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Unified symmetry of the nonholonomic system of non-Chetaev type with unilateral constraints in event space

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This paper studies the unified symmetry of a nonholonomic system of non-Chetaev type with unilateral constraints in event space under infinitesimal transformations of group. Firstly, it gives the differential equations of motion of the system. Secondly, it obtains the definition and the criterion of the unified symmetry for the system. Thirdly, a new conserved quantity, besides the Noether conserved quantity and the Hojman conserved quantity, is deduced from the unified symmetry of a nonholonomic system of non-Chetaev type with unilateral constraints. Finally, an example is given to illustrate the application of the results.

Keywords: event space, unilateral constraint, unified symmetry, conserved quantity
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1. Introduction

The theory of symmetries and conserved quantities is a very important research field in mathematics and physics, especially in analytical mechanics. There are three main methods used to seek the conserved quantities of a mechanical system: namely, the Noether symmetry,[1−3] the Lie symmetry[4−6] and the form invariance.[7−10] As to the conserved quantity, there are mainly the Noether conserved quantity,[1−3] the Hojman conserved quantity[11−14] and a new type of conserved quantity.[15−17] Previous studies on symmetries and conserved quantities are all carried out from a certain kind of symmetry to seek a type of conserved quantity. Recently, Mei *et al* gave a unified symmetry of a Lagrange system[18] that could lead to the three kinds of conserved quantities mentioned above. Subsequently, significant achievements have been obtained in the study of this method.[19−21]

The study of symmetry and conserved quantities has become a popular subject in event space. In recent years, some important results have been obtained in event space.[22−28] In this paper, we generalize Mei's results to a nonholonomic system of non-Chetaev type with unilateral constraints in event space and obtain the three kinds of conserved quantities mentioned above.

2. Differential equations of motion

Here we consider a mechanical system whose configuration is determined by the n generalized coordinates $q_s(s = 1, \ldots, n)$. Let its motion be subjected to the g ideal nonlinear unilateral nonholonomic constraints of non-Chetaev type

$$\phi_\beta(q_s, \dot{q}_s, t) \geq 0, \quad (\beta = 1, \ldots, g). \tag{1}$$

Limitations of the nonholonomic constraints (1) added on the virtual displacements are

$$\psi_{\beta s}(q_s, \dot{q}_s, t)\delta q_s = 0, \quad (\beta = 1, \ldots, g), \tag{2}$$

the repeated subscripts represent the summation. Generally speaking, $\psi_{\beta s}(\beta = 1, \ldots, g)$ have no relations with $\partial\phi_\beta/\partial\dot{q}_s$. If $\psi_{\beta s} = \partial\phi_\beta/\partial\dot{q}_s$, constraints of non-Chetaev type become constraints of Chetaev type. Then, the differential equations of the system can be expressed in the form of configuration space

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = Q_s + \lambda_\beta \psi_{\beta s}, \\ &\lambda_\beta \geq 0, \quad \phi_\beta \geq 0, \quad \lambda_\beta \phi_\beta = 0, \quad (\beta = 1, \ldots, g), \end{aligned} \tag{3}$$

where $L = L(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ is the Lagrangian, $Q_s = Q_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ are the nonpotential generalized forces, λ_β are the multipliers of constraint.

We construct a space $\boldsymbol{R}^{n+1}$ (event space) in which the coordinates of a point are $q_s (s = 1, \ldots, n)$ and t. We introduce the notion

$$x_1 = t, \quad x_{s+1} = q_s, \quad (s = 1, \ldots, n), \tag{4}$$

then all the variables $x_\alpha (\alpha = 1, \ldots, n+1)$ may be given as a function of some parameter τ. Let $x_\alpha = x_\alpha(\tau)$ be such curves of class C^2 that $\mathrm{d}x_\alpha/\mathrm{d}\tau = x'_\alpha$ where the x'_α are not all zero simultaneously. We set $\dot{x}_\alpha = \mathrm{d}x_\alpha/\mathrm{d}t = x'_\alpha/x'_1$, where $\dot{x}_\alpha$ denotes the derivative with respect to t and x'_α denotes the derivative with respect to τ. In event space, the Lagrangian $\Lambda(x_\alpha, x'_\alpha)$ and the nonpotential generalized forces P_α are defined by the following equations

