only equilibrium point: $O(0,\ 0,\ 0)$. The corresponding characteristic equation is

$$\lambda^3 - k_3 \lambda^2 + k_2 \lambda + k_1 = 0. \tag{12}$$

According to the Routh–Hurwitz criterion, the real eigenvalues and all the real parts of complex conjugate eigenvalues are negative when $k_i\,(i = 1,\ 2,\ 3)$ meets the conditions (10). Therefore, the origin is asymptotically stable. Meanwhile, the RGS relationship is given as follows:

$$\begin{pmatrix} x_1 \\ x_3 \\ x_4 \end{pmatrix} = \begin{bmatrix} 1 & 2/a & 1/a^2 \\ 1 & 1/a & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}. \tag{13}$$

5. Numerical simulations

To demonstrate and verify the proposed theoretical results, some numerical simulations are presented in this section. Fourth-order Runge-Kutta method is applied to solve the system of differential equations, with a time step size 0.001s in what follows. The system parameters (a, b, c) are chosen to be (3.1, 4, 2.45) with which the Chen–Liao system behaves chaotically. The initial conditions are set at

$$x_1(0) = x_2(0) = x_3(0) = x_4(0) = 1.0.$$

In the simulations of controlling the chaotic motion to any specified point in the phase space, the control input is switched on at $t = 15$ s. To display the control results, we select parameter $r = 6$ then the control target is $P = (6,\ 9,\ 6,\ 6)$. Figures 2(a) and 2(b) presents the numerical simulation results, showing the process that the chaotic trajectories of Chen–Liao system converge to the control target quickly.

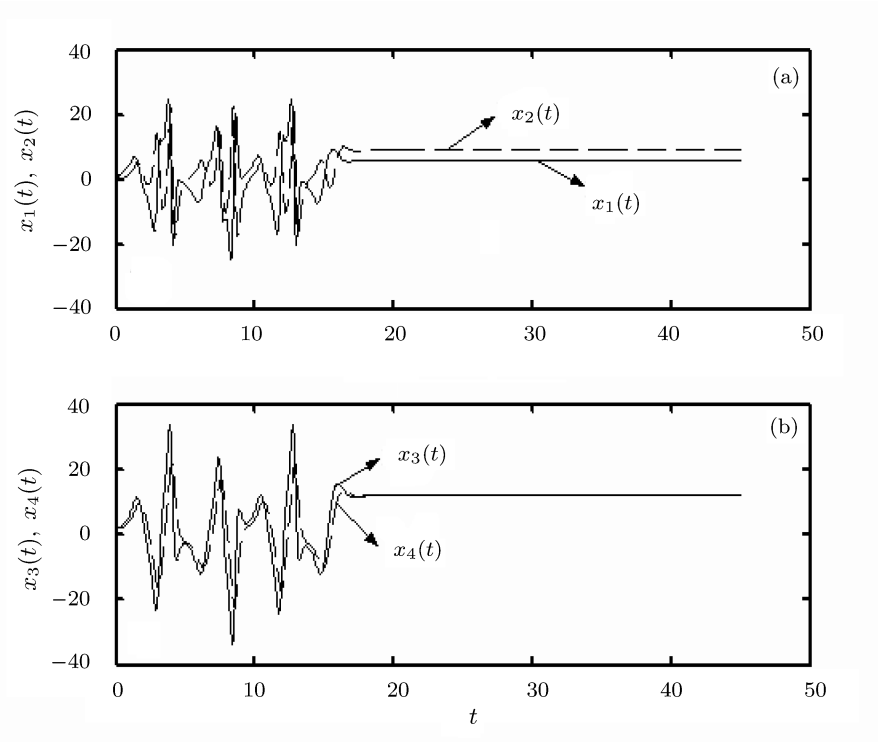


Fig.2. Controlling the chaotic Chen–Liao system to the point (6, 9, 6, 6). (a) the controlled trajectories of $x_1(t)$ and $x_2(t)$; (b) the controlled trajectories of $x_3(t)$ and $x_4(t)$.

