Vol 16 No 6, June 2007
1009-1963/2007/16(06)/1534-11

Chinese Physics

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Lie point symmetry algebras and finite transformation groups of the general Broer–Kaup system

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(Received 19 September 2006; revised manuscript received 21 December 2006)

Using a new symmetry group theory, the transformation groups and symmetries of the general Broer–Kaup system are obtained. The results are much simpler than those obtained via the standard approaches.

Keywords: Lie point symmetry, finite transformation group, new symmetry group theory
PACC: 0340K

1. Introduction

The study of symmetries plays an important role in branches of some natural sciences especially in integrable systems. In a standard method, the first step is to find infinitesimal transformations, i.e. to find the symmetry algebra of the studied model and then to find a generalized finite transformation, a symmetry group, by using the well-known first fundamental theorem of Lie.[1] However, it is still quite complicated and difficult to find Lie point symmetry groups via the standard method. In Ref.[2], Lou and Ma have introduced a method of deriving symmetry groups of a nonlinear system. The method is much simpler than the standard approach. Though the new method takes a similar ansatz as the Clarkson–Kruskal (CK) direct method, [3,4] the two methods are completely different from each other at least in two aspects. Firstly, the new method is used to study the symmetry group while the CK direct method is used to reduce a high-dimensional system to lower-dimensional ones. Secondly, all the results from the first method are related to strict symmetry transformations while some results obtained from the second method may be related to only the conditional symmetries.[5]

In the remainder of this paper, we firstly apply the standard method to a $(2+1)$-dimensional model equation, the general Broer–Kaup (GBK) system, to obtain the Lie point symmetries, then use the Lou–Ma method to acquire a general symmetry group (finite transformation) of the system.

2. Lie point symmetries by the standard approach

In this section, we use the standard approach to obtain the Lie point symmetries of the GBK system[6]

$$h_t - h_{xx} + 2hh_x + u_x + Au + Bg = 0, \tag{1}$$

$$\begin{aligned} &g_t + 2(gh)_x + g_{xx} + 4A(g_x - h_{xy}) \\ &+4B(g_y - h_{yy}) + C(g - 2h_y) = 0, \end{aligned} \tag{2}$$

$$u_y - g_x = 0, \tag{3}$$

where A, B and C are arbitrary constants.

According to the traditional method, the first step is to find the Lie symmetry algebras. Based on the symmetry definition, the symmetry of the GBK system $\sigma \equiv (\sigma^h, \sigma^g, \sigma^u)^{\mathrm{T}}$, where the superscript T means the transposition of the matrix, satisfies

$$\sigma_t^h - \sigma_{xx}^h + 2(h\sigma^h)_x + \sigma_x^u + A\sigma^u + B\sigma^g = 0, \tag{4}$$

$$\begin{aligned} &\sigma_t^g + 2(g\sigma^h)_x + 2(h\sigma^g)x + \sigma_{xx}^g + 4A(\sigma^g - \sigma_y^h)_x \\ &+4B(\sigma^g - \sigma_y^h)_y + C(\sigma^g - 2\sigma_y^h) = 0, \end{aligned} \tag{5}$$

$$\sigma_y^u - \sigma_x^g = 0. \tag{6}$$

For the Lie point symmetries, the general solution of Eqs.(4)–(6) can be written as[7]

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$$\sigma(x,y,t,h,g,u)=\begin{pmatrix}\sigma^h\\ \sigma^g\\ \sigma^u\end{pmatrix}=X\begin{pmatrix}h_x\\ g_x\\ u_x\end{pmatrix}+Y\begin{pmatrix}h_y\\ g_y\\ u_y\end{pmatrix}+T\begin{pmatrix}h_y\\ g_y\\ u_y\end{pmatrix}+\begin{pmatrix}H\\ G\\ U\end{pmatrix}, \tag{7}$$

where $G(x,\ y,\ t,\ h,\ g,\ u)\equiv G,\ H(x,\ y,\ t,\ h,\ g,\ u)\equiv H,\ U(x,\ y,\ t,\ h,\ g,\ u)\equiv U,\ X(x,\ y,\ t,\ h,\ g,\ u)\equiv X,\ Y(x,\ y,\ t,\ h,\ g,\ u)\equiv Y$ and $T(x,\ y,\ t,\ h,\ g,\ u)\equiv T$, i.e. $H,\ G,\ U,\ X,\ Y,\ T$ are all functions of $x,\ y,\ t,\ h,\ g$ and u.

Substituting expression (7) into Eq.(6) and letting $g(x,y,t),\ h(x,y,t)$ and $u(x,y,t)$ be a solution of the GBK system yield

$$\begin{aligned}&2gT_hh_x^2+[2hT_uu_x+(4AT_h+2hT_h-X_h+2gT_g)g_x-4BT_hh_{yy}-G_h+4BT_hg_y\\&-g_yY_h-4AT_hh_{xy}+T_hg_{xx}+2gT_x+CuT_h-2CT_hh_y]h_x+[2(2A+h)T_ug_x\\&-4AT_uh_{xy}+X_gg_y+X_hh_y-4BT_uh_{yy}+T_ug_{xx}-G_u+X_y+CgT_u+4BT_ug_y\\&-Y_uG_y-2CT_uh_y]u_x+(2hT_g+Y_u-X_g+4AT_g)g_x^2+[2hT_x+T_uu_t-2CT_gh_y\\&+4AT_x-4BT_gh_{yy}+4BT_gg_y+Y_hh_y+Y_y+CgT_g-G_g-4AT_gh_{xy}+T_gg_{xx}-X_x\\&+U_u]g_x-Y_xg_y+(T_gg_y+T_hh_y+T_y)u_t-2CT_xh_y-4AT_xh_{xy}-4BT_xh_{yy}\\&-G_x+T_xg_{xx}+CgT_x+U_y+U_hh_y+u_gg_y+4BT_xg_y=0\end{aligned} \tag{8}$$

where $h_t,\ g_t,\ u_y$ and their higher-order derivatives have been eliminated by use of the GBK system. Equation (8) holds true for an arbitrary solution ($h,\ g$ and u) only for all coefficients of the polynomials of the different derivatives of the functions $h,\ g$ and u with respect to space x and time t being zero.