$$\Lambda(x_\alpha, x'_\alpha) = x'_1 L(x_\alpha, x'_2/x'_1, \ldots, x'_{n+1}/x'_1), \tag{5}$$

$$P_1 = -Q_s x'_{x+1},$$

$$P_{s+1} = x'_1 Q_s(x_\alpha, x'_2/x'_1, \ldots, x'_{n+1}/x'_1). \tag{6}$$

Equations (1) and (2) can be expressed in the form

$$\Phi_\beta(x_\alpha, x'_\alpha) \geq 0, \quad (\beta = 1, \ldots, g), \quad (\alpha = 1, \ldots, n+1), \tag{7}$$

$$\Psi_{\beta\alpha}(x_\alpha, x'_\alpha)\delta x_\alpha = 0, \quad (\beta = 1, \ldots, g), \quad (\alpha = 1, \ldots, n+1), \tag{8}$$

then the motion of the mechanical system can be described by the differential equations in event space as follows:[29]

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial \Lambda}{\partial x'_\alpha} - \frac{\partial \Lambda}{\partial x_\alpha} = P_\alpha + \lambda_\beta \Psi_{\beta\alpha},$$

$$\lambda_\beta \geq 0, \quad \Phi_\beta \geq 0, \quad \lambda_\beta \Phi_\beta = 0,$$

$$(\beta = 1, \ldots, g), (\alpha = 1, \ldots, n+1). \tag{9}$$

Suppose that the system is not singular. If the system is constrained, i.e. constrains (7) take the sign of equality, we have

$$\Phi_\beta = 0, (\beta = 1, \ldots, g), \tag{10}$$

λ_β can be expressed as function of x_α and x'_α. Thus, Eq.(9) can be expressed as follows:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial \Lambda}{\partial x'_\alpha} - \frac{\partial \Lambda}{\partial x_\alpha} = P_\alpha + \Pi_\alpha, \quad (\alpha = 1, \ldots, n+1), \tag{11}$$

where

$$\Pi_\alpha = \lambda_\beta \Psi_{\beta\alpha} = \Pi_\alpha(x_\alpha, x'_\alpha). \tag{12}$$

Though $(n+1)$ equations of Eq.(11) are not respective, we can find the last n x''_{s+1}. In summary

$$x''_{s+1} = A_s(x_\alpha, x'_\alpha, x''_1), \quad (\alpha = 1, \ldots, n+1). \tag{13}$$

If the system is free from the constraints, that is, the sign of inequality in constraints (7) holds strictly, we have

$$\Phi_\beta(x_\alpha, x'_\alpha) > 0, \quad (\beta = 1, \ldots, g), \quad (\alpha = 1, \ldots, n+1). \tag{14}$$

The Eq.(9) can be expressed as follows

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial \Lambda}{\partial x'_\alpha} - \frac{\partial \Lambda}{\partial x_\alpha} = P_\alpha, \quad (\alpha = 1, \ldots, n+1). \tag{15}$$

Though $(n+1)$ equations of Eq.(15) are not respective, we can find the last n x''_{s+1}. In summary

$$x''_{s+1} = B_s(x_\alpha, x'_\alpha, x''_1), \quad (\alpha = 1, \ldots, n+1), \tag{16}$$

3. Definition and criterion of unified symmetry

Definition If a symmetry is a Noether one as well as a Lie one and is also form invariant at one time, the symmetry is called a unified symmetry.

Take the following infinitesimal transformations:

$$\tau^* = \tau, \quad x^*_\alpha(\tau^*) = x_\alpha(\tau) + \varepsilon\xi_\alpha(x_\alpha, x'_\alpha), \quad (\alpha = 1, \ldots, n+1), \tag{17}$$

where ε is an infinitesimal parameter and ξ_α are infinitesimal generators.

