Vol 16 No 6, June 2007
1009-1963/2007/16(06)/1586-05

Chinese Physics

© 2007 Chin. Phys. Soc.
and IOP Publishing Ltd

# Realization of fractional-order Liu chaotic system by circuit

Lu Jun-Jie(逯俊杰) and Liu Chong-Xin(刘崇新)

*Institute of Electrical Engineering, Xi'an Jiaotong University, Xi'an* 710049, *China*

(Received 21 April 2006; revised manuscript received 21 December 2006)

In this paper, chaotic behaviours in the fractional-order Liu system are studied. Based on the approximation theory of fractional-order operator, circuits are designed to simulate the fractional- order Liu system with $q = 0.1 - 0.9$ in a step of 0.1, and an experiment has demonstrated the 2.7-order Liu system. The simulation results prove that the chaos exists indeed in the fractional-order Liu system with an order as low as 0.3. The experimental results prove that the fractional-order chaotic system can be realized by using hardware devices, which lays the foundation for its practical applications.

**Keywords:** fractional-order Liu system, circuit simulation, circuit experiment
**PACC:** 0545

## 1. Introduction

The study of fractional-order chaotic systems has been a focal point in nonlinear science. As is well known, the behaviours of fractional-order chaotic systems have been extensively studied. For example, it is shown that the fractional-order Chua circuit with an appropriate cubic nonlinearity and order as low as 2.7 can produce a chaotic attractor.[1] In Ref.[2], the fractional-order Genesio-Tesi system showing chaotic behaviours is introduced. In Ref.[3], the chaotic behaviours in a fractional-order chaotic electronic oscillator are studied. In Ref.[4], chaotic behaviours of a fractional-order 'jerk' model are studied, and the chaotic control of this fractional-order chaotic system is investigated in Ref.[5]. In Ref.[6], chaos and hyperchaos in fractional-order Rŏssler equations are discussed. In Ref.[7], a critical chaotic system with fractional-order is studied.

In this paper, the chaotic behaviours in the fractional-order Liu system[8] are studied.

## 2. Fractional derivative and its approximation

There have existed a number of definitions of fractional derivative.[9] Of them, the best definition perhaps is the Riemann–Liouville definition, which is given by

$$\frac{\mathrm{d}^\alpha f(t)}{\mathrm{d}t^\alpha} = \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}}\mathrm{d}\tau, \quad (1)$$

where $\Gamma(\cdot)$ is the gamma function, and $n-1 \le \alpha \le n$. Assuming all the initial values to be zero, the Laplace transform of the Riemann–Liouville fractional derivative is

$$L\left\{\frac{\mathrm{d}^\alpha f(t)}{\mathrm{d}t^\alpha}\right\} = s^\alpha L\{f(t)\}. \quad (2)$$

Thus, the fractional integral operator of order "$\alpha$" can be represented by the transfer function $F(s) = 1/s^\alpha$ in the frequency domain.

The standard definition of fractional differintegral does not allow the direct implementation of the fractional operators in time-domain simulations. An efficient method of solving this problem is to approximate the fractional operators by using the standard integer-order operators. In Ref.[10], an effective algorithm is developed to approximate the fractional-order transfer functions. In Table 1 of Ref.[4], approximations of $1/s^\alpha$ with $\alpha = 0.1 - 0.9$ in a step of 0.1 are given with an error of about 2dB. We will utilize these approximations to realize the fractional-order Liu system by using a circuit.

## 3. The fractional-order Liu chaotic system and its circuit realization

In Ref.[8], a new chaotic attractor is presented and we call it the 'Liu chaotic system' for short. Here, we consider the fractional-order Liu system. The standard derivative is replaced by a fractional derivative as follows:

$$\begin{aligned}\frac{\mathrm{d}^q x}{\mathrm{d}t^q} &= a(y-x),\\ \frac{\mathrm{d}^q y}{\mathrm{d}t^q} &= bx - cxz,\\ \frac{\mathrm{d}^q z}{\mathrm{d}t^q} &= dz + hx^2,\end{aligned} \tag{3}$$

where $q$ is the fractional order satisfying $0 < q \leq 1$, and $a, b, c, d$ and $h$ are system parameters. When $q = 1$, system (3) is the original integer-order Liu chaotic system, which is chaotic when $a = 10, b = 40, c = 1, d = -2.5$ and $h = 4$. In this paper, we take $c = 10$ and $q = 0.9$ in circuit design.

