Vol 16 No 6, June 2007
1009-1963/2007/16(06)/1545-04
Chinese Physics
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and IOP Publishing Ltd

Exact periodic solution in coupled nonlinear Schrödinger equations

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(Received 9 April 2006; revised manuscript received 12 October 2006)

The coupled nonlinear Schrödinger equations (CNLSEs) of two symmetrical optical fibres are nonintegrable, however the transformed CNLSEs have integrability. Integrability of the transformed CNLSEs is proved by the Hamilton dynamics theory and Galilei transform. Making use of a transform for CNLSEs and using the ansatz with Jacobi elliptic function form, this paper obtains the exact optical pulse solutions.

Keywords: dual core fibre, the coupled nonlinear Schrödinger equations, elliptic function
PACC: 0340K, 0220

1. Introduction

The optical fibre transmission network can support high-speed communication because of tremendous bandwidth of the optical fibre. The optical soliton has the characteristic of a particle; neglecting the loss of the optical fibre, the wave form and energy of the optical soliton will remain intact during the process of transmission, thus the optical soliton communication technology is a kind of effective technology to realize high-speed communication. In recent years, research on optical soliton technology has received wide attention.[1−9] For the optical soliton transmission technology to become practical, people must develop photonic devices which are applicable for the all-optical exchanging and switching in high-speed communication. Because the response speed of a device in the electric domain limits the information processing speed, it further limits the capacity and speed of communication. But the optical fibre coupler based on nonlinear effect has extremely fast information processing capability, thus it is a candidate device for all-optical communication, and people have carried out a large amount of research on it.[10−13] The nonlinear response has the characteristic of fast speed, thus if there is a nonlinear effect in optical fibre cores, the optical fibre coupler can be used as the all-optical switching device with high speed.

The nonlinear optical fibre coupler is regarded as a directional coupler, where there are soliton pairs whose form and energy remain intact, thus the coupled nonlinear Schrödinger equations (CNLSEs) of two cores should have accurate soliton solution. But in general, the CNLSEs of two symmetrical cores in an optical fibre coupler are nonintegrable; thus far, the analytical solution has not been obtained in the literature; numerical solutions and approximate solutions have been obtained in correlated literature.[10−15]

In this paper, after performing transform for the CNLSEs in the optical fibre coupler, we identify that the system of the transformed CNLSEs is a Hamilton system with conservation quantities, and it does not change in Galilei transform. Using the separation variables method for the transformed CNLSEs, we find their exact solutions, and finally obtain the exact solutions of the CNLSEs in optical fibre coupler with two symmetrical cores.

2. Theory

In the dual core of coupler, the propagating equations in optical fibre coupler with two symmetrical cores are[16]

$$\begin{aligned} \mathrm{i}\frac{\partial u}{\partial \xi} - \frac{s}{2}\frac{\partial^2 u}{\partial \tau^2} + |u|^2 u + \sigma v = 0, \\ \mathrm{i}\frac{\partial v}{\partial \xi} - \frac{s}{2}\frac{\partial^2 v}{\partial \tau^2} + |v|^2 v + \sigma u = 0, \end{aligned} \tag{1}$$

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where $s = \mathrm{sgn}(\beta_2) = \pm 1$, β_2 is the second-order dispersion coefficient, and $\sigma = \kappa L_{\mathrm{D}}$ is the zero-order coupled coefficient, other normalized quantities are as follows:

$$\xi = z/L_{\mathrm{D}}, \quad \tau = t/T_0,$$
$$u = (\gamma L_{\mathrm{D}})^{1/2} Q_1,$$
$$v = (\gamma L_{\mathrm{D}})^{1/2} Q_2,$$

where z is the transmission distance, t is time delay, $L_{\mathrm{D}} = T_0^2/|\beta_2|$ is the second-order dispersion length, T_0 is the pulse width, Q_1, Q_2 are the slowly-varying envelope amplitude of the field, and γ is the nonlinear coefficient.

Equation (1) is nonintegrable, but by performing the transform

$$u = \frac{1}{2}(\psi_1 + \mathrm{i}\psi_2), \quad v = \frac{1}{2}(\psi_1 - \mathrm{i}\psi_2),$$

we get the following equations:

$$\begin{aligned}
&\mathrm{i}\frac{\partial \psi_1}{\partial \xi} - \frac{s}{2}\frac{\partial^2 \psi_1}{\partial \tau^2} + \frac{1}{4}(|\psi_1|^2 + 2|\psi_2|^2)\psi_1 \\
&- \frac{1}{4}\psi_2^2 \psi_1^* + \sigma\psi_1 = 0, \\
&\mathrm{i}\frac{\partial \psi_2}{\partial \xi} - \frac{s}{2}\frac{\partial^2 \psi_2}{\partial \tau^2} + \frac{1}{4}(|\psi_2|^2 + 2|\psi_1|^2)\psi_2 \\
&- \frac{1}{4}\psi_1^2 \psi_2^* - \sigma\psi_2 = 0.
\end{aligned} \tag{2}$$

Through the following identifications, the above mentioned equations have integrability.