Letting the coefficients of the polynomials of the derivatives of $h,\ g$ and u be zero, we have

$$\begin{aligned}&T(x,y,t,h,g,u)=T(t),\ Y(x,y,t,h,g,u)\equiv Y(x,y,t,g,u),\\&X(x,y,t,h,g,u)=X(x,y,t,h,u),\ G(x,y,t,h,g,u)=G(x,y,t,h,u),\\&U(x,y,t,h,g,u)=U(x,y,t,h,u),\frac{\partial X}{\partial g}=\frac{\partial Y}{\partial u},\ \frac{\partial U}{\partial g}=\frac{\partial Y}{\partial x},\\&\frac{\partial G}{\partial u}=\frac{\partial X}{\partial y},\quad \frac{\partial G}{\partial x}=\frac{\partial U}{\partial y},\ \frac{\partial G}{\partial g}=\frac{\partial Y}{\partial y}-\frac{\partial X}{\partial x}+\frac{\partial U}{\partial u}.\end{aligned} \tag{9}$$

Under these conditions, substituting expression (7) into Eq.(4) we have

$$\begin{aligned}&-4h^2Th_{xx}+2Ch_gh_y-(Au+Bg)H_h-Y_{xx}h_y+2hH_x+AU+BG+2(-H_{xg}\\&-2AY_gh_y-Y_uh_{xx}+Y_x-2AH_g-Y_{xg}h_y-Y_gh_{xy})g_x+Y_uh_yu_t-CGH_g-CgY_gh_y\\&-4BY_gg_yh_y+Y_th_y+U_x+T_th_{xx}-(Au+Bg)T_t-2H_gg_{xx}+H_t-H_{xx}-2Y_gh_yg_{xx}\\&-Y_uh_yu_{xx}-2X_xh_{xx}+(-H_{uu}+X_u-Y_{uu}+X_u)u_x^2+(-2gY_u-H_{gg})h_x^2+[-X_{uu}u_x^2\\&+2(-Y_{uu}g_x+hX_u-X_{xu}-H_{hu})u_x-Y_{gu}g_x^2+2(2Y_u-H_{gh}-Y_{xu})+4BgT+2CY_uh_y\\&-CgY_u2hT_t-2gH_g+2hX_x-2AUT+2H-2Y_ug_{xx}-X_{xx}-2H_{xh}-4BY_ug_y+X_t\\&-X_uU_{xx}+4AY_uH_{xy}+2AuT+4BY_uh_{yy}-4BgT+X_uu_t-2gX_gh_y]h_x+2(Y_x\\&+2AY_gh_y+2AH_g)h_{xy}+4BY_gh_yh_{yy}+4h^2Th_{xx}-4BH_gg_y+2CY_gh_{yy}+2hY_xh_y\\&+H_u(u_t-u_{xx})+4BH_gh_{yy}+[2(H_{gu}+Y_u-Y_{gu}h_y)g_x-2Y_u(h_{xy}-hh_y)-2Y_{xu}h_y\\&-H_h+2hH_u-T_t-2H_{xu}+X_x+U_u-2X_uh_{xx}]u_x-(Y_{gg}h_y+H_{gg}-Y_g)g_x^2=0,\end{aligned} \tag{10}$$

which has the following solutions for the determining equation system, obtained by vanishing the coefficients of the polynomials for the derivatives of $h,\ g$ and u:

$$Y(x,y,t,g,u)\equiv Y(y),\quad U(x,y,t,h,u)=U(x,y,t),$$

$$
\begin{aligned}
&G(x,y,t,h,u)=gG(x,y,t)+G_0(x,y,t),\quad G(x,y,t)=\frac{\mathrm{d}}{\mathrm{d}y}Y(y)+G(t),\\
&H(x,y,t,h,g,u)=H(x,y,t,h)=hG(t)+H(x,y,t),\\
&X(x,y,t,h,u)=X(x,y,t)=\frac{1}{2}x\frac{\mathrm{d}}{\mathrm{d}t}T(t)+x_0(y,t),\\
&H(x,y,t)=-\frac{1}{4}x\frac{\mathrm{d}^2}{\mathrm{d}t^2}T(t)-\frac{1}{2}\frac{\partial}{\partial t}x_0(y,t),\quad G(t)=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}T(t),
\end{aligned}\tag{11}
$$

where $x_0(y,t)$ is an arbitrary function of y and t. Then the Lie point symmetry of the GBK system is reduced to

$$
\begin{aligned}
\sigma(x,y,t,h,g,u)=&\begin{pmatrix}\sigma^h\\ \sigma^g\\ \sigma^u\end{pmatrix}\\
=&\begin{pmatrix}
\frac{1}{2}\frac{\mathrm{d}T(t)}{\mathrm{d}t}(hx)_x+x_0(y,t)h_x+Y(y)h_y+T(t)h_t-\frac{1}{4}\frac{\mathrm{d}^2T(t)}{\mathrm{d}t^2}x-\frac{1}{2}\frac{\mathrm{d}x_0(y,t)}{\mathrm{d}t}\\
\frac{1}{2}\frac{\mathrm{d}T(t)}{\mathrm{d}t}g_x x+x_0(y,t)g_x+Y(y)g_y+T(t)g_t+\frac{\mathrm{d}Y(y)}{\mathrm{d}y}g+\frac{1}{2}\frac{\mathrm{d}T(t)}{\mathrm{d}t}g\\
\frac{1}{2}\frac{\mathrm{d}T(t)}{\mathrm{d}t}u_x x+x_0(y,t)u_x+Y(y)u_y+T(t)u_t+\frac{\mathrm{d}T(t)}{\mathrm{d}t}u+U(x,y,t)
\end{pmatrix},
\end{aligned}
$$

while Eqs.(4)–(6) are reduced to

$$
\begin{aligned}
&Cx_{0y}u+2G_0h_x+4AG_{0x}+4BG_{0y}+CG_0+2hG_{0x}+(x_{0y}u)_t+g_{0t}+2Bx_{0yyt}+CX_{0yt}\\
&+2hu_xx_{0y}+2h_xx_{0y}u-2Ag_xT_t+4Au_xx_{0y}+8Bg_xx_{0y}+4BY_yg_y-4BT_tg_y-2Cx_{0y}h_x\\
&+2CT_th_y+x_{0y}u_{xx}+2AT_th_{xy}+G_{0xx}-CT_tg-4Ax_{0y}h_{xx}+4Bx_{0yy}u+4BgY_{yy}\\
&-8Bx_{0y}h_{xy}-4Bx_{0yy}h_x-4B(Y_yh_y)_y+4BT_th_{yy}=0,
\end{aligned}\tag{12}
$$

$$
-\frac{1}{4}T_{ttt}x+U_x-BgT_t-\frac{1}{2}AuT_t-\frac{1}{2}x_{0tt}+Bux_{0y}+BgY+BG_0+AU=0,\tag{13}
$$

$$
U_y-G_{0x}=0.\tag{14}
$$

Vanishing the coefficients of u_{xx} and h_x of Eq.(12) leads to

$$
x_0(y,t)=x_0(t),\quad G_0(x,y,t)=0\tag{15}
$$

and then

$$
U(x,y,t)=U(x,t),\tag{16}
$$

due to Eq. (14).

To study further, according to the different choices of the constants A and B and the arbitrary functions, $Y(y)$, $T(t)$ and $x_0(t)$, there exist five possible cases of the symmetries which have to be considered separately as follows:

Case a $A\neq 0, B\neq 0$.

