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Perturbation to symmetries and Hojman adiabatic invariant for nonholonomic controllable mechanical systems with non-Chetaev type constraints*

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This paper studies the perturbation to symmetries and adiabatic invariant for nonholonomic controllable mechanical systems with non-Chetaev type constraints. It gives the exact invariants introduced by the Lie symmetries of the nonholonomic controllable mechanical system with non-Chetaev type constraints without perturbation. Based on the definition of high-order adiabatic invariants of mechanical system, the perturbation of Lie symmetries for nonholonomic controllable mechanical system with non-Chetaev type constraints with the action of small disturbances is investigated, and a new type of adiabatic invariant of system are obtained. In the end of this paper, an example is given to illustrate the application of the results.

Keywords: nonholonomic controllable mechanical system, symmetry, perturbation, adiabatic invariant

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1. Introduction

The study of adiabatic invariants has become a popular subject after Burgers first proposed adiabatic invariants which point to a special type of Hamilton system in 1917.[1] The adiabatic invariants mean that they are almost unchanged when the parameter varies very slowly and they play a very important role in the research into quasi-integrability of a mechanical system. In fact the very slowly varying parameter is equivalent to the action of a small perturbation.[2,3] Research on adiabatic invariant has become an important and active field of mechanics,[4,5] atomic and molecular physics,[6] etc., and obtained series of important results.[7−15] But the previous studies on adiabatic invariants are Noether conserved quantities. In Ref.[16], Zhang *et al* gave a type of Hojman adiabatic invariant for Lagrangian system, in this paper we shall conduct our research by the latter model.

There are plenty of dynamical problems of the controllable systems in nature as well as in engineering, many scholars study the controllable systems and obtained some important development.[17−24] But people did not study the perturbation to symmetries and adiabatic invariants of controllable mechanical systems. In this paper, we study the perturbation to symmetries and Hojman adiabatic invariants for nonholonomic controllable mechanical systems with non-Chetaev type constraints, firstly, we construct the equations of this system, and propose the concept of higher order adiabatic invariants. Secondly, we prove the conditions for existence of the exact invariants and adiabatic invariants, and give their forms. In the end of this paper, an example is given to illustrate the application of the results.

2. The differential equations of motion of the system

Assume that the configuration of the mechanical system is determined by the n generalized coordinates $q_s(s = 1, \ldots, n)$. Let its motion be subjected to the g independent nonholonomic constraints of non-Chetaev type

$$F_\beta(t, q_s, \dot{q}_s, \mu_r, \dot{\mu}_r) = 0 \quad (\beta = 1, \ldots, g), \qquad (1)$$

where $\mu_r(r = 1, \ldots, p)$ are controlled parameters. The motion of the mechanical system is controlled by using

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the controlling parameters μ_r, which are important for a mechanical system.

The condition for the constraints (1) acting on the virtual displacements is

$$\sum_{\beta=1}^{g} f_{\beta s}(t, q_s, \dot{q}_s, \mu_r, \dot{\mu}_r)\delta q_s = 0 \quad (\beta = 1, \ldots, g)\,. \quad (1)$$

Generally speaking, $f_{\beta s}$ is independent on $\partial F_\beta/\partial\dot{q}_s$, especially when

$$f_{\beta s} = \partial F_\beta/\partial\dot{q}_s, \quad (2)$$

the constraint will be nonholonomic constraints of Chetaev type.

From the d'Alembert–Lagrange principle and virtual displacements of Eq.(2) and using the method of Lagrange multiplier we can obtain the differential equation of motion of the system as

$$E_s(L) = Q_s + \sum_{\beta=1}^{g} f_{\beta s}\lambda_\beta$$
$$(s = 1, \ldots, n; \beta = 1, \ldots, g), \quad (3)$$

where $L = L(t, q_s, \dot{q}_s)$ is the Lagrangian, $Q_s = Q_s(t, q_s, \dot{q}_s)$ are the nonpotential generalized forces, E_s are the Euler operators

$$E_s = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial\dot{q}_s} - \frac{\partial}{\partial q_s}. \quad (4)$$

In general, it is supposed that the system (3) is nonsingular, i.e.

$$\det\Big(\frac{\partial^2 L}{\partial\dot{q}_s\partial\dot{q}_k}\Big) \neq 0\,, \quad (5)$$

before integrating Eq.(3), we can find λ_β as functions of t, q_s, $\dot{q}_s$, μ_r, $\dot{\mu}_r$ i.e.

$$\lambda = \lambda(t, q_s, \dot{q}_s, \mu_r, \dot{\mu}_r) \quad (6)$$

the equations of motion of the holonomic system corresponding to the nonholonomic system expressed by Eqs.(1) and (3) are

$$E_s(L) = Q_s + \Lambda_s \quad (s = 1, \ldots, n), \quad (7)$$
$$\Lambda_s = \sum_{\beta=1}^{g} f_{\beta s}\lambda_\beta. \quad (8)$$

From Eq.(7), we can find all the generalized accelerations

$$\ddot{q}_s = \alpha_s(t, q_s, \dot{q}_s, \mu_r, \dot{\mu}_r) \quad (s = 1, \ldots, n). \quad (9)$$