For the nonholonomic system of non-Chetaev type with unilateral constraints in event space, the Noether identity can be written in the form

$$\tilde{X}^{(1)}(\Lambda) + (P_\alpha + \Pi_\alpha)\xi_\alpha + \frac{\bar{\mathrm{d}}}{\mathrm{d}\tau}G_{\mathrm{N}} = 0, \quad \Psi_{\beta\alpha}\xi_\alpha = 0, \quad \Phi_\beta = 0;$$

$$\tilde{X}^{(1)}(\Lambda) + P_\alpha\xi_\alpha + \frac{\bar{\mathrm{d}}}{\mathrm{d}\tau}G_{\mathrm{N}} = 0, \quad \Phi_\beta > 0. \tag{18}$$

The determining equations of the Lie symmetry are

$$\begin{aligned}&\tilde{X}^{(2)}\left[\tilde{E}_\alpha(\Lambda)\right]-\tilde{X}^{(1)}(P_\alpha+\Pi_\alpha)=0,\quad \tilde{X}^{(1)}\left[\Phi_\beta(x_\alpha,x'_\alpha)\right]=0,\quad \Psi_{\beta\alpha}\xi_\alpha=0,\quad \Phi_\beta=0;\\&\tilde{X}^{(2)}\left[\tilde{E}_\alpha(\Lambda)\right]-\tilde{X}^{(1)}(P_\alpha)=0,\quad \Phi_\beta>0.\end{aligned}\tag{19}$$

The criterion equations of the form invariance are

$$\begin{aligned}&\tilde{X}_\alpha\left[\tilde{X}^{(1)}(\Lambda)\right]-\tilde{X}^{(1)}(P_\alpha+\Pi_\alpha)=0,\quad \tilde{X}^{(1)}\left[\Phi_\beta(x_\alpha,x'_\alpha)\right]=0,\quad \Phi_\beta=0;\\&\tilde{E}_\alpha\left[\tilde{X}^{(1)}(\Lambda)\right]-\tilde{X}^{(1)}(P_\alpha)=0,\quad \Phi_\beta>0,\end{aligned}\tag{20}$$

where

$$\begin{aligned}\tilde{X}^{(1)}&=\xi_\alpha\frac{\partial}{\partial x_\alpha}+\frac{\bar{\mathrm{d}}}{\mathrm{d}\tau}\xi_\alpha\frac{\partial}{\partial x'_\alpha},\\\tilde{X}^{(2)}&=\tilde{X}^{(1)}+\frac{\bar{\mathrm{d}}}{\mathrm{d}\tau}\frac{\bar{\mathrm{d}}}{\mathrm{d}\tau}\xi_\alpha\frac{\partial}{\partial x''_\alpha},\\\tilde{E}_\alpha&=\frac{\bar{\mathrm{d}}}{\mathrm{d}\tau}\frac{\partial}{\partial x'_\alpha}-\frac{\partial}{\partial x_\alpha},\end{aligned}\tag{21}$$

and

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}\tau}=x'_\alpha\frac{\partial}{\partial x_\alpha}+x''_1\frac{\partial}{\partial x'_1}+A_s\frac{\partial}{\partial x'_{s+1}},\quad \Phi_\beta=0;$$

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}\tau}=x'_\alpha\frac{\partial}{\partial x_\alpha}+x''_1\frac{\partial}{\partial x'_1}+B_s\frac{\partial}{\partial x'_{s+1}},\quad \Phi_\beta>0.\tag{22}$$

According to the above definition, the following result can be obtained easily.

Criterion If there exists such a gauge function $G_{\mathrm{N}}=G_{\mathrm{N}}(x_\alpha,x'_\alpha)$ that the infinitesimal generators ξ_α satisfy

$$\begin{aligned}&\left\{\tilde{X}^{(1)}(\Lambda)+(P_\alpha+\Pi_\alpha)\xi_\alpha+\frac{\bar{\mathrm{d}}}{\mathrm{d}\tau}G_{\mathrm{N}}\right\}^2+\left\{\tilde{X}^{(2)}\left[\tilde{E}_\alpha(\Lambda)\right]-\tilde{X}^{(1)}(P_\alpha+\Pi_\alpha)\right\}^2\\&+\left\{\tilde{E}_\alpha\left[\tilde{X}^{(1)}(\Lambda)\right]-\tilde{X}^{(1)}(P_\alpha+\Pi_\alpha)\right\}^2=0,\\&\tilde{X}^{(1)}(\Phi_\beta)=0,\quad \Psi_{\beta\alpha}\xi_\alpha=0,\quad \Phi_\beta=0;\\&\left\{\tilde{X}^{(1)}(\Lambda)+P_\alpha\xi_\alpha+\frac{\bar{\mathrm{d}}}{\mathrm{d}\tau}G_{\mathrm{N}}\right\}^2+\left\{\tilde{X}^{(2)}\left[\tilde{E}_\alpha(\Lambda)\right]-\tilde{X}^{(1)}(P_\alpha)\right\}^2\\&+\left\{\tilde{E}_\alpha\left[\tilde{X}^{(1)}(\Lambda)\right]-\tilde{X}^{(1)}(P_\alpha)\right\}^2=0,\\&\Phi_\beta>0,\end{aligned}\tag{23}$$