In this case, the symmetry is

$$
\begin{aligned}
\sigma^h&=x_0(t)h_x+y_0h_y+t_0h_t-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}x_0(t),\\
\sigma^g&=x_0(t)g_x+y_0h_y+t_0g_t,\\
\sigma^u&=x_0(t)u_x+y_0u_y+t_0u_t+\mathrm{e}^{-Ax}u_0(t)\\
&\quad+\frac{1}{2A}\frac{\mathrm{d}^2}{\mathrm{d}t^2}x_0(t),
\end{aligned}\tag{17}
$$

with $x_0(t)$ and $u_0(t)$ being arbitrary functions of t, and y_0 and t_0 being arbitrary constants.

Case b $A\neq 0, B=0$.

In this case, the symmetry is

$$
\begin{aligned}
\sigma^h&=x_0(t)h_x+Y(y)h_y+t_0h_t-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}x_0(t),\\
\sigma^g&=x_0(t)g_x+Y(y)h_y+t_0g_t+g\frac{\mathrm{d}}{\mathrm{d}y}Y(y),
\end{aligned}
$$

$$\sigma^u = x_0(t)u_x + Y(y)u_y + t_0u_t + \mathrm{e}^{-Ax}u_0(t) + \frac{1}{2A}\frac{\mathrm{d}^2}{\mathrm{d}t^2}x_0(t), \tag{18}$$

where $x_0(t)$ and $u_0(t)$ are arbitrary functions of t, $Y(y)$ is an arbitrary function of y, and t_0 is an arbitrary constant.

Case c $A = 0, B \neq 0$.
In this case, the symmetry is

$$\begin{aligned}\sigma^h &= x_0(t)h_x + y_0h_y + t_0h_t - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}x_0(t),\\ \sigma^g &= x_0(t)g_x + y_0h_y + t_0g_t,\\ \sigma^u &= x_0(t)u_x + y_0u_y + t_0u_t + u_0(t) + \frac{1}{2}x\frac{\mathrm{d}^2}{\mathrm{d}t^2}x_0(t),\end{aligned} \tag{19}$$

with $x_0(t)$ and $u_0(t)$ being arbitrary functions of t, y_0 and t_0 being arbitrary constants.

Case d $A = B = 0, C \neq 0$.
In this case, the symmetry is

$$\begin{aligned}\sigma^h &= x_0(t)h_x + Y(y)h_y + t_0h_t - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}x_0(t),\\ \sigma^g &= x_0(t)g_x + Y(y)h_y + t_0g_t + g\frac{d}{dy}Y(y),\\ \sigma^u &= x_0(t)u_x + Y(y)u_y + t_0u_t + u_0(t) + \frac{1}{2}x\frac{\mathrm{d}^2}{\mathrm{d}t^2}x_0(t),\end{aligned} \tag{20}$$

with $u_0(t)$ and $x_0(t)$ being arbitrary functions of t, $Y(y)$ being arbitrary function of y, and t_0 arbitrary constant.

Case e $A = B = C = 0$.
In this case, the symmetry is

$$\begin{aligned}\sigma^h &= (\frac{1}{2}x\frac{\mathrm{d}}{\mathrm{d}t}T(t) + x_0(t))h_x + Y(y)h_y + T(t)h_t + \frac{1}{2}h\frac{\mathrm{d}}{\mathrm{d}t}T(t) - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}x_0(t) - \frac{1}{4}x\frac{\mathrm{d}^2}{\mathrm{d}t^2}T(t),\\ \sigma^g &= (\frac{1}{2}x\frac{\mathrm{d}}{\mathrm{d}t}T(t) + x_0(t))g_x + Y(y)g_y + T(t)g_t + g\frac{\mathrm{d}}{\mathrm{d}y}Y(y),\\ \sigma^u &= (\frac{1}{2}x\frac{\mathrm{d}}{\mathrm{d}t}T(t) + x_0(t))u_x + Y(y)u_y + T(t)u_t + u_0(t) + \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}x_0(t) + \frac{1}{8}x^2\frac{\mathrm{d}^3}{\mathrm{d}t^3}T(t),\end{aligned} \tag{21}$$

with $u_0(t)$, $x_0(t)$ and $T(t)$ being arbitrary functions of t, and $Y(y)$ being arbitrary function of y.

Once a symmetry (a generator of the symmetry group) is known, the corresponding finite transformation can be obtained by Lie's first fundamental theorem.[8] However, it is still difficult to find the finite transformation due to the complex calculation, and the final expressions are very complicated and difficult to apply for practical purposes. Here, the direct method is applied to acquiring the transformation groups of the GBK system.

3. Transformation group by the direct method

According to Ref. [2], it is straightforwardly assumed that

$$\begin{aligned}g &= \alpha_1 + \beta_1G(\xi,\eta,\tau),\\ h &= \alpha_2 + \beta_2H(\xi,\eta,\tau),\\ u &= \alpha_3 + \beta_3U(\xi,\eta,\tau),\end{aligned} \tag{22}$$

which is a solution of the GBK system. Here, α_1, α_2, α_3, β_1, β_2, β_3, ξ, η and τ are functions of x, y and t. Substituting expressions (22) into Eq.(1), letting $G(\xi,\eta,\tau)$, $H(\xi,\eta,\tau)$ and $U(\xi,\eta,\tau)$ also be a solution of the GBK system but with independent variables, and eliminating H_τ, G_τ, U_η and their higher-order derivatives by use of the GBK system yield

$$-\beta_2\tau_x^2H_{\xi\xi\xi\xi} + W(x,y,t,H,H_\xi,\ldots,H_\eta,\ldots,U_\xi,U_\eta,\ldots,U_{\eta\eta},\ldots,G_\xi,G_\eta,\ldots,G_{\eta\eta}) = 0, \tag{23}$$

where W is a complicated independent function of $H_{\xi\xi\xi\xi}$ (and its higher order derivatives). Equation (23) holds true for an arbitrary solution H only for all coefficients of the polynomials for the derivatives of H, G and U being zero.

Obviously, β_2 should not be zero and there exists no nontrivial solution for $\tau_x = 0$, so

$$\tau_x = 0, \quad \text{i.e.} \quad \tau \equiv \tau(y,t). \tag{24}$$

Under condition (24), Eq.(23) is reduced to

$$\begin{aligned}&\beta_2(\tau_t - \xi_x^2)H_{\xi\xi} - \beta_2\eta_x^2 H_{\eta\eta} - 2\beta_2\xi_x\eta_x H_{\xi\eta} + (\beta_2\xi_t - 2\beta_{2x}\xi_x + 2\alpha_2\beta_2\xi_x - \beta_2\xi_{xx})H_\xi\\ &+(\beta_2\eta_t - 2\beta_{2x}\eta_x + 2\alpha_2\beta_2\eta_x - \beta_2\eta_{xx})H_\eta + 2\beta_2[(\beta_2\xi_x - \tau_t)H_\xi + \beta_2\eta_x H_\eta + \beta_{2x}H]H\\ &+(2(\beta_2\alpha_2)_x + \beta_{2t} - \beta_{2xx})H + (\beta_3\xi_x - \beta_2\tau_t)U_\xi + (A\beta_3 - A\beta_2\tau_t + \beta_{3x})U + \beta_3\eta_x G_\xi\\ &-B(-\beta_1 + \beta_2\tau_t)G + \beta_{2t} - \alpha_{2xx} + \alpha_{2t} + \alpha_{3x} + 2\alpha_2\alpha_{2x} + A\alpha_3 + B\alpha_1 = 0.\end{aligned} \tag{25}$$