From Table 1 in Ref.[4], we can obtain an approximation of $1/s^{0.9}$ with an error of about 2 dB as follows:

$$\frac{1}{S^{0.9}} \approx \frac{2.2675(s+1.292)(s+215.4)}{(s+0.01292)(s+2.154)(s+359.4)}. \tag{4}$$

We utilize the circuit unit as shown in Fig.1 to implement Eq.(4) in the Laplace domain.

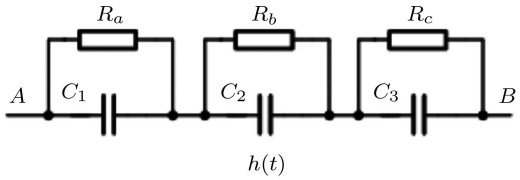



**Fig.1.** The circuit unit to realize $1/s^{0.9}$.

Thus, we can obtain the transfer function $H(s)$ between $A$ and $B$ as follows:

$$\begin{aligned}H(s) &= \frac{1/C_1}{s+1/R_aC_1} + \frac{1/C_2}{s+1/R_bC_2} + \frac{1/C_3}{s+1/R_cC_3}\\ &= \frac{1}{C_0}\frac{\left(\frac{C_0}{C_1}+\frac{C_0}{C_2}+\frac{C_0}{C_3}\right)\left[s^2 + \frac{\left(\frac{C_2+C_3}{R_a}+\frac{C_1+C_3}{R_b}+\frac{C_1+C_2}{R_c}\right)s+\frac{(R_a+R_b+R_c)}{R_aR_bR_c}}{(C_1C_2+C_2C_3+C_1C_3)}\right]}{(s+1/R_aC_1)(s+1/R_bC_2)(s+1/R_cC_3)},\end{aligned} \tag{5}$$

where $C_0$ is a unit parameter. Let $C_0 = 1\mu$F and $F(s) = H(s)\cdot C_0 = 1/s^{0.9}$. Compare $F(s)$ with Eq.(4) and we can obtain $R_a = 62.84\,\mathrm{M\Omega}, R_b = 250\,\mathrm{k\Omega}, R_c = 2.5\,\mathrm{k\Omega}, R_c = 2.5\,\mathrm{k\Omega}, C_1 = 1.232\,\mu\mathrm{F}$, $C_2 = 1.835\,\mu\mathrm{F}$ and $C_3 = 1.1\,\mu\mathrm{F}$.

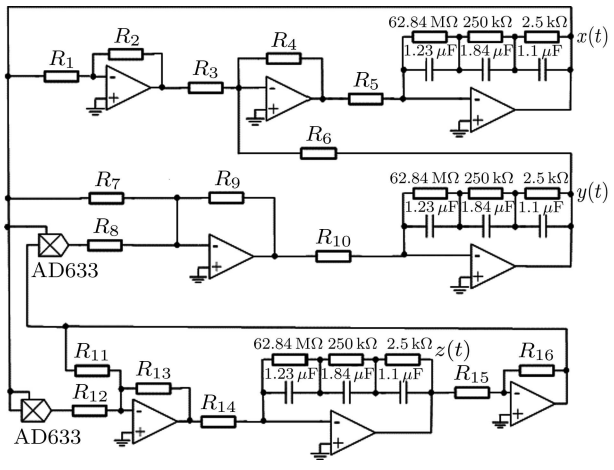



**Fig.2.** Circuit to realize the 2.7-order Liu system.

Then, a circuit is designed to realize the 2.7-order Liu system as shown in Fig.2, where AD633 is the multiplier with an output coefficient of 0.1 and LM741 is the operational amplifier.

From the circuit as shown in Fig.2, we can acquire the 2.7-order Liu system equation in the Laplace domain as follows:

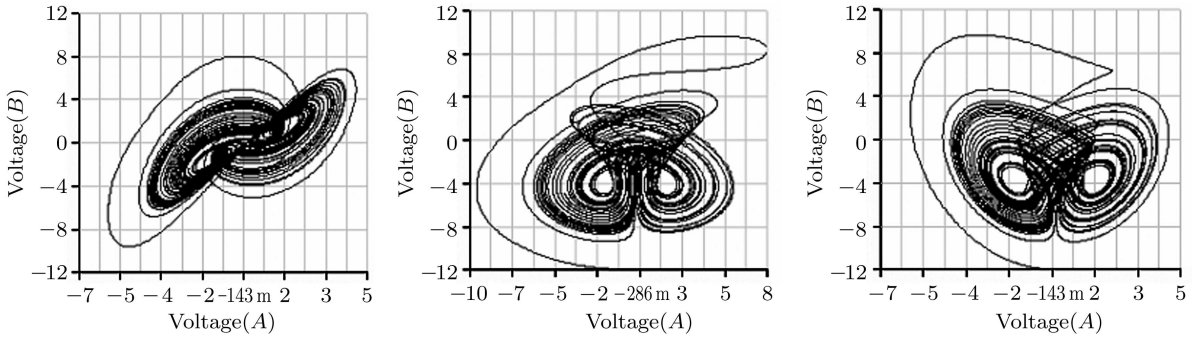
$$\begin{aligned}\frac{X(s)}{F(s)} &= \frac{R_4}{C_0R_5}\left[\frac{Y(s)}{R_6} - \frac{R_2X(s)}{R_1R_3}\right],\\ \frac{Y(s)}{F(s)} &= \frac{R_9}{C_0R_{10}}\left[\frac{X(s)}{R_7} - \frac{kR_{16}X(s)Z(s)}{R_8R_{15}}\right],\\ \frac{Z(s)}{F(s)} &= \frac{R_{13}}{C_0R_{14}}\left[-\frac{R_{16}Z(s)}{R_{11}R_{15}} + \frac{kX(s)X(s)}{R_{12}}\right],\end{aligned} \tag{6}$$

where $k$ is the AD633's output coefficient, and here $k = 0.1$. Let $R_8 = 4\mathrm{k\Omega}$; $R_{12} = 2.5\mathrm{k\Omega}$; $R_1$, $R_2$, $R_3$, $R_4$, $R_6$, $R_7$, $R_{13}$, $R_{15}$ and $R_{16}$ each equal $10\mathrm{k\Omega}$; $R_5, R_{10}$ and $R_{14}$ each equal $100\mathrm{k\Omega}$; $R_9$ and $R_{11}$ each equal $40\mathrm{k\Omega}$, because of $F(s) = 1/s^{0.9}$, then the frequency form of system (6) will be obtained as fol-

lows:

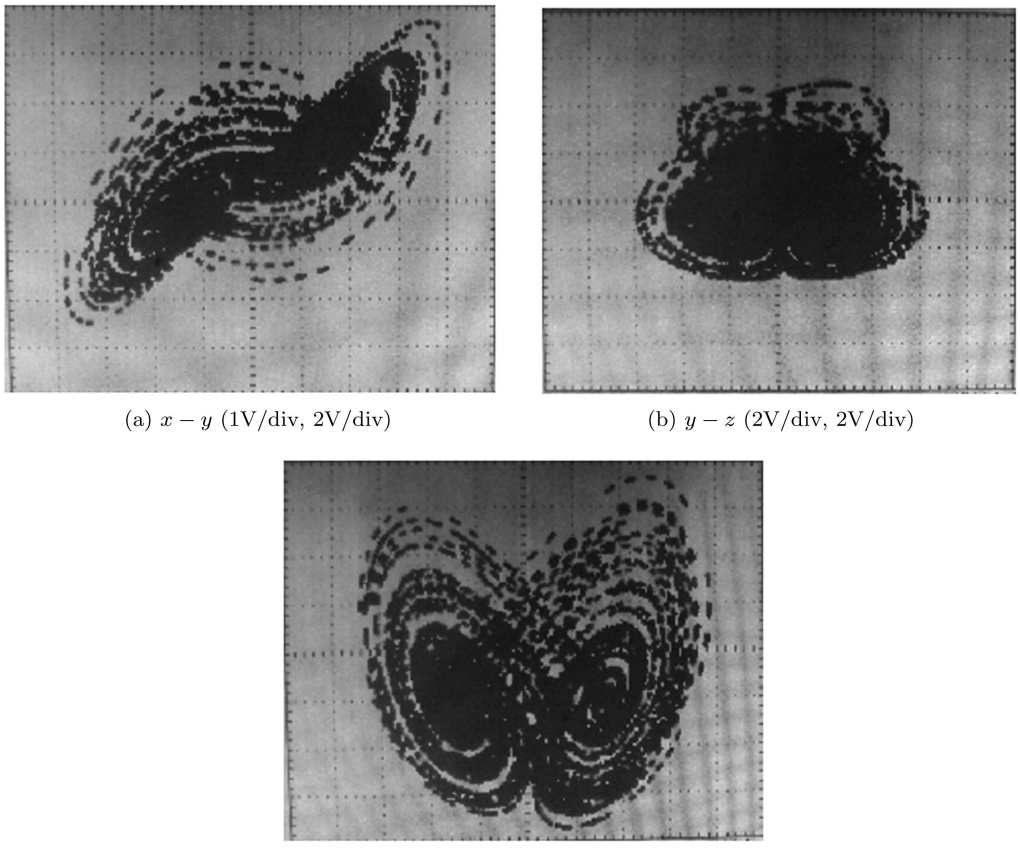
$$\begin{aligned}\frac{\mathrm{d}^{0.9}x}{\mathrm{d}t^{0.9}} &= 10(y-x),\\ \frac{\mathrm{d}^{0.9}y}{\mathrm{d}t^{0.9}} &= 40x-10xz,\\ \frac{\mathrm{d}^{0.9}z}{\mathrm{d}t^{0.9}} &= -2.5z+4x^2.\end{aligned} \tag{7}$$

