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Wigner function of coherent state of N components*

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In this paper, we study the Wigner function of coherent state of N components, especially two components and three components. This function consists of two terms: the Gaussian term and the interference term with the negativity. The first term comprises N Gaussian surfaces evenly centred on a circle of radius $|\beta| = |\alpha|$ with a separate angle of $2\pi/N$, and the second term is composed of $\frac{1}{2}N(N-1)$ Gaussian-cosine surfaces evenly centred in a circular region of radius $|\beta| < |\alpha|$. Here, α is the eigenvalue of the annihilation operator a, and β is a variable in some complex space in which the Wigner function is defined. We have proved that the essential condition to eliminate the negativity of the Wigner function is that the mean photon count of the coherent state is equal to that of the Glouber coherent state.

Keywords: multi-component coherent states, Wigner function, non-classicality
PACC: 0365, 4250

1. Introduction

Since Glouber introduced the concept of coherent state, there has been deep and extensive study of the coherent state in its own right and its applications in physics.[1] The coherent state of two components, namely the even and odd coherent states for harmonic oscillators, and later the so-called 'Schrödinger cat' states,[2] have been extensively studied in recent years because of their wide applications in quantum optics.[3−7] So far, however, multi-component coherent states have been little studied. The coherent state of N components constructed in this paper is a generalization of the Glouber coherent state (when $N = 1$) and even–odd coherent states (when $N = 2$). Therefore, the study on the coherent state of N components may have a universal significance. All the components of the coherent state are orthogonal to each other and they themselves form a complete set. Combining the orthogonality with the non-classicality of multi-component coherent states, we believe that multi-component coherent states may have important applications in quantum information processing.

The Wigner function is an important representation of electromagnetic field. A very convenient way to describe the quantum state is to utilize the Wigner function. The Wigner function is analogous to a classical probability distribution, but it can take negative values, which is regarded as a signature of quantum interference. The Wigner function of the Glouber coherent state is a Gaussian surface function, and is positive in any part of complex space. The coherent state is of minimum-uncertainty. Because of this property, it may be regarded as a classical state. The positivity of Wigner function is closely related to its classicality.

The aim of this paper is to study the Wigner function of the coherent state of N components and its negativity. We first calculate the Wigner function of the coherent state of N components, and then prove that the negativity of the Wigner function may be eliminated under a certain condition. Finally, we give a brief discussion on results.

2. Wigner function of coherent state of N components

Let a^+ and a denote the creation operator and annihilation operator of electromagnetic field respectively. They satisfy the commutation relation $[a, a^+] = 1$. According to Ref.[8], the coherent state of N components may be defined as the eigenstate $|\alpha\rangle_k (k = 1, 2, \ldots, N)$ of a^N with the eigenvalue α^N: $a^N|\alpha\rangle_k = \alpha^N|\alpha\rangle_k$, where $|\alpha\rangle_k$ is expressed as

$$|\alpha\rangle_k = N_k^{-1/2} \sum_{n=0}^{\infty} \frac{\alpha^{Nn+k-1}}{\sqrt{(Nn+k-1)!}} \times |Nn+k-1\rangle, \quad (1)$$

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$N_k^{-1/2}$ is a normalized constant, and N_k is expressed as follows:

$$N_k = \sum_{n=0}^{\infty} \frac{|\alpha|^{2(Nn+k-1)}}{(Nn+k-1)!}. \tag{2}$$

All the components of the coherent state are orthogonal to each other, i.e. ${}_l\langle\alpha|\alpha'\rangle_k = 0$ $(k \neq l)$, and satisfy the circulation relation $a\|\alpha\rangle_k = \alpha\|\alpha\rangle_{k-1} (k = 2, 3, \ldots, N), a\|\alpha\rangle_1 = \alpha\|\alpha\rangle_N$, where $\|\alpha\rangle_k = N_k^{1/2}|\alpha\rangle_k (k = 1, 2, \ldots, N)$.

Following its definition in Refs.[9–11], the Wigner function is expressed as

$$W(\beta) = \frac{1}{\pi^2} \int \mathrm{Tr}\{\rho \mathrm{e}^{\eta a^+ - \eta^* a}\} \mathrm{e}^{\eta^* \beta - \eta \beta^*} \mathrm{d}^2\eta, \tag{3}$$

where ρ is the density operator of the field; both η and β are complex variables. This function can be interpreted as a pseudo-probability distribution, and satisfies the normalization condition $\int W(\beta)\mathrm{d}^2\beta = 1$.