It is easy to see that

$$\xi_x^2 - \tau_t = 0, \quad \eta_x = 0, \tag{26}$$

to eliminate $H_{\xi\xi}$ and $H_{\eta\eta}$, or

$$\xi = x\xi_1 + \xi_2, \quad \tau_t = \xi_1^2, \quad \eta \equiv \eta(y,t), \tag{27}$$

where ξ_1 and ξ_2 are only functions of y and t. Substituting expression (27) into Eq.(25) we have a simplified equation as follows:

$$\begin{aligned}&[\beta_2(\xi_{1t}x + \xi_{2t}) + 2\alpha_2\beta_2\xi_1 - 2\beta_{2x}\xi_1]H_\xi + (\beta_3 - \beta_2\xi_1)U_\xi + \beta_2\eta_t H_\eta + 2\beta_2(\beta_2 - \xi_1)HH_\xi\\ &+2\beta_2\beta_{2x}H^2 + [2(\alpha_2\beta_2)_x - \beta_{2xx} + \beta_{2t}]H + B(\beta_1 - \beta_2\xi_1^2)G + (A\beta_3 - A\beta_2\xi_1^1 + \beta_{3x})U\\ &+\alpha_{2t} + \alpha_{3x} + 2\alpha_2\alpha_{2x} + A\alpha_3 + B\alpha_1 - \alpha_{2xx} = 0.\end{aligned} \tag{28}$$

By vanishing the coefficients of the polynomials of H, U, G and their derivatives, the remaining determining equations of the functions read

$$\beta_2(\xi_{1t}x + \xi_{2t}) + 2\alpha_2\beta_2\xi_1 - 2\beta_{2x}\xi_1 = 0, \tag{29}$$
$$-\beta_2\xi_1 + \beta_3 = 0, \tag{30}$$
$$\beta_2\eta_t = 0, \tag{31}$$
$$2\beta_2(-\xi_1 + \beta_2) = 0, \tag{32}$$
$$2\beta_2\beta_{2x} = 0, \tag{33}$$
$$2(\alpha_2\beta_2)_x - \beta_{2xx} + \beta_{2t} = 0, \tag{34}$$
$$B(\beta_1 - \beta_2\xi_1^2) = 0, \tag{35}$$
$$A\beta_3 - A\beta_2\xi_1^2 + \beta_{3x} = 0, \tag{36}$$
$$\alpha_{2t} + \alpha_{3x} + 2\alpha_2\alpha_{2x} + A\alpha_3 + B\alpha_1 - \alpha_{2xx} = 0. \tag{37}$$

From Eqs.(29)–(34) we can have the following results:

$$\begin{aligned}&\eta \equiv \eta(y), \quad \beta_2 = \xi_1,\\ &\beta_3 = \beta_2\xi_1 = \xi_1^2, \quad \alpha_2 = -\frac{\xi_{1t}x + \xi_{2t}}{2\xi_1}.\end{aligned} \tag{38}$$

As to Eq.(35), there appear two cases due to the different choices of the constant B.

Case 1 $B \neq 0$.

In this case, to vanish the coefficient of G or Eq.(35) there is no choice but

$$\beta_1 = \xi_1^3, \ \beta_2 = \xi_1. \tag{39}$$

Under conditions (38) and (39), Eq.(28) is reduced to

$$\begin{aligned}&-2A(\xi_1 - 1)\xi_1^4 U - \xi_1\xi_{1tt} - \xi_1\xi_{2tt} + 2\xi_{1t}^2 x\\ &+2\xi_{1t}\xi_{2t} + 2B\xi_1^2\alpha_1 + 2\xi_1^2\alpha_{3x} = 0.\end{aligned} \tag{40}$$

To lose the coefficient of U, there are also two subcases:

(i) $A \neq 0$.

In this situation, the required equation is

$$(\xi_1 - 1)\xi_1^4 = 0.$$

Obviously, ξ_1 should not be zero due to ξ being the transformation of x (27), so

$$\xi_1 = 1. \tag{41}$$

Substituting the known results into Eq.(2) and Eq.(3) we have

$$
\begin{aligned}
&-4B\tau_y^2 H_{\xi\xi\xi\xi} - 8B\tau_y\eta_y H_{\xi\xi\eta} - 4B\tau_y(2B\xi_{2y}+A)H_{\xi\xi\xi} + 4B\tau_y H H_{\xi\xi\xi} + 2(2B\tau_{yy}\\
&+2B\xi_{2y}^2 + 2A\xi_{2y} + C\tau_y)H_{\xi\xi} + 8\tau_y(A+2B\xi_{2y})(H_\xi^2 + HH_{\xi\xi}) - 16B\tau_y^2 H^2 H_{\xi\xi}\\
&+32B\tau_y^2(H_{\xi\xi}H_\xi - HH_\xi^2) + 4(A - A\eta_y + 4AB^2\tau_y^2 - 2B\eta_y\xi_{2y})H_{\xi\eta} + 16B\eta_y\tau_y H H_{\xi\eta}\\
&+2(C - 2AB\eta_{yy} + 4B^2C\tau_y^2 + 4BC\tau_y - C\eta_y)H_\eta + 4(C\tau_y + 2B\tau_{yy})HH_\xi\\
&+16B\eta_y\tau_y H_\xi H_\eta + 2(\alpha_1 - C\xi_{2y} - 2B\xi_{2yy})H_\xi - B\tau_y^2 H_\xi U_\xi - 8B\tau_y(1+2B\tau_y)GH_\xi\\
&-8AB\tau_y^2 UH_\xi + 2\alpha_1 H + 4B[(1+2B\tau_y)^2 - \eta_y^2]H_{\eta\eta} + 4B\tau_y^2 U_{\xi\xi\xi} + 4B\tau_y^2 U_{\xi\xi\eta}\\
&+4\tau_y(A + 2B\xi_{2y} + AB\tau_y)U_{\xi\xi} + 4AB\tau_y^2 U_\tau + 2(2A^2\tau_y + 2B\tau_{yy} + 4AB\tau_y\xi_{2y}\\
&+C\tau_y)U_\xi - 8AB\tau_y^2 HU_\xi + 2A(B\tau_{yy} + C\tau_y)U + 4B\tau_y(2\eta_y - 1)G_{\xi\xi} + 4B(-3A\tau_y\\
&+2B\tau_y\xi_{2y} + 2A\eta_y\tau_y - 4AB\tau_y^2 + B\xi_{2y})G_\xi + 4B(2B\eta_y\tau_y - 4B\tau_y + \eta_y - 4B^2\tau_y^2\\
&-1)G_\eta - 8B\tau_y(2B\tau_y + 1)HG_\xi + 2(2B^2\tau_{yy} - 2B^2\tau_y^2 - BC\tau_y)G\\
&+\alpha_{1t} + C\xi_{2yt} + 4B\alpha_{1yu} + C\alpha_1 + 4A\alpha_{1x} - \alpha_{1x}\xi_{2t} + \alpha_{1xx} = 0,
\end{aligned}
\tag{42}
$$

and

$$
(\eta_y - 1)G_\xi - \beta_{1x}G + \xi_{2y}U_\xi + \tau_y U_\tau + \alpha_{3y} - \alpha_{1x} = 0. \tag{43}
$$