3. Analysis of equation's integrability

3.1. Conservation Hamiltonian of the coupled equation

We can obtain the following Lagrangian density from Eq.(2):

$$\begin{aligned}
L = &\frac{\mathrm{i}}{2}\Big(\psi_1^* \frac{\partial \psi_1}{\partial \xi} - \psi_1 \frac{\partial \psi_1^*}{\partial \xi}\Big) \\
&+ \frac{\mathrm{i}}{2}\Big(\psi_2^* \frac{\partial \psi_2}{\partial \xi} - \psi_2 \frac{\partial \psi_2^*}{\partial \xi}\Big) \\
&- \frac{s}{2}\Big(\Big|\frac{\partial \psi_1}{\partial \tau}\Big|^2 + \Big|\frac{\partial \psi_2}{\partial \tau}\Big|^2\Big) \\
&+ \frac{1}{8}\Big(|\psi_1|^4 + |\psi_2|^4\Big) + \frac{1}{2}|\psi_1|^2|\psi_2|^2 \\
&- \frac{1}{8}\psi_1^2\psi_2^{*2} - \frac{1}{8}\psi_2^2\psi_1^{*2} + \sigma(|\psi_1|^2 - |\psi_2|^2),
\end{aligned} \tag{3}$$

where $*$ accounts for the conjugation.

Also we can get the simple law of conservation as follows:

$$N = \int_{-\infty}^{\infty} (|\psi_1|^2 + |\psi_2|^2)\mathrm{d}\tau, \quad \frac{\mathrm{d}N}{\mathrm{d}\xi} = 0, \tag{4}$$

then we can easily obtain the following quantities of conservation from Eqs.(3) and (4):

$$\frac{\mathrm{d}P}{\mathrm{d}\xi} = 0, \quad P = \frac{1}{\mathrm{i}}\int_{-\infty}^{\infty}\Big[\Big(\psi_1^*\frac{\partial \psi_1}{\partial \xi} - \psi_1\frac{\partial \psi_1^*}{\partial \xi}\Big) + \Big(\psi_2^*\frac{\partial \psi_2}{\partial \xi} - \psi_2\frac{\partial \psi_2^*}{\partial \xi}\Big)\Big]\mathrm{d}\tau, \tag{5}$$

$$\frac{\mathrm{d}H}{\mathrm{d}\xi} = 0, \quad H = \frac{\mathrm{i}}{2}\int_{-\infty}^{\infty}\Big[\psi_1^*\frac{\partial \psi_1}{\partial \xi} - \psi_1\frac{\partial \psi_1^*}{\partial \xi} + \psi_2^*\frac{\partial \psi_2}{\partial \xi} - \psi_2\frac{\partial \psi_2^*}{\partial \xi} - L\Big]\mathrm{d}\tau, \tag{6}$$

where P is considered as the momentum of system , H is the Hamiltonian of system.

Substituting Eq.(3) into Eq.(6), we can get the conversation Hamiltonian

$$\begin{aligned}
H\{\psi_1, \psi_2, \psi_1^*, \psi_2^*\} = &\frac{\mathrm{i}}{2}\int_{-\infty}^{\infty}\Big[\frac{s}{2}\Big(\Big|\frac{\partial \psi_1}{\partial \tau}\Big|^2 + \Big|\frac{\partial \psi_2}{\partial \tau}\Big|^2\Big) - \frac{1}{8}(|\psi_1|^4 + |\psi_2|^4) - \frac{1}{2}|\psi_1|^2|\psi_2|^2 \\
&+ \frac{1}{8}\psi_1^2\psi_2^{*2} + \frac{1}{8}\psi_2^2\psi_1^{*2} - \sigma(|\psi_1|^2 - |\psi_2|^2)]\mathrm{d}\tau.
\end{aligned} \tag{7}$$

In this way, Eq.(2) can be expressed as the variation of $H\{\psi_1, \psi_2, \psi_1^*, \psi_2^*\}$

$$\mathrm{i}\frac{\partial \psi_1}{\partial \xi} = \frac{\delta H\{\psi_1, \psi_2, \psi_1^*, \psi_2^*\}}{\delta \psi_1^*}, \tag{8}$$

$$\mathrm{i}\frac{\partial \psi_2}{\partial \xi} = \frac{\delta H\{\psi_1, \psi_2, \psi_1^*, \psi_2^*\}}{\delta \psi_2^*}. \tag{9}$$