then the symmetry is a unified symmetry of the nonholonomic system of non-Chetaev type with unilateral constraints in event space.

4. Conserved quantities deduced from unified symmetry

The unified symmetry of the nonholonomic system of the non-Chetaev type with unilateral constraints in event space can lead to all of the Noether conserved quantity, the Hojman conserved quantity and the new conserved quantity under certain conditions.

Proposition 1 For the nonholonomic system of the non-Chetaev type with unilateral constraints in event space, the unified symmetry will lead to the Noether conserved quantity

$$I_{\mathrm{N}}=\frac{\partial\Lambda}{\partial x'_\alpha}\xi_\alpha+G_{\mathrm{N}}=\mathrm{const.}\tag{24}$$

Proof Because the unified symmetry is certainly a Noether symmetry, there exists such a gauge function $G_{\mathrm{N}}=G_{\mathrm{N}}(x_\alpha,x'_\alpha)$ satisfying the Noether identity (18). According to Ref.[29], the system has the con-

served quantity (24).

Proposition 2 For the nonholonomic system of the non-Chetaev type with unilateral constraints in event space, under the special infinitesimal transformation in which x_1 ($\xi_1 = 0$) remains unchanged, if there exists such a function $\mu = \mu(x_\alpha, x'_\alpha)$ that

$$\frac{\partial A_s}{\partial x'_{s+1}} + \frac{\bar{\mathrm{d}}}{\mathrm{d}\tau}\ln\mu = 0, \quad \Phi_\beta = 0;$$
$$\frac{\partial B_s}{\partial x'_{s+1}} + \frac{\bar{\mathrm{d}}}{\mathrm{d}\tau}\ln\mu = 0, \quad \Phi_\beta > 0, \tag{25}$$

then the unified symmetry will lead to the Hojman conserved quantity

$$I_{\mathrm{H}} = \frac{1}{\mu}\frac{\partial}{\partial x_{s+1}}(\mu\xi_{s+1}) + \frac{1}{\mu}\frac{\partial}{\partial x'_{s+1}}\left(\mu\frac{\bar{\mathrm{d}}}{\mathrm{d}\tau}\xi_{s+1}\right) = \text{const.} \tag{26}$$

Proof Because the unified symmetry is certainly a Lie symmetry, by using Hojman's proof method, Proposition 2 is easily verified.

Proposition 3 For the nonholonomic system of the non-Chetaev type with unilateral constraints in event space, if there exists such a function $G_{\mathrm{F}} = G_{\mathrm{F}}(x_\alpha, x'_\alpha)$ that

$$\tilde{X}^{(1)}\left\{\tilde{X}^{(1)}(\Lambda)\right\} + \tilde{X}^{(1)}(P_\alpha + \Pi_\alpha)\xi_\alpha + \frac{\bar{\mathrm{d}}}{\mathrm{d}\tau}G_{\mathrm{F}} = 0,$$
$$\Phi_\beta = 0;$$
$$\tilde{X}^{(1)}\left\{\tilde{X}^{(1)}(\Lambda)\right\} + \tilde{X}^{(1)}(P_\alpha)\xi_\alpha + \frac{\bar{\mathrm{d}}}{\mathrm{d}\tau}G_{\mathrm{F}} = 0,$$
$$\Phi_\beta > 0. \tag{27}$$

then the unified symmetry will lead to the new conserved quantity

$$I_{\mathrm{F}} = \frac{\partial\tilde{X}^{(1)}(\Lambda)}{\partial x'_\alpha}\xi_\alpha + G_{\mathrm{F}} = \text{const.} \tag{28}$$

Proof Because the unified symmetry is certainly a form invariance. When a form invariance satisfies condition (27), there is a new type of conserved quantity (28).