Vanishing the coefficient of $H_{\xi\xi\xi\xi}$ in Eq.(42) requires

$$
-4B\tau_y^2 = 0, \quad \text{i.e.} \quad \tau = \tau(t), \tag{44}
$$

so it is easy to obtain the result from expressions (27) and (41)

$$
\tau = t - t_0, \tag{45}
$$

where t_0 is a constant. Vanishing the coefficients of G_ξ and U_ξ in Eq.(43) we have

$$
\eta = y - y_0, \quad \xi_2(y,t) = \xi_2(t), \tag{46}
$$

where y_0 is a constant.

Now the remaining determining reduced equations are

$$
\begin{aligned}
&-\xi_{2t} + 2\alpha_{3x} + 2A\alpha_3 + 2B\alpha_1 = 0,\\
&2\alpha_1 H_\xi + 2\alpha_{1x}H + \alpha_{1t} + \alpha_{1xx} + 4A\alpha_{1x}\\
&+4B\alpha_{1y} + C\alpha_1 - \alpha_{1x}\xi_{2t} = 0,\\
&\alpha_3 y - \alpha_{1x} = 0.
\end{aligned}
\tag{47}
$$

The general solution of these equations reads

$$
\begin{aligned}
&\alpha_1 = 0,\\
&\alpha_3 = \frac{1}{2A}\xi_{2tt} + e^{-Ax} + f_1,
\end{aligned}
\tag{48}
$$

where f_1 is an arbitrary function of t.

In summary, the following theorem holds.

Theorem 1 If $\{G = G(x,y,t),\ H = H(x,y,t),\ U = U(x,y,t)\}$ is a solution of the GBK system (1)–(3) when $A \neq 0$ and $B \neq 0$ then so is

$$
\begin{aligned}
g =&\ G(x+\xi_2, y-y_0, t-t_0),\\
h =&\ -\frac{\xi_{2t}}{2} + H(x+\xi_2, y-y_0, t-t_0),\\
u =&\ \frac{1}{2A}\xi_{2tt} + e^{-Ax} + f_1(t)\\
&+U(x+\xi_2, y-y_0, t-t_0),
\end{aligned}
\tag{49}
$$

with ξ_2 and f_1 being arbitrary functions of t, and y_0 and t_0 being constants.

It is obvious that once the symmetry group is obtained, to find its symmetry algebra, the infinitesimal transformations, is quite a straightforward work. The generators of the symmetry algebra related to the symmetry group theorem 1 can be obtained by setting

$$
\begin{aligned}
&\xi_2 = \epsilon X_0(t), \quad y_0 = -\epsilon y_0,\\
&t_0 = -\epsilon t_0, \quad f_1(t) = \epsilon f(t),
\end{aligned}
\tag{50}
$$

with ϵ being an infinitesimal parameter, then expression (49) can be written as

$$
\begin{aligned}
g =&\ G + \epsilon(X_0(t)G_x + y_0 G_y + t_0 G_t),\\
h =&\ H + \epsilon(X_0(t)H_x + y_0 H_y + t_0 H_t - \frac{1}{2}X_{0t}(t)),\\
u =&\ U + \epsilon[\frac{1}{A}(X_0(t)U_x + y_0 U_y + t_0 U_t)\\
&+\frac{1}{2A}X_{0tt}(t) + e^{-Ax}f(t)].
\end{aligned}
\tag{51}
$$

which is exactly the same as that for Case a in Section 2.

(ii) $A=0$.

If $A=0$, the corresponding reduction equation is

$$\begin{aligned}&-\xi_1\xi_{1tt}x-\xi_1\xi_{2tt}+2\xi_{1t}^2x\\&+2\xi_{1t}\xi_{2t}+2\xi_1^2\alpha_{3x}+2B\xi_1^2\alpha_1=0,\end{aligned}\tag{52}$$

which is still difficult to solve. Now substituting expressions (22) and (24), (27), (38) and (39) into Eq.(3) yields

$$\begin{aligned}&\beta_2\xi_1(\xi_{1y}x+\xi_{2y})U_\xi+\beta_2\xi_1\tau_yU_\tau+\xi_1(\beta_2\eta_y-\beta_1)G_\xi\\&+(\beta_2\xi_1)_yU+\alpha_{3y}-\alpha_{1x}=0,\end{aligned}\tag{53}$$

with the determining equations

$$\begin{aligned}&\beta_2\xi_1(\xi_{1y}x+\xi_{2y})=0,\\&\beta_2\xi_1\tau_y=0,\\&\xi_1(\beta_2\eta_y-\beta_1)=0,\\&(\beta_2\xi_1)_y=0,\\&\alpha_{3y}-\alpha_{1x}=0,\end{aligned}\tag{54}$$

which have the general solutions

$$\begin{aligned}&\tau=\tau(t),\quad \xi_1=\xi_0,\quad \xi_2=\xi_2(t),\\&\beta_2=b_2(t),\quad \eta=\xi_0^2y+y_0,\end{aligned}\tag{55}$$

where ξ_0 and y_0 are arbitrary constants and $\tau(t)$, $\xi_2(t)$ and $b_2(t)$ are arbitrary functions of t. Under these conditions, Eq.(2) is reduced to

$$\begin{aligned}&4B\xi_0^5(b_2-\xi_0)H_{\eta\eta}+4A\xi_0^5(b_2-1)H_{\xi\eta}+2C\xi_0^4(\xi_0b_2-1)H_\eta+2\xi_0^3\alpha_1H_\xi+2\xi_0^2\alpha_{1x}H\\&-4A\xi_0^5b_2G_\xi+\xi_0^3(-Cb_2\xi_0^2+Cb_2+b_{2t})G-\alpha_{1x}\xi_{2t}+\xi_0\alpha_{1xx}+\xi_0\alpha_{1t}+4A\xi_0\alpha_{1x}\\&+4B\xi_0\alpha_{1y}+C\xi_0\alpha_1=0,\end{aligned}\tag{56}$$

By vanishing the coefficients of the polynomials of H, U and their derivatives, we obtain the following results:

$$\begin{aligned}&\alpha_1=0,\quad \alpha_3=\frac{1}{2}\frac{\mathrm{d}^2\xi_2}{\mathrm{d}t^2}x+a_3,\\&b_2(t)=1,\quad \tau=t-t_0,\quad \xi_0=1.\end{aligned}\tag{57}$$

We can arrive at the following final transformation group theorem of the GBK system when $A=0$:

Theorem 2 If $\{G=G(x,y,t),\ H=H(x,y,t),\ U=U(x,y,t)\}$ is a solution of the GBK system (1)–(3) when $A=0$ while $B\neq 0$, so is {g, h, u} with

$$\begin{aligned}&g=G(x+\xi_2,y+y_0,t-t_0),\\&h=-\frac{1}{2}\frac{\mathrm{d}\xi_2}{\mathrm{d}t}+H(x+\xi_2,y+y_0,t-t_0),\\&u=\frac{1}{2}x\frac{\mathrm{d}^2\xi_2}{\mathrm{d}t^2}+a_3+U(x+\xi_2,y+y_0,t-t_0),\end{aligned}\tag{58}$$

where $\xi_2\equiv\xi_2(t)$ and $a_3\equiv a_3(t)$ are arbitrary functions of t, and y_0, t_0 are arbitrary constants.