We consider the equation of mechanical quantity $A\{\psi_1, \psi_2, \psi_1^*, \psi_2^*\}$ as

$$\mathrm{i}\frac{\partial A}{\partial \xi} = [H, A], \tag{10}$$

where $[A\{\psi_1, \psi_2, \psi_1^*, \psi_2^*\}, B\{\psi_1, \psi_2, \psi_1^*, \psi_2^*\}]$ is Poisson bracket, which is defined as

$$\begin{aligned}&[A\{\psi_1, \psi_2, \psi_1^*, \psi_2^*\}, B\{\psi_1, \psi_2, \psi_1^*, \psi_2^*\}]\\ &= \int_{-\infty}^{\infty}\Big[\frac{\delta A}{\delta \psi_1}\frac{\delta B}{\delta \psi_1^*} - \frac{\delta A}{\delta \psi_1^*}\frac{\delta B}{\delta \psi_1}\\ &\quad + \frac{\delta A}{\delta \psi_2}\frac{\delta B}{\delta \psi_2^*} - \frac{\delta A}{\delta \psi_2^*}\frac{\delta B}{\delta \psi_2}\Big]\mathrm{d}\tau.\end{aligned} \tag{11}$$

So we can consider the system of Eq.(2) as a Hamilton system.

3.2. Galilei transform

Substituting the following transforms into Eq.(2):

$$\begin{aligned}\psi_1(\xi, \tau) &= U(\xi', \tau')\mathrm{e}^{\mathrm{i}(\mu\xi - \Omega\tau)},\\ \psi_2(\xi, \tau) &= V(\xi', \tau')\mathrm{e}^{\mathrm{i}(\mu\xi - \Omega\tau)},\end{aligned} \tag{12}$$

where $\xi' = \xi$, $\tau' = \tau - b\xi$ are new coordinates, we can get b and μ from Eq.(2)

$$b = -s\Omega, \quad \mu = \frac{s}{2}\Omega^2\,. \tag{13}$$

New equations after Galilei transform can be easily obtained as follows:

$$\begin{aligned}&\mathrm{i}\frac{\partial U}{\partial \xi'} - \frac{s}{2}\frac{\partial^2 U}{\partial \tau'^2} + \frac{1}{4}(|U|^2 + 2|V|^2)U\\ &- \frac{1}{4}V^2U^* + \sigma U = 0,\\ &\mathrm{i}\frac{\partial V}{\partial \xi'} - \frac{s}{2}\frac{\partial^2 V}{\partial \tau'^2} + \frac{1}{4}(|V|^2 + 2|U|^2)V\\ &- \frac{1}{4}U^2V^* - \sigma V = 0.\end{aligned} \tag{14}$$

So, Eq.(14) has the same form as Eq.(2), which shows that the system of Eq.(2) has invariability of Galilei transform.

According to the above analysis, we can find that Eq.(2) are integrable.

4. Periodic solution of equations

In the case of abnormal dispersion $s = -1$, considering Eq.(2), we assume

$$\psi_1 = \mathrm{e}^{\mathrm{i}p\xi}\phi_1(\tau), \quad \psi_2 = \mathrm{e}^{\mathrm{i}p\xi}\phi_2(\tau), \tag{15}$$

where p is a constant. Substituting ψ_1, ψ_2 into Eq.(2), we get the following expressions:

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}^2\phi_1}{\mathrm{d}\tau^2} + \frac{1}{4}(\phi_1^2 + \phi_2^2)\phi_1 - (p - \sigma)\phi_1 = 0,\\ &\frac{1}{2}\frac{\mathrm{d}^2\phi_2}{\mathrm{d}\tau^2} + \frac{1}{4}(\phi_1^2 + \phi_2^2)\phi_2 - (\sigma + p)\phi_1 = 0.\end{aligned} \tag{16}$$

Coupled equations of Eq.(16) are multiplied by $\mathrm{d}\phi_1/\mathrm{d}\tau$, $\mathrm{d}\phi_2/\mathrm{d}\tau$ respectively, and integrated with respect to τ, we obtain

$$\begin{aligned}\Big(\frac{\mathrm{d}\phi_1}{\mathrm{d}\tau}\Big)^2 &= C_1 + [2(p - \sigma) - 2\phi_2^2]\phi_1^2 - \frac{1}{4}\phi_1^4,\\ \Big(\frac{\mathrm{d}\phi_2}{\mathrm{d}\tau}\Big)^2 &= C_2 + [2(p + \sigma) - 2\phi_1^2]\phi_2^2 - \frac{1}{4}\phi_2^4,\end{aligned}$$

where C_1 and C_2 are constants. So we have proved that Eq.(16) have solutions of elliptic function form. Considering $(p+\sigma) > (p-\sigma)$, we obtain the solutions of coupled equations as follows:

$$\phi_1 = A_1 k\mathrm{sn}(\alpha\tau, k), \quad \phi_2 = A_2\mathrm{cn}(\alpha\tau, k), \tag{17}$$

where $\mathrm{sn}(\alpha\tau, k)$ and $\mathrm{cn}(\alpha\tau, k)$ are Jacobi elliptic sine and cosine function, A_1, A_2, α, p and the elliptical function modulus k satisfy the following dispersion relations:

$$\begin{aligned}&\alpha^2 = \frac{4\sigma}{k^2}, \quad A_1^2 = \frac{2\alpha^2(1 - 2k^2) + 4(p + \sigma)}{k^2},\\ &A_2^2 = 2\alpha^2(1 + k^2) + 4(p - \sigma)\,.\end{aligned}$$