We now calculate the Wigner function of the coherent state of N components. When $N = 1$, the coherent state is of one component, namely the Glouber coherent state

$$|\alpha\rangle = \mathrm{e}^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle. \tag{4}$$

Its Wigner function is known as

$$W^{(1)}(\beta) = \frac{2}{\pi} e^{-2|\beta-\alpha|^2},$$

which is a Gaussian surface, a round hill centred at α in complex space. For the Glouber coherent state, the mean photon count reads $\overline{n^{(1)}} = |\alpha|^2$.

When $N = 2$ and $k = 1, 2$, the coherent state is of two components. This is just what we call even-odd coherent states

$$\begin{aligned} |\alpha\rangle_1 &= N_1^{-1/2} \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{\sqrt{(2n)!}} |2n\rangle, \\ |\alpha\rangle_2 &= N_2^{-1/2} \sum_{n=0}^{\infty} \frac{\alpha^{2n+1}}{\sqrt{(2n+1)!}} |2n+1\rangle, \end{aligned} \tag{6}$$

where $N_1 = \cosh|\alpha|^2$ and $N_2 = \sinh|\alpha|^2$ are normalized constants.

For a quantum state including even-odd coherent states, its density operator may be expressed as

$$\rho = P_1|\alpha\rangle_{11}\langle\alpha| + P_2|\alpha\rangle_{22}\langle\alpha|, \tag{7}$$

where P_1 and P_2 are the probabilities of the field being in $|\alpha\rangle_1$ and $|\alpha\rangle_2$ respectively, and satisfy $P_1 + P_2 = 1$. The mean photon count and the variance are

$$\begin{aligned} \overline{n^{(2)}} &= K|\alpha|^2, \\ \overline{(\Delta n^{(2)})^2} &= (1 - K^2)|\alpha|^4 + K|\alpha|^2, \end{aligned} \tag{8}$$

where $K = P_1(N_2/N_1) + P_2(N_1/N_2)$.

Using the definition (3) and expressions $a|\alpha\rangle_2 = \alpha(\coth|\alpha|^2)^{1/2}|\alpha\rangle_1$ and $a|\alpha\rangle_1 = \alpha(\tanh|\alpha|^2)^{1/2}|\alpha\rangle_2$, the Wigner function of the coherent state can be obtained as

$$W^{(2)}(\beta) = W_{\mathrm{G}}^{(2)}(\beta) + W_{\mathrm{I}}^{(2)}(\beta), \tag{9}$$

where

$$\begin{aligned} W_{\mathrm{G}}^{(2)}(\beta) = \frac{1}{2\pi}(1+K)(e^{-2|\beta-\alpha|^2} \\ + \mathrm{e}^{-2|\beta+\alpha|^2}), \end{aligned} \tag{10}$$

$$\begin{aligned} W_{\mathrm{I}}^{(2)}(\beta) = \frac{1}{\pi}(1-K)\mathrm{e}^{-2(|\beta|^2-|\alpha|^2)} \\ \times \cos 4(\alpha_1\beta_2 - \alpha_2\beta_1). \end{aligned} \tag{11}$$

Here we have set $\alpha = \alpha_1 + \mathrm{i}\alpha_2$, and $\beta = \beta_1 + \mathrm{i}\beta_2$. As expected, this function consists of two terms: the first term $W_{\mathrm{G}}^{(2)}(\beta)$, which comprises two Gaussian surfaces centred at $-\alpha$ and α respectively, similar to those of the Glouber coherent state, representing each state of the coherent state, and the second term $W_{\mathrm{I}}^{(2)}(\beta)$, which is composed of the quantum interference between these two states and may be regarded as a Gaussian function multiplied by a cosine function, i.e. a Gaussian-cosine surface with the negativity.

Figure 1 shows the variations of the Wigner function $W^{(2)}(\beta)$ with $|\alpha|$ for several values of separation $|\alpha|$ when the following values are taken: $P_1 = 0.3$, and $P_2 = 0.7$. One can clearly see the formation of the two round hills associated with each state of the coherent state and the quantum interference structure with negative regions at the halfway between the two hills as the separation value $|\alpha|$ increases. Figure 2 is contour plots corresponding to Fig.1, and clearly exhibits the distribution of the negativities of $W^{(2)}(\beta)$.