In the simulations of the RGS, the Genesio chaotic system is used as a response system, that is $f(y_1, y_2, y_3) =$

$-my_1-ny_2-ky_3+y_1^2$. The initial values and the parameters of the Genesio system are $y_1(0) = 0.2$, $y_2(0) = -0.3$, $y_3(0) = 0.1$ and $m = 6$, $n = 2.92$, $k = 1.2$, which ensures the chaotic behaviour. The errors of RGS are shown in Fig.3, from which we can see that the RGS is achieved rapidly using the proposed controller.

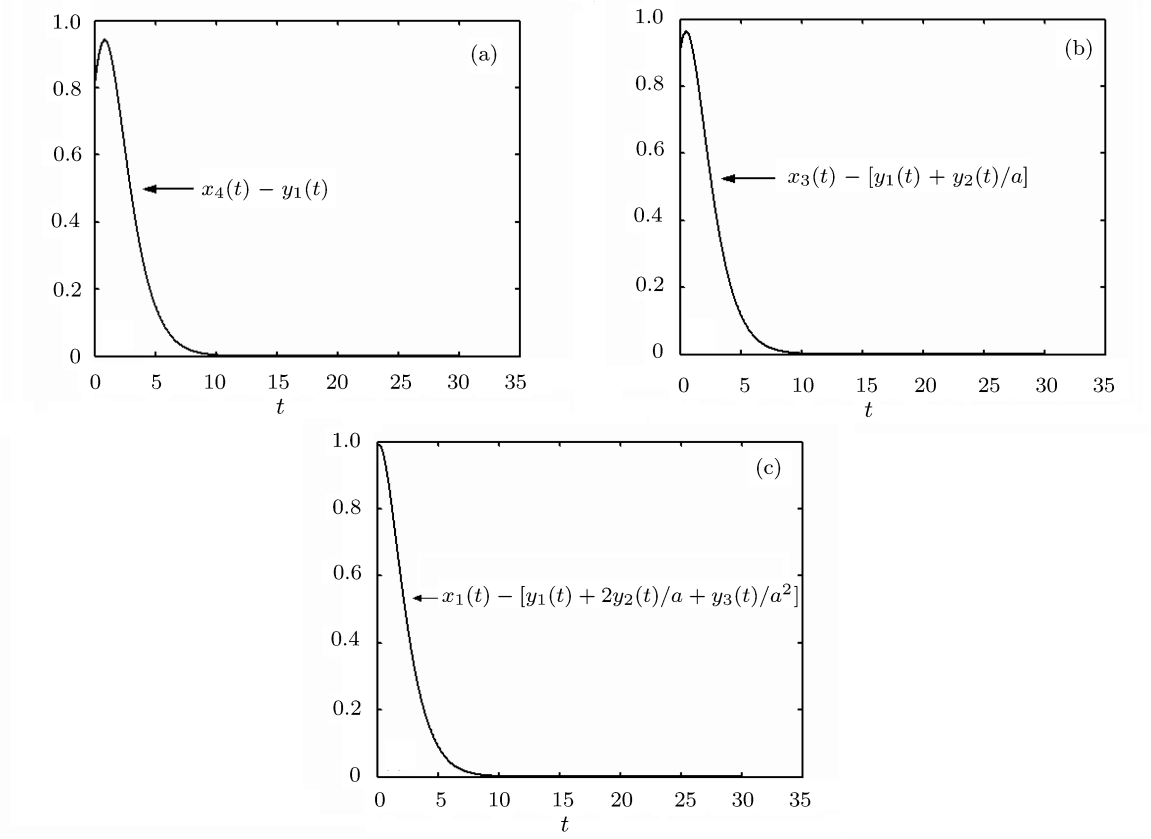


Fig.3. The RGS errors between the partial master system variables (x_1, x_3, x_4) and the response system variables (y_1, y_2, y_3).