Utilizing electronic workbench (EWB) to simulate system (7), we obtain its phase portraits as shown in Fig.3. Considering that the practical values of resistances and capacitances are not exactly the same as those in Fig.1, we choose $R_a = 66\mathrm{M}\Omega$, $R_b = 250\mathrm{k}\Omega$, $R_c = 2.5\mathrm{k}\Omega$, $C_1 = 1.2\mu\mathrm{F}$, $C_2 = 2\mu\mathrm{F}$ and $C_3 = 1.1\mu\mathrm{F}$ to do the hardware circuit experiment, and obtain its phase portraits as shown in Fig.4.




(a) $x-y$ (b) $y-z$ (c) $x-z$

**Fig.3.** Chaotic attractor of the 2.7-order Liu system simulated by EWB.



(a) $x-y$ (1V/div, 2V/div) (b) $y-z$ (2V/div, 2V/div)

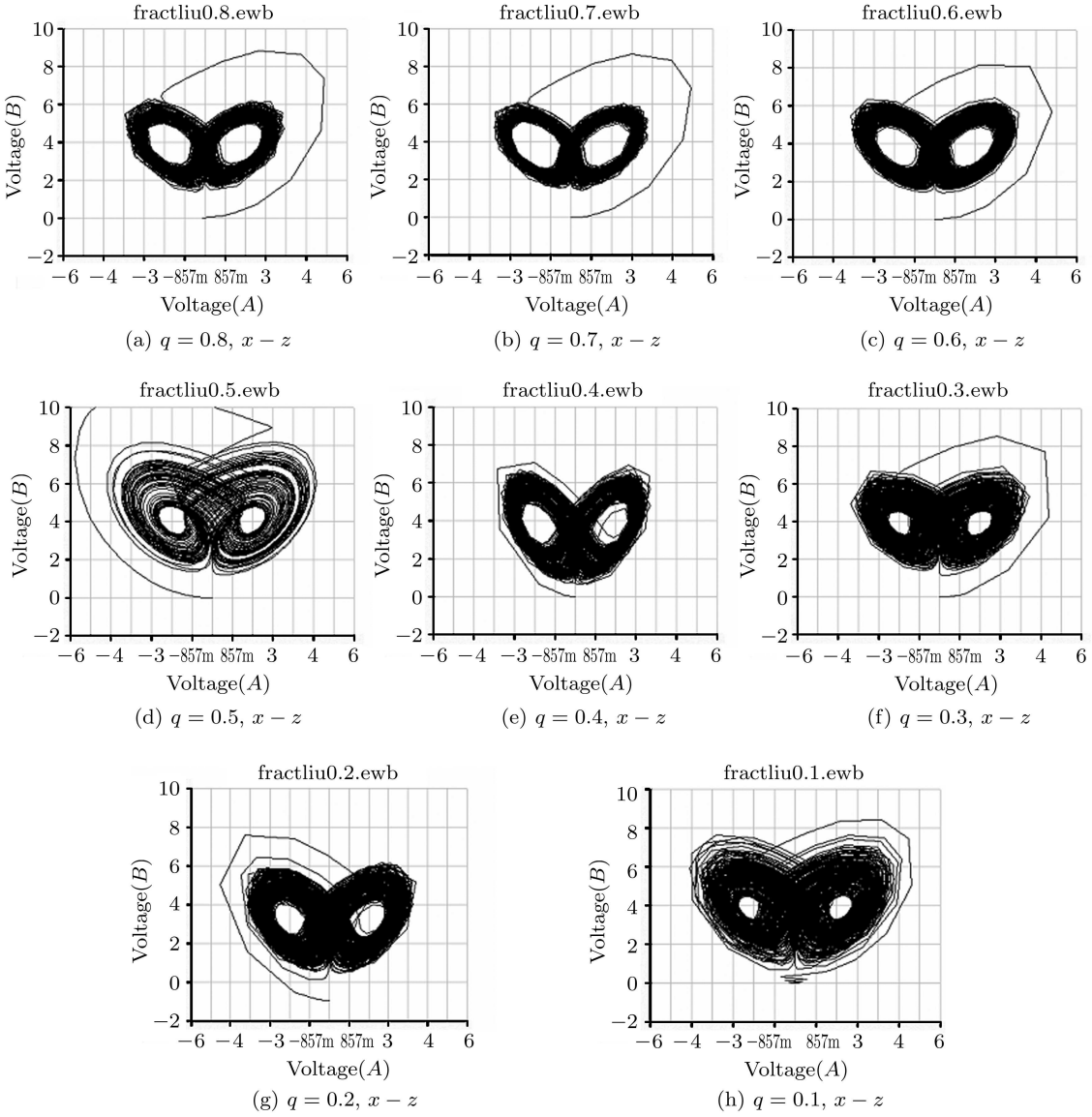
(c) $x-z$ (1V/div, 1V/div)