By setting

$$\begin{aligned}&\xi_2=\epsilon X_0(t),\quad y_0=-\epsilon y_0,\\&t_0=-\epsilon t_0,\quad a_3(t)=\epsilon a(t),\end{aligned}\tag{59}$$

with ϵ being an infinitesimal parameter, the corresponding symmetry algebra can be obtained as follows:

$$\begin{aligned}g=\ & G+\epsilon(X_0(t)G_x+y_0G_y+t_0G_t),\\h=\ & H+\epsilon(X_0(t)H_x+y_0H_y+t_0H_t-\frac{1}{2}X_{0t}(t)),\\u=\ & U+\epsilon(X_0(t)U_x+y_0U_y+t_0U_t\\&+\frac{1}{2}X_{0tt}(t)x+a(t)).\end{aligned}\tag{60}$$

Case 2 $B=0$.

If $B=0$, Eq.(35) is identically satisfied. Then in this case, the reduced equation is

$$\begin{aligned}&[\beta_2(\xi_{1t}x+\xi_{2t})+2\alpha_2\beta_2\xi_1-2\beta_{2x}\xi_1]H_\xi+(\beta_3-\beta_2\xi_1)U_\xi+2\beta_2(\beta_2-\xi_1)HH_\xi\\&+\beta_2\eta_tH_\eta+2\beta_2\beta_{2x}H^2+[2(\alpha_2\beta_2)_x-\beta_{2xx}+\beta_{2t}]H+(A\beta_3-A\beta_2\xi_1^2+\beta_{3x})U\\&+\alpha_{2t}+\alpha_{3x}+2\alpha_2\alpha_{2x}+A\alpha_3-\alpha_{2xx}=0,\end{aligned}\tag{61}$$

and it is straightforward to obtain the determining equations by vanishing the coefficients of H, G and their derivatives as follows:

$$\beta_2(\xi_{1t}x+\xi_{2t})+2\alpha_2\beta_2\xi_1-2\beta_{2x}\xi_1=0, \quad (62)$$
$$-\beta_2\xi_1+\beta_3=0, \quad (63)$$
$$\beta_2\eta_t=0, \quad (64)$$
$$2\beta_2(-\xi_1+\beta_2)=0, \quad (65)$$
$$2\beta_2\beta_{2x}=0, \quad (66)$$
$$2(\alpha_2\beta_2)_x-\beta_{2xx}+\beta_{2t}=0, \quad (67)$$
$$A\beta_3-A\beta_2\xi_1^2+\beta_{3x}=0, \quad (68)$$
$$\alpha_{2t}+\alpha_{3x}+2\alpha_2\alpha_{2x}+A\alpha_3-\alpha_{2xx}=0, \quad (69)$$

with the general solutions of Eqs.(62)–(67) being

$$\eta=\eta(y), \quad \beta_2=\xi_1,$$
$$\beta_3=\xi_1^2, \quad \alpha_2=-\frac{1}{2}\frac{\xi_{1t}x+\xi_{2t}}{\xi_1}. \quad (70)$$

Under these conditions, Eq.(68) becomes

$$A\xi_1^2(1-\xi_1)=0. \quad (71)$$

Similarly, there are two subcases due to the difference of constant A:

(i) $A\neq 0$.

If $A\neq 0$, then we have

$$\xi_1=1, \quad \tau=t-t_1, \quad (72)$$

where t_1 is an arbitrary function of y. From Eq.(69) it is found that

$$\alpha_3=\frac{1}{2A}\xi_{2tt}+\mathrm{e}^{-Ax}a_3, \quad (73)$$

with a_3 being an arbitrary function of y and t. And then the reduced equations of Eqs.(2) and (3) become

$$-4At_{1y}H_{\xi\xi\xi}-2(2A\xi_{2y}+Ct_{1y})H_{\xi\xi}+8At_{1y}HH_{\xi\xi}+4A(\beta_1-\eta_y)H_{\xi\eta}+2(\alpha_1-C\xi_{2y})H_\xi$$
$$+2C(\beta_1-\eta_y)H_\eta+4Ct_{1y}HH_\xi+At_{1y}H_\xi^2+2\alpha_{1x}H+4At_{1y}U_{\xi\xi}+4A^2t_{1y}U_\xi+2ACt_{1y}U$$
$$+2\beta_{1x}G_\xi+(\beta_{1t}-\xi_{2t}\beta_{1x}+\beta_{1xx}+4A\beta_{1x})G+2\beta_{1x}GH+\alpha_{1xx}+4A\alpha_{1x}+C\xi_{2yt}+\alpha_{1t}$$
$$+C\alpha_1-\alpha_{1x}\xi_{2t}=0, \quad (74)$$

and

$$2A(\eta_y-\beta_1)G_\xi+2A\xi_{2y}U_\xi+2At_{1y}U_\tau-2A\beta_{1x}G+\xi_{2ytt}+2A\mathrm{e}^{-Ax}a_{3y}=0. \quad (75)$$

Thus it is easy to obtain the solutions of the determining equations as follows:

$$t_1=-t_0, \quad \alpha_1=0, \quad \beta_1=\frac{\mathrm{d}\eta}{\mathrm{d}y},$$
$$\xi_2=\xi_2(t), \quad a_3=a_3(t), \quad (76)$$

with t_0 being an arbitrary constant and ξ_2, a_3 being arbitrary functions of t. So the following theorem holds.

Theorem 3 If $\{G=G(x,y,t),\ H=H(x,y,t),\ U=U(x,y,t)\}$ is a solution of the GBK system (1)–(3) when $A\neq 0$ while $B=0$, so is {g, h, u} with

$$g=\frac{\mathrm{d}\eta}{\mathrm{d}y}G(x+\xi_2,\eta,t-t_0),$$
$$h=-\frac{1}{2}\frac{\mathrm{d}\xi_2}{\mathrm{d}t}+H(x+\xi_2,\eta,t-t_0),$$
$$u=\frac{1}{2A}\frac{\mathrm{d}^2\xi_2}{\mathrm{d}t^2}+\mathrm{e}^{-Ax}a_3+U(x+\xi_2,\eta,t-t_0), \quad (77)$$

where $\xi_2\equiv\xi_2(t)$ and $a_3\equiv a_3(t)$ are arbitrary functions of t, and $\eta\equiv\eta(y)$ is an arbitrary function of y.