When $N = 3$, and $k = 1, 2, 3$, we obtain the coherent states of three components

$$\begin{aligned} |\alpha\rangle_1 &= N_1^{-1/2} \sum_{n=0}^{\infty} \frac{\alpha^{3n}}{\sqrt{(3n)!}} |3n\rangle, \\ |\alpha\rangle_2 &= N_2^{-1/2} \sum_{n=0}^{\infty} \frac{\alpha^{3n+1}}{\sqrt{(3n+1)!}} |3n+1\rangle, \\ |\alpha\rangle_3 &= N_3^{-1/2} \sum_{n=0}^{\infty} \frac{\alpha^{3n+2}}{\sqrt{(3n+2)!}} |3n+2\rangle, \end{aligned} \tag{12}$$

where the normalized constants are

$$\begin{aligned} N_1 &= \frac{1}{3}\Big[\mathrm{e}^{|\alpha|^2} + 2\mathrm{e}^{-\frac{1}{2}|\alpha|^2} \cos\Big(\frac{\sqrt{3}}{2}|\alpha|^2\Big)\Big], \\ N_2 &= \frac{1}{3}\Big[\mathrm{e}^{|\alpha|^2} - 2\mathrm{e}^{-\frac{1}{2}|\alpha|^2} \cos\Big(\frac{\sqrt{3}}{2}|\alpha|^2 + \frac{\pi}{3}\Big)\Big], \\ N_3 &= \frac{1}{3}\Big[\mathrm{e}^{|\alpha|^2} - 2\mathrm{e}^{-\frac{1}{2}|\alpha|^2} \cos\Big(\frac{\sqrt{3}}{2}|\alpha|^2 - \frac{\pi}{3}\Big)\Big]. \end{aligned} \tag{13}$$

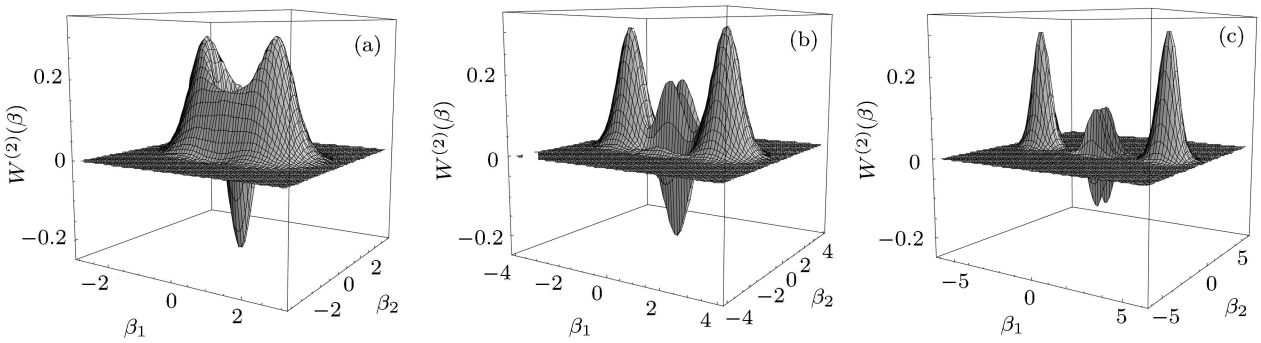


Fig.1. The variations of $W^{(2)}(\beta)$ with $|\alpha|$: (a) $|\alpha|=1$, (b) $|\alpha|=2$, (c) $|\alpha|=4$.