6. Conclusions

In this paper, we have investigated the chaos control and synchronization for the Chen–Liao system. By the Lyapunov stability theory, the chaotic trajectories can be exactly led to some specified point, whose coordinates have the relation $x_1 = x_3 = x_4 = \pm\sqrt{bx_2}$. At the same time, a new synchronization mode (RGS) is proposed and achieved between the Chen–Liao system and a class of chaotic system. All the results are proved and validated by rigorous reasoning and detailed numerical simulations, respectively. In practical applications, the less the number of controllers designed in control process, the better the control method. Therefore, our controller has the advantage of efficiency and easy implementation.

Appendix

Proof According to the solution of ordinary differential equations, we can easily obtain the following results for system (6):

$$\begin{aligned} X(t) &= \exp[\Lambda(t-t_0)]c_0 + \int_{t_0}^{t} \exp[\Lambda(t-s)]f(X(s))\mathrm{d}s \\ &= \exp[\Lambda(t-t_0)]c_0 + \int_{t_0}^{t} \exp[\Lambda(t-t_0)+\Lambda(t_0-s)]f(X(s))\,\mathrm{d}s \\ &= \exp[\Lambda(t-t_0)]c_0 + \exp[\Lambda(t-t_0)]\int_{t_0}^{t} \exp[\Lambda(t_0-s)]f(X(s))\,\mathrm{d}s, \end{aligned} \tag{14}$$

where $c_0 = [c_{i0}]_{n\times 1}$ $(i = 1, 2, \ldots, n)$ is the constant vector which is determined by the initial state of system (6).

In consequence of condition (1) in Theorem 1, $\exp[\Lambda(t - t_0)]c_0 \to [0]_{n\times 1}$ as $t \to \infty$. In addition, because of $\lim_{t\to\infty} f(X(t)) = A$, we can deduce that

$$\int_{t_0}^{t} \exp[\Lambda(t - s)]f(X(s))\mathrm{d}s \to \infty$$

as $t \to \infty$ according to the basic calculus method. Then taking the limit of $X(t)$ based on the expression (14), we obtain

$$\lim_{t\to\infty} X(t) = \lim_{t\to\infty} \exp[\Lambda(t - t_0)] \times \int_{t_0}^{t} \exp[\Lambda(t_0 - s)]f(X(s))\,\mathrm{d}s. \quad (15)$$

To find the solution of Eq.(15), first we consider any element of the vector space $X(t)$, denoted by $x_i(t)$ $(i = 1,\ 2,\ \ldots,\ n)$.

Since

$$\exp[\Lambda t] = \sum_{k=0}^{\infty} \frac{\Lambda^k t^k}{k\,!},$$

with Λ being a diagonal matrix, we can easily obtain the following expression:

$$x_i(t) = \mathrm{e}^{a_i(t-t_0)} \int_{t_0}^{t} f_i(X(s))\mathrm{e}^{a_i(t_0-s)}\mathrm{d}s, \quad i = 1,\ 2, \ldots, n. \quad (16)$$

Then for Eq.(16), we can use the L'Hospital rule directly to seek the limit of $x_i(t)$ ($i = 1,\ 2,\ \ldots,\ n$),

$$\begin{aligned}\lim_{t\to\infty} x_i(t) &= \lim_{t\to\infty} \frac{\int_{t_0}^{t} f_i(X(s))\mathrm{e}^{a_i(t_0-s)}\mathrm{d}s}{\mathrm{e}^{a_i(t_0-t)}} \\ &= \lim_{t\to\infty} \frac{f_i(X(s))\mathrm{e}^{a_i(t_0-t)}}{-a_i\mathrm{e}^{a_i(t_0-t)}} \\ &= -\frac{b_i}{a_i},\ \ (\,i = 1,\ 2, \ldots, n).\end{aligned}$$

In accordance with the matrix operation rule, Theorem 1 has been proved.

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