By setting

$$\xi_2=\epsilon X_0(t), \quad \eta(y)=y+\epsilon Y(y),$$
$$t_0=-\epsilon t_0, \quad a_3(t)=\epsilon a(t), \quad (78)$$

with ϵ being an infinitesimal parameter, the related symmetry algebra can be obtained as follows:

$$g=\ G+\epsilon(X_0(t)G_x+Y(y)G_y+t_0G_t+Y_y(y)G),$$
$$h=\ H+\epsilon(X_0(t)H_x+Y(y)H_y+t_0H_t-\frac{1}{2}X_{0t}(t)),$$
$$u=\ U+\epsilon(X_0(t)U_x+Y(y)U_y+t_0U_t+\frac{1}{2A}X_{0tt}(t)$$
$$+\mathrm{e}^{-Ax}a(t)). \quad (79)$$

(ii) $A=0$.

In this case, the reduced equations are

$$-\xi_1(\xi_{1tt}x+\xi_{2tt})+2\xi_{1t}^2x+2\xi_{1t}\xi_{2t}+2\alpha_{3x}\xi_1^2=0, \tag{80}$$

$$\begin{aligned}&-2C\xi_1^3\tau_yH_{\xi\xi}+4C\xi_1^3\tau_yHH_\xi-2\xi_1^3(C\xi_{2y}x+C\xi_{1y}-\xi_1\alpha_{1x})H_\xi+2C\xi_1^3(\beta_1\xi_1-\eta_y)H_\eta\\&-2\xi_1^2(C\xi_{1y}-\xi_1\alpha_{1x})H+2\xi_1^3\beta_{1x}G_\xi+[-\xi_1\beta_{1x}(\xi_{1t}x+\xi_{2t})+C\xi_1^2\beta_1(1-\xi_1^2)+\xi_1^2(\beta_{1xx}\\&+\beta_{1t})]G+2\xi_1^3\beta_{1x}GH+2C\xi_1^3\tau_yU_\xi+C(\xi_1\xi_{1yt}x+\xi_{2yt})-(\xi_{1t}x+\xi_{2t})(\xi_1\alpha_{1x}+C\xi_{1y})\\&+\xi_1^2\alpha_{1xx}-\xi_1\xi_{1t}\alpha_1+\xi_1^2\alpha_{1t}+C\xi_1^2\alpha_1=0,\end{aligned} \tag{81}$$

$$\xi_1^2(\xi_{1y}x+\xi_{2y})U_\xi+\xi_1^2\tau_yU_\tau+\xi_1(\xi_1\eta_y-\beta_1)G_\xi+2\xi_1\xi_{1y}U-\beta_{1x}G+\alpha_{3y}-\alpha_{1x}=0, \tag{82}$$

To vanish the coefficient of $H_{\xi\xi}$ in Eq.(81) we have

$$-2C\xi_1^3\tau_y=0, \tag{83}$$

which means that there exist two different cases because of the different choices of the constant C:

(a) $C\neq 0$.

If $C\neq 0$, the determining equations are

$$\begin{aligned}&-\xi_1(\xi_{1tt}x+\xi_{2tt})+2\xi_{1t}^2x+2\xi_{1t}\xi_{2t}+2\alpha_{3x}\xi_1^2=0,\\&\xi_1^3\tau_y=0,\\&-2\xi_1^3(C\xi_{2y}x+C\xi_{1y}-\xi_1\alpha_{1x})=0,\\&2C\xi_1^3(\beta_1\xi_1-\eta_y)=0,\\&-2\xi_1^2(C\xi_{1y}-\xi_1\alpha_{1x})=0,\\&2\xi_1^3\beta_{1x}=0,\\&-\xi_1\beta_{1x}(\xi_{1t}x+\xi_{2t})+C\xi_1^2\beta_1(1-\xi_1^2)\\&+\xi_1^2(\beta_{1xx}+\beta_{1t})=0,\\&2C\xi_1^3\tau_y=0,\\&C(\xi_1\xi_{1yt}x+\xi_{2yt})-(\xi_{1t}x+\xi_{2t})(\xi_1\alpha_{1x}+C\xi_{1y})\\&+\xi_1^2(\alpha_{1xx}+\alpha_{1t}+C\alpha_1)-\xi_1\xi_{1t}\alpha_1=0,\\&\xi_1^2(\xi_{1y}x+\xi_{2y})=0,\\&\xi_1^2\tau_y=0,\\&\xi_1(\xi_1\eta_y-\beta_1)=0,\\&2\xi_1\xi_{1y}=0,\\&-\beta_{1x}=0,\\&\alpha_{3y}-\alpha_{1x}=0.\end{aligned} \tag{84}$$

The general solutions are as follows:

$$\begin{aligned}&\xi_1=1,\quad \tau=t+t_0,\quad \eta=\eta(y),\\&\beta_1=\frac{\mathrm{d}\eta}{\mathrm{d}y},\quad \xi_2=\xi_2(t)\\&\alpha_1=-\frac{1}{2}\frac{\mathrm{d}\xi_2}{\mathrm{d}t},\quad \alpha_3=-\frac{1}{2}\frac{\mathrm{d}^2\xi_2}{\mathrm{d}t^2}x+a_3(t),\end{aligned} \tag{85}$$

where η is an arbitrary function of y, ξ_2 and a_3 are arbitrary functions of t, and t_0 is an arbitrary constant.

In summary, the following theorem is obtained.

Theorem 4 If $\{G=G(x,y,t),\ H=H(x,y,t),\ U=U(x,y,t)\}$ is a solution of the GBK system (1)–(3) when $A=0$, $B=0$ and $C\neq 0$, then so is

$$\begin{aligned}g&=\frac{\mathrm{d}\eta}{\mathrm{d}y}G(x+\xi_2,\eta,t+t_0),\\h&=-\frac{1}{2}\frac{\mathrm{d}\xi_2}{\mathrm{d}t}H(x+\xi_2,\eta,t+t_0),\\u&=-\frac{1}{2}\frac{\mathrm{d}^2\xi_2}{\mathrm{d}t^2}x+a_3(t)+U(x+\xi_2,\eta,t+t_0),\end{aligned} \tag{86}$$

with ξ_2 and a_3 being arbitrary functions of t, η being an arbitrary function of y, and t_0 being arbitrary constant.

By setting

$$\begin{aligned}&\xi_2=\epsilon X_0(t),\quad \eta(y)=y+\epsilon Y(y),\\&t_0=-\epsilon t_0,\quad a_3(t)=\epsilon a(t),\end{aligned} \tag{87}$$

with ϵ being an infinitesimal parameter, the symmetry algebra in this case can be obtained as follows:

$$\begin{aligned}g=&\ G+\epsilon(X_0(t)G_x+Y(y)G_y+t_0G_t),\\h=&\ H+\epsilon(X_0(t)H_x+Y(y)H_y+t_0H_t-\frac{1}{2}X_{0t}(t)),\\u=&\ U+\epsilon(X_0(t)U_x+Y(y)U_y+t_0U_t\\&+\frac{1}{2}X_{0tt}(t)x+a(t)).\end{aligned} \tag{88}$$

(b) $C=0$.