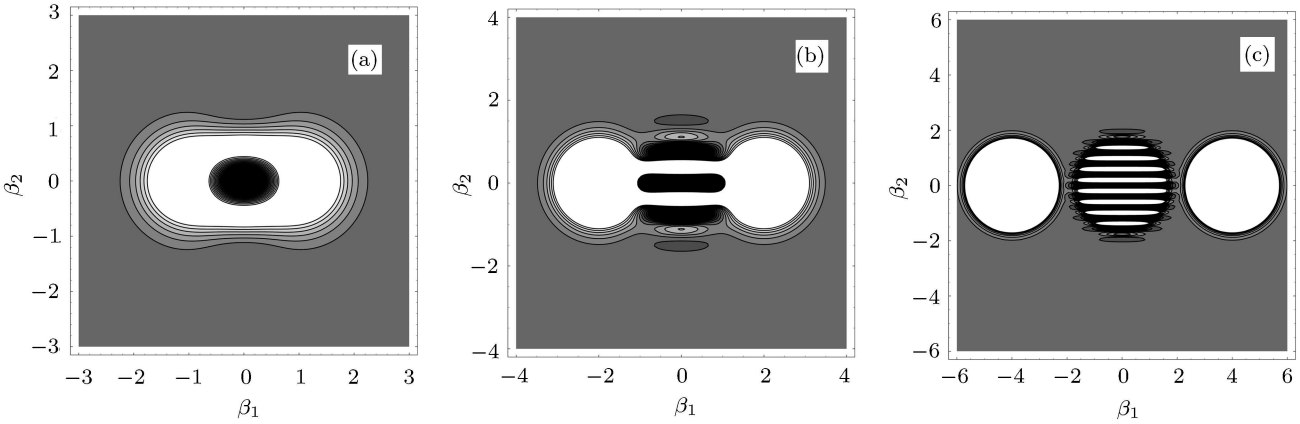


Fig.2. Contour plots corresponding to Fig.1.

For a quantum state including the coherent states of three components, its density operator may be expressed as

$$\rho = P_1|\alpha\rangle_{11}\langle\alpha| + P_2|\alpha\rangle_{22}\langle\alpha| + P_3|\alpha\rangle_{33}\langle\alpha|, \tag{14}$$

where $P_i (i=1,2,3)$ are the probabilities at which the field appears in the state $|\alpha\rangle_i$ correspondingly, and satisfy $P_1+P_2+P_3=1$. For this quantum state, the mean photon count and the variance are

$$\overline{n^{(3)}} = K_1|\alpha|^2 \tag{15}$$

and

$$\overline{(\Delta n^{(3)})^2} = (K_2 - K_1^2)|\alpha|^4 + K_1|\alpha|^2 \tag{16}$$

respectively, where

$$K_1 = P_1(N_3/N_1) + P_2\frac{N_1}{N_2} + P_3(N_2/N_3),$$

$$K_2 = P_1(N_2/N_1) + P_2\frac{N_3}{N_2} + P_3(N_1/N_3).$$

Using Eq.(3) and expressions

$$a|\alpha\rangle_3 = \alpha(N_2/N_3)^{1/2}|\alpha\rangle_2, \quad a|\alpha\rangle_2 = \alpha(N_1/N_2)^{1/2}|\alpha\rangle_1$$

and $a|\alpha\rangle_1 = \alpha(N_3/N_1)^{1/2}|\alpha\rangle_3$, the Wigner function of the coherent states of three components can be obtained as follows:

$$W^{(3)}(\beta) = W_{\mathrm{G}}^{(3)}(\beta) + W_{\mathrm{I}}^{(3)}(\beta), \tag{17}$$

where

$$W_{\mathrm{G}}^{(3)}(\beta) = \frac{2}{9\pi}(1+K_1+K_2)(\mathrm{e}^{-2|\beta-\alpha|^2} + \mathrm{e}^{-2|\beta-\mathrm{e}^{\mathrm{i}4\pi/3}\alpha|^2} + \mathrm{e}^{-2|\beta-\mathrm{e}^{\mathrm{i}2\pi/3}\alpha|^2}), \tag{18}$$

$$W_{\mathrm{I}}^{(3)}(\beta) = \frac{4}{9\pi}\Big\{\mathrm{e}^{-2|\beta+\frac{1}{2}\alpha|^2+\frac{3}{2}|\alpha|^2}\Big[\Big(1-\frac{1}{2}K_1-\frac{1}{2}K_2\Big)\cos\theta_1 + \frac{\sqrt{3}}{2}(K_2-K_1)\sin\theta_1\Big]$$

$$+\mathrm{e}^{-2|\beta+\frac{1}{2}\mathrm{e}^{\mathrm{i}4\pi/3}\alpha|^2+\frac{3}{2}|\alpha|^2}\left[\left(1-\frac{1}{2}K_1-\frac{1}{2}K_2\right)\cos\theta_2+\frac{\sqrt{3}}{2}(K_2-K_1)\sin\theta_2\right]$$

$$+\mathrm{e}^{-2|\beta+\frac{1}{2}\mathrm{e}^{\mathrm{i}2\pi/3}\alpha|^2+\frac{3}{2}|\alpha|^2}\left[\left(1-\frac{1}{2}K_1-\frac{1}{2}K_2\right)\cos\theta_3+\frac{\sqrt{3}}{2}(K_2-K_1)\sin\theta_3\right]\Big\}, \tag{19}$$