5. An illustrative example

The Lagrangian, the nonpotential generalized forces, the unilateral constraint equation of a nonholonomic system and limitations of the nonholonomic constraints added on the virtual displacements are, respectively,

$$L = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) + q_1, \quad Q_1 = 0, \quad Q_2 = \dot{q}_1, \tag{29}$$
$$\phi = \dot{q}_2 - t\dot{q}_1 \geq 0, \tag{30}$$
$$\delta q_1 - \delta q_2 = 0. \tag{31}$$

Let us study the unified symmetry of the system in event space.

In event space, we have

$$\Lambda = x'_1\left\{\frac{1}{2}\left[\left(\frac{x'_2}{x'_1}\right)^2 + \left(\frac{x'_3}{x'_1}\right)^2\right] + x_2\right\}, \tag{32}$$
$$P_1 = -\frac{x'_2}{x'_1}x'_3, \quad P_2 = 0, \quad P_3 = x'_2, \tag{33}$$
$$\Phi = \frac{1}{x'_1}(x'_3 - x_1x'_2) \geq 0, \tag{34}$$
$$\delta x_2 - \delta x_3 = 0. \tag{35}$$

The differential equations of motion of the system can be expressed as

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}\tau}\left(\frac{x'_2}{x'_1}\right) - x'_1 = \lambda, \quad \frac{\bar{\mathrm{d}}}{\mathrm{d}\tau}\left(\frac{x'_3}{x'_1}\right) = x'_2 - \lambda,$$
$$\lambda \geq 0, \quad \Phi \geq 0, \quad \lambda\Phi = 0, \tag{36}$$

where λ is the multiplier of constraint.

If the system is subjected to the constraint, from Eqs.(34) and (36) we have

$$\lambda = -\frac{x'_1x_1}{1+x_1}, \quad A_1 = \frac{x'^2_1}{x_1+1} + \frac{x'_2}{x'_1}x''_1,$$
$$A_2 = \frac{x'^2_1x_1}{x_1+1} + \frac{x'_3}{x'_1}x''_1 + x'_2x'_1. \tag{37}$$

If the system is free from the constraint, from Eq.(36) we obtain

$$B_1 = x'^2_1 + \frac{x'_2}{x'_1}x'', \quad B_2 = \frac{x'_3}{x'_1}x''_1 + x'_2x'_1. \tag{38}$$

Choose the special infinitesimal generators as

$$\xi_1 = 0, \quad \xi_2 = \xi_3 = 1, \tag{39}$$

it is very easy to verify that the generators (39) have unified symmetry.

Substituting Eq.(39) into Eq.(23) ,we obtain the gauge function

$$G_{\mathrm{N}} = -(x_1 + x_2), \tag{40}$$

and Proposition 1 gives

$$I_{\mathrm{N}} = \frac{x'_2}{x'_1} + \frac{x'_3}{x'_1} - x_1 - x_2 = \text{const.}, \tag{41}$$

which is a Noether conserved quantity deduced from the unified symmetry (39).

Equation (25) gives

$$2\frac{x''_1}{x'_1} + \frac{\bar{\mathrm{d}}}{\mathrm{d}\tau}\ln\mu = 0 \tag{42}$$

from which we obtain

$$\mu = (x_1')^{-2}, \tag{43}$$

$$\mu = (x_1')^{-2}\left(\frac{x_2'}{x_1'} + \frac{x_3'}{x_1'} - x_1 - x_2\right). \tag{44}$$

From Eqs.(39) and (44), by using Proposition 2, we obtain

$$I_{\rm H} = -\left(\frac{x_2'}{x_1'} + \frac{x_3'}{x_1'} - x_1 - x_2\right)^{-1} = \text{const.}, \tag{45}$$

which is a Hojman conserved quantity deduced from the unified symmetry (39).

Substituting Eq.(39) into Eq.(27) ,we obtain

$$G_{\rm F} = \frac{x_2'}{x_1'} + \frac{x_3'}{x_1'} - x_1 - x_2, \tag{46}$$

and Proposition 3 gives

$$I_{\rm F} = \frac{x_2'}{x_1'} + \frac{x_3'}{x_1'} - x_1 - x_2 = \text{const.}, \tag{47}$$

which is a new conserved quantity deduced from the unified symmetry (39).

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