Finally, substituting Eq.(17) into Eq.(15), and using the transform relation $u = 1/2(\psi_1 + \mathrm{i}\psi_2)$ and $v = 1/2(\psi_1 - \mathrm{i}\psi_2)$, we can get the solution of Eq.(1) in the case of abnormal dispersion

$$\begin{aligned}u = \frac{1}{2}\Big[&\sqrt{\frac{8\sigma}{k^2}(1 - 2k^2) + 4(p + \sigma)}\,\mathrm{sn}\Big(\sqrt{\frac{4\sigma}{k^2}}\tau, k\Big)\\ &+ \mathrm{i}\sqrt{\frac{8\sigma}{k^2}(1 + k^2) + 4(p - \sigma)}\mathrm{cn}\Big(\sqrt{\frac{4\sigma}{k^2}}\tau, k\Big)\Big]\\ &\times \exp(\mathrm{i}p\xi)\,,\\ v = \frac{1}{2}\Big[&\sqrt{\frac{8\sigma}{k^2}(1 - 2k^2) + 4(p + \sigma)}\,\mathrm{sn}\Big(\sqrt{\frac{4\sigma}{k^2}}\tau, k\Big)\\ &- \mathrm{i}\sqrt{\frac{8\sigma}{k^2}(1 + k^2) + 4(p - \sigma)}\,\mathrm{cn}\Big(\sqrt{\frac{4\sigma}{k^2}}\tau, k\Big)\Big]\\ &\times \exp(\mathrm{i}p\xi),\end{aligned} \tag{18}$$

where $0 \le k \le 1$, k is a free parameter.

5. Conclusion

In conclusion, using the Hamilton dynamics theory and Galilei transform, we have found that the transformed CNLSEs have integrability and they have the solutions of Jacobi elliptic function form. So we start with the ansatz with Jacobi elliptic function

form, and obtain the periodic solutions of the CNLSEs in dual core fibre coupler. The solutions exist in a certain dispersion relation among the parameters of the coupled equations.

Acknowledgment

The author Li Q. L. would like to thanks anonymous referee and Professor Li Chengjia about their beneficial advice and help.

References

[1] Mookherjea S and Yariv A 2001 *J. Opt. Soc. Am.* B **18** 1150

[2] Kruglov V I, Peacock A C and Harvey J D 2005 *Phys. Rev.* E**71** 056619
Porsezian K 2003 *Phys. Rev.* E **68** 066607
Soto-Crespo J M and Pesquera L 1997 *Phys. Rev.* E **56** 7288

[3] Li Q L, Zhu H D, Tang X H, Li C J, Wang X J and Lin L B 2004 *Acta Phys. Sin.* **53** 1623 (in Chinese)

[4] Zhou Z J and Li Z B 2003 *Acta Phys. Sin.* **52** 262 (in Chinese)

[5] Zhao X Q, Zhi H Y and Zhang H Q 2006 *Chin. Phys.* **10** 3202

[6] [Wang Z and Zhang H Q 2006 *Chin. Phys.* **16** 3210

[7] Zhang H B, Chen L Q and Liu W R 2005 *Chin. Phys.* **14** 1063

[8] Liu S C 2005 *Acta Phys. Sin.* **54** 4506 (in Chinese)

[9] Li H M 2005 *Chin. Phys.* **14** 0251

[10] Liu M and Shum P 2003 *Opt. Expr.* **11** 116

[11] Shum P, Chiang K S and Gambling W A 1999 *IEEE J. Quantum Electron.* **35** 79

[12] Ramos P M and Pavia C R 1999 *IEEE J. Quantum Electron.* **35** 983

[13] Valkering T P, Boer P T D and Hoekstra H J W M 1998 *Phys.* D **123** 223

[14] Wang W F, Wang Y F, Allaart K and Lenstra D 2005 *Opt. Comm.* **253** 164

[15] Ramos P M and Paiva C R J 2000 *J. Opt. Soc. Am.* B **17** 1125

[16] Agrawal G P 2001 *Nonlinear Fiber Optics* 3rd ed. San Diego (CA: Academic)