with $\theta_1=\sqrt{3}(|\beta+\alpha|^2-|\beta|^2)$, $\theta_2=\sqrt{3}(|\beta+\mathrm{e}^{\mathrm{i}4\pi/3}\alpha|^2-|\beta|^2)$, $\theta_3=\sqrt{3}(|\beta+\mathrm{e}^{\mathrm{i}2\pi/3}\alpha|^2-|\beta|^2)$. As expected, this function consists of two terms: the Gaussians term $W_{\mathrm{G}}^{(3)}(\beta)$ and the interference term $W_{\mathrm{I}}^{(3)}(\beta)$. $W_{\mathrm{G}}^{(3)}(\beta)$ comprises three Gaussian surfaces centred at the points $\mathrm{e}^{\mathrm{i}2\pi/3}\alpha$, $\mathrm{e}^{\mathrm{i}4\pi/3}\alpha$ and α in complex space respectively. They are evenly distributed on a circle of radius $|\beta|=|\alpha|$ with a separate angle of $2\pi/3$. $W_{\mathrm{I}}^{(3)}(\beta)$ is composed of three Gaussian-cosine surfaces, of which the centres are all on the same circle, and it exhibits the quantum interference among the coherent states.

Figure 3 shows the variations of Wigner function $W^{(3)}(\beta)$ of the coherent state of three components with $|\alpha|$ for several values of the separation $|\alpha|$ when the following values are chosen: $P_1=0.3$, $P_2=0.6$, and $P_3=0.1$. We see three round hills each associated with one of the coherent states, a finely structured interference pattern with a negativity region surrounded by the three round hills (Fig.3(a), and such patterns each located between two of the three round hills (Figs.3(b) and 3(c)). Figure 4 shows the contour plots corresponding to Fig.3, and clearly exhibits the distribution of the negativities of $W^{(3)}(\beta)$.

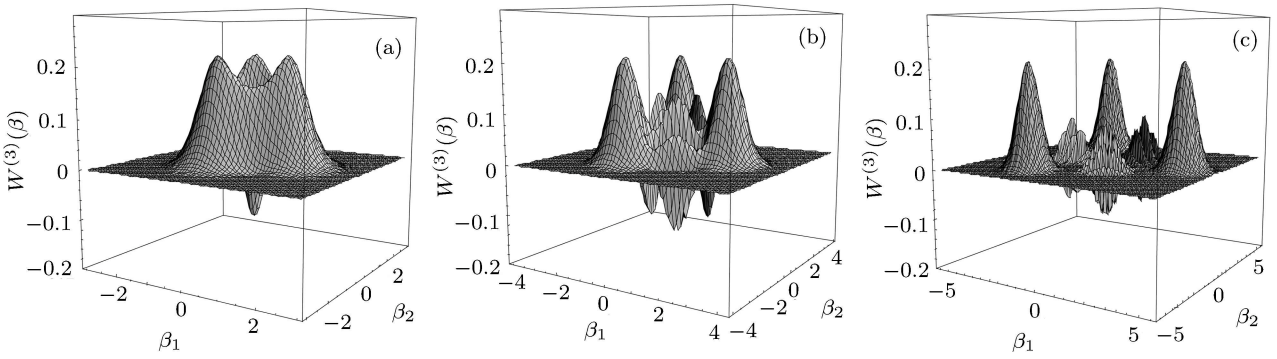


Fig.3. The variations of $W^{(3)}(\beta)$ with $|\alpha|$: (*a*) $|\alpha|=1$, (*b*) $|\alpha|=2$, (*c*) $|\alpha|=4$.