**Fig.4.** Chaotic attractor of the 2.7-order Liu system realized in experiment.

## 4. Other fractional-order Liu systems simulated by EWB

Using the same method, other circuits are designed to realize other fractional-order Liu systems, with $q$ ranging from 0.8 down to 0.1. Their phase portraits are shown in Fig.5.

From Fig.5, we can conclude that the chaos exists in the fractional-order Liu system with an order as low as 0.3, and Liu system has abundant chaotic dynamical behaviours.




(a) $q = 0.8$, $x - z$ (b) $q = 0.7$, $x - z$ (c) $q = 0.6$, $x - z$

(d) $q = 0.5$, $x - z$ (e) $q = 0.4$, $x - z$ (f) $q = 0.3$, $x - z$

(g) $q = 0.2$, $x - z$ (h) $q = 0.1$, $x - z$

**Fig.5.** Chaotic attractors of fractional-order Liu systems with $q$ ranging from 0.8 to 0.1.

## 5. Conclusions

By utilizing frequency domain techniques based on Bode diagrams, one can obtain a linear approximation of the fractional-order operator. Based on the approach, Ref.[4] presented an approximation of $1/S^{\alpha}$ with $\alpha = 0.1 - 0.9$ in a step of 0.1 with errors of about 2dB and 3dB. We have utilized the approximation of $1/S^{\alpha}$ to design a circuit, and realized the fractional-order Liu chaotic system. The simulation results prove that the chaos exists indeed in the fractional-order Liu system with an order as low as 0.3, and the circuit experiment proves that we can realize the fractional-order chaotic system by using hardware devices and lays the foundation for its practical applications.

# References


[1] Hartley TT, Lorenzo C F and Qammer H K 1995 *IEEE Trans CAS-I* **42** 485

[2] Wu Z M, Lu J G and Xie J Y 2006 *Chin. Phys.* **15** 1201

[3] Gao X and Yu J B 2005 *Chin. Phys.* **14** 908

[4] Ahmad W M and Sprott J C 2003 *Chaos, Solitons and Fractals* **16** 339

[5] Ahmad W M and Harb W M 2003 *Chaos, Solitons and Fractals* **18** 693

[6] Li C G and Chen G 2004 *Physica* A **341** 55

[7] Wang F Q and Liu C X 2006 *Acta Phys. Sin.* **55** 3922 (in Chinese)

[8] Liu C X, Liu T, Liu L and Liu K 2004 *Chaos, Solitons and Fractals* **22** 1031

[9] Podlubny I 1999 (New York: Academic)

[10] Charef A, Sun H H, Tsao Y Y and Onaral B 1992 *IEEE Trans. Auto. Contr.* **37** 9