If $C=0$, the GBK system is reduced to the usual BK system

$$h_t-h_{xx}+2hh_x+u_x=0, \tag{89}$$

$$g_t+2(gh)_x+g_{xx}=0, \tag{90}$$

$$u_y-g_x=0, \tag{91}$$

then substituting expression (22) and the known results into Eq.(89)–Eq.(91) yields

$$-\xi_1(\xi_{1tt}x+\xi_{2tt})+2\xi_{1t}^2x+2\xi_{1t}\xi_{2t}+2\alpha_{3x}\xi_1^2=0, \tag{92}$$

$$\begin{aligned}&2\xi_1^4\alpha_{1x}H_\xi+2\xi_1^3\alpha_{1x}H+2\xi_1^3\beta_{1x}G_\xi+[-\xi_1\beta_{1x}(\xi_{1t}x+\xi_{2t})+\xi_1^2(\beta_{1xx}+\beta_{1t})]G\\&+2\xi_1^3\beta_{1x}GH-\xi_1\alpha_{1x}(\xi_{1t}x+\xi_{2t})+\xi_1^2\alpha_{1xx}-\xi_1\xi_{1t}\alpha_1+\xi_1^2\alpha_{1t}=0,\end{aligned} \tag{93}$$

$$\xi_1^2(\xi_{1y}x+\xi_{2y})U_\xi+\xi_1^2\tau_yU_\tau+\xi_1(\xi_1\eta_y-\beta_1)G_\xi+2\xi_1\xi_{1y}U-\beta_{1x}G+\alpha_{3y}-\alpha_{1x}=0, \tag{94}$$

with the determining equations

$$\begin{aligned}&-\xi_1(\xi_{1tt}x+\xi_{2tt})+2\xi_{1t}^2x+2\xi_{1t}\xi_{2t}+2\alpha_{3x}\xi_1^2=0,\\&2\xi_1^4\alpha_{1x}=0,\\&2\xi_1^3\beta_{1x}=0,\\&-\xi_1\beta_{1x}(\xi_{1t}x+\xi_{2t})+\xi_1^2(\beta_{1xx}+\beta_{1t})=0,\\&-\xi_1\alpha_{1x}(\xi_{1t}x+\xi_{2t})+\xi_1^2\alpha_{1xx}-\xi_1\xi_{1t}\alpha_1+\xi_1^2\alpha_{1t}=0,\\&\xi_1^2(\xi_{1y}x+\xi_{2y})=0,\\&\xi_1^2\tau_y=0,\\&\xi_1(\xi_1\eta_y-\beta_1)=0,\\&2\xi_1\xi_{1y}=0,\\&-\beta_{1x}=0,\\&\alpha_{3y}-\alpha_{1x}=0,\end{aligned} \tag{95}$$

which have the general solutions:

$$\begin{aligned}&\alpha_1=0,\quad \tau=\tau(t),\quad \tau_t=\xi_1^2\\&\xi_2=\xi_2(t),\quad \beta_1=\frac{\mathrm{d}\eta}{\mathrm{d}y}\xi_1,\end{aligned}$$

$$\begin{aligned}\alpha_3=&\frac{\xi_1\xi_{1tt}x+2\xi_1\xi_{2tt}-2\xi_{1t}^2x-4\xi_{1t}\xi_{2t}}{4\xi_1^2}\\&+a_3(t),\end{aligned} \tag{96}$$

where τ, ξ_1, ξ_2 and a_3 are arbitrary functions of t.

Theorem 5 If $\{G=G(x,y,t),\ H=H(x,y,t),\ U=U(x,y,t)\}$ is a solution of the GBK system (1)–(3) when $A=B=C=0$, so is $\{g,h,u\}$ with

$$\begin{aligned}g=&\ \frac{\mathrm{d}\eta}{\mathrm{d}y}\xi_1G(\xi_1x+\xi_2,\eta,\tau),\\h=&\ -\frac{\xi_{1t}x+\xi_{2t}}{2\xi_1}+H(\xi_1x+\xi_2,\eta,\tau),\\u=&\ \frac{\xi_1\xi_{1tt}x+2\xi_1\xi_{2tt}-2\xi_{1t}^2x-4\xi_{1t}\xi_{2t}}{4\xi_1^2}\\&+a_3(t)+\xi_1^2U(\xi_1x+\xi_2,\eta,\tau),\end{aligned} \tag{97}$$

where τ, ξ_2 and a_3 are arbitrary functions of t, $\xi_1^2=\tau_t$ and η is an arbitrary function of y.

By setting

$$\begin{aligned}&\xi_1(t)=\sqrt{\tau_t},\quad \xi_2(t)=\epsilon t_2(t),\\&\eta(y)=y+\epsilon Y(y),\quad \tau(t)=t+\epsilon T(t),\\&a_3(t)=\epsilon a(t),\end{aligned} \tag{98}$$

with ϵ being an infinitesimal parameter, the relevant generators of the symmetry algebra can be obtained as follows:

$$\begin{aligned}g=&\ G+\epsilon((xt_1(t)+t_2(t))G_x+Y(y)G_y+T(t)G_t+(t_1(t)+Y_y(y))G),\\h=&\ H+\epsilon((xt_1(t)+t_2(t))H_x+Y(y)H_y+T(t)U_t+t_1(t)H-\frac{1}{2}xt_{1t}(t)-\frac{1}{2}t_{2t}(t)),\\u=&\ U+\epsilon((xt_1(t)+t_2(t))U_x+Y(y)U_y+T(t)U_t+2t_1(t)U\\&+\frac{1}{4}t_{1tt}(t)x^2+\frac{1}{2}t_{2t}(t)x+a(t)).\end{aligned} \tag{99}$$

The result for this case is just that for case e in the last section. Furthermore, the finite transformations given in Theorem 5 are related to the non-Lie sector of the symmetry group.

4. Summary and discussion

In this paper, using the Lie symmetry method developed by Lou, the general Lie point symmetries, the generators of the Lie point symmetry algebra, of the GBK are studied. It is found that the Lie point symmetry algebra structures for five different cases are quite different.

In principle, once the Lie point symmetry algebra of a model is known, one can find its Lie point symmetry group constructed by finite symmetry transformations by the Lie's first fundamental theorem. However, the results obtained by Lie's first fundamental theorem are difficult not only to acquire but also to understand. In this paper, we solve the problem via a direct method established by Lou and Ma. Using the direct method, we successfully find not only all possible Lie point symmetry groups for five different cases of the GBK system but also the non-Lie point symmetry group.

Acknowledgments

The author is indebted to Professor S. Y. Lou and Dr X. Y. Tang for helpful discussions.

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