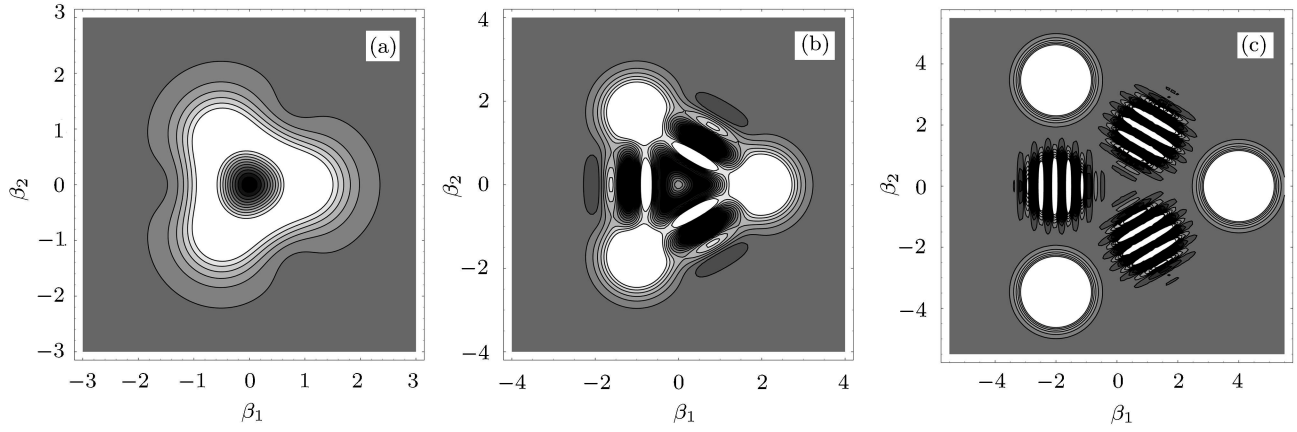


Fig.4. Contour plots corresponding to Fig.3.

For the coherent state of N components, we can obtain the same results. the Wigner function consists of two terms: the Gaussians term and the interference term, i.e.

$$W^{(N)}(\beta) = W_{\mathrm{G}}^{(N)}(\beta) + W_{\mathrm{I}}^{(N)}(\beta). \qquad (20)$$

$W_{\mathrm{G}}^{(N)}(\beta)$ comprises N Gaussian surfaces centred at $\mathrm{e}^{\mathrm{i}2\pi/N}\alpha$, $\mathrm{e}^{\mathrm{i}4\pi/N}\alpha$, ..., $\mathrm{e}^{\mathrm{i}2k\pi/N}\alpha$, ..., α respectively. They are evenly distributed on the same circle of radius $|\beta| = |\alpha|$ with a separate angle of $2\pi/N$. $W_{\mathrm{I}}^{(N)}(\beta)$ is composed of $\frac{1}{2}N(N-1)$ Gaussian-cosine surfaces centred in a circular region of radius $|\beta| < |\alpha|$, where a complicated interference pattern with negative regions appears.

3. Elimination of the negativity of Wigner function

From the above, we naturally ask in what condition the negativity of the Wigner function may be eliminated. One can see from expression (11) that when $1 - K = 0$, then

$$\overline{n^{(2)}} = \overline{n^{(1)}} = |\alpha|^2, \quad \overline{(\Delta n^{(2)})^2} = |\alpha|^2, \qquad (21)$$

thus, the negativity of the Wigner function may be eliminated. Expression (21) means that the mean photon count for the coherent state of two components is equal to that for the Glouber coherent state, and the variance is the same as that of the Glouber coherent state.

From expression (19), one can see that when $1 - \frac{1}{2}K_1 - \frac{1}{2}K_2 = 0$, and $K_2 - K_1 = 0$, then

$$\overline{n^{(3)}} = \overline{n^{(1)}} = |\alpha|^2, \quad \overline{(\Delta n^{(3)})^2} = |\alpha|^2, \qquad (22)$$

thus, the negativity of the Wigner function may be eliminated. Expression (22) means that the mean photon count for the coherent states of three components is equal to that for the Glouber coherent state, and the variance is also the same as that of the Glouber coherent state.

4. Summary

The Wigner function of coherent state of N components consists of two terms: the Gaussians term and the interference term with the negativity. The first term comprises N Gaussian surfaces evenly centred on a circle of radius $|\beta| = |\alpha|$ with a separate angle of $2\pi/N$ and the second term is composed of $\frac{1}{2}N(N-1)$ Gaussian-cosine surfaces centred in a circular region of radius $|\beta| < |\alpha|$, where the Wigner function exhibits a highly non-classical feature: a finely structured interference pattern with the negativity. We call this circular region the non-classical circle. As shown in Figs.1–4, the Wigner function changes its structure with $|\alpha|$. As $|\alpha|$ increases, the interference patterns gradually separate from the Gaussian surfaces, and the frequency of the interference structure increases, while the degree of non-classical feature decreases. The essential condition for eliminating the negativity of the Wigner function is that the mean photon count for the coherent state is equal to that of the Glouber coherent state. If $N \geq 3$, some other conditions may be required. The non-classical feature of multi-component coherent states such as squeezing and antibunching will be studied in the future.

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