3. Lie symmetries and exact invariants

We first introduce infinitesimal transformation in terms of generalized coordinates and unchanged time ($\tau^0 = 0$)

$$t^* = t, \quad q_s^*(t^*) = q_s(t) + \Delta q_s \quad (s = 1, \ldots, n), \quad (10)$$

or their expansion formulas

$$t^* = t,$$
$$q_s^*(t^*) = q_s(t) + \varepsilon\xi_s^0(t, q_s, \dot{q}_s), \quad (s = 1, \ldots, n), \quad (11)$$

where ε is an infinitesimal parameter and ξ_s^0 are infinitesimal generators. Introduce an infinitesimal generator vector and its first extension

$$X^{(0)} = \xi_s^0\frac{\partial}{\partial q_s}, \quad X^{(1)} = \xi_s^0\frac{\partial}{\partial q_s} + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s^0\frac{\partial}{\partial\dot{q}_s}. \quad (12)$$

Where

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \dot{q}_k\frac{\partial}{\partial q_k} + \alpha_k\frac{\partial}{\partial\dot{q}_k}. \quad (13)$$

The invariance of Eq.(9) under the infinitesimal transformation (11) leads to the following determining equations

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s^0 = X^{(1)}(\alpha_s). \quad (14)$$

Definition 1 Under the special infinitesimal transformation with unchanged time ($\tau^0 = 0$), if the generators ξ_s^0 satisfy the determining equations (14), then the corresponding transformation is called the Lie symmetrical transformation, and the corresponding symmetries are called Lie symmetries.

Theorem 1 Under the special infinitesimal transformation with unchanged time ($\tau^0 = 0$), if the generators ξ_s^0 satisfy the determining equations (14), and there exists a gauge function $\tilde{\mu}^0 = \tilde{\mu}^0(t, q_s, \dot{q}_s, \mu_r, \dot{\mu}_r)$ satisfying the following equation:

$$\frac{\partial\alpha_s}{\partial\dot{q}_s} + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\tilde{\mu}^0 = 0, \quad (15)$$

then the exact invariants introduced by the Lie symmetries will lead to a Hojman conserved quantity

$$I_0 = \frac{1}{\tilde{\mu}^0}\frac{\partial}{\partial q_s}(\tilde{\mu}^0\xi_s^0) + \frac{1}{\tilde{\mu}^0}\frac{\partial}{\partial\dot{q}_s}\Big(\tilde{\mu}^0\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s^0\Big)$$
$$= \text{const.} \quad (16)$$

Proof Under the special infinitesimal transformation with unchanged time ($\tau^0 = 0$), where controlled parameters μ_r do not appear obviously, according to the definition, the generators ξ_s^0 have Lie symmetry, i.e. they satisfy

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s^0 = \sum_{k=1}^{n}\frac{\partial\alpha_s}{\partial q_k}\xi_k^0 + \sum_{k=1}^{n}\frac{\partial\alpha_s}{\partial\dot{q}_k}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k^0\,. \quad (17)$$

From expressions (14)–(17), by using Hojman's proof method, we have

$$\begin{aligned}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}I_0 &= \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial\xi_s^0}{\partial q_s}+\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial}{\partial\dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s^0+\frac{\bar{\mathrm{d}}}{\mathrm{d}t}X^{(0)}(\ln\tilde{\mu}^0)\\ &= \frac{\partial}{\partial\dot{q}_s}\Big[\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s^0-X_0^{(1)}(\alpha_s)\Big]\\ &\quad+X_0^{(1)}\Big(\frac{\partial\alpha_s}{\partial\dot{q}_s}+\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\tilde{\mu}^0\Big)\\ &= 0,\end{aligned}$$

thus Theorem 1 can be proved, hence I_0 is an exact invariant of the system.

4. Perturbation to symmetries and Hojman adiabatic invariant

Firstly we give the concept of higher adiabatic invariants.

Definition 2 If $I_z(t,q_s,\dot{q}_s,\mu_r,\dot{\mu}_r)$ is a physical quantity including ε for which the highest power is z in a mechanical system, and its derivative with respect to time t is directly proportional to ε^{z+1}, then I_z is a zth order adiabatic invariant of the mechanical system.

Suppose the nonholonomic mechanical system corresponding to Eq.(7) is perturbed by small quantities εW_s, then the equations of motion of the system become

$$E_s(L)=Q_s+\Lambda_s+\varepsilon W_s\quad(s=1,\ldots,n). \tag{18}$$

Equation (5) becomes

$$\begin{aligned}&\ddot{q}_s=\alpha_s(t,q_s,\dot{q}_s,\mu_r,\dot{\mu}_r)+\varepsilon\frac{M_{sk}}{D}W_k\\ &(s=1,\ldots,n),\end{aligned} \tag{19}$$

where

$$D=\det\Big[\frac{\partial^2L}{\partial\dot{q}_s\partial\dot{q}_k}\Big],$$

M_{sk} is the algebraic complement of $\partial^2L/\partial\dot{q}_s\partial\dot{q}_k$ of D.

Due to the action of εW_s, the primary symmetries and invariants of system may vary. Suppose the variation is a small perturbation based on the symmetrical transformation of the system without perturbation, and ξ_s express the generators of space after being perturbed, then

$$\xi_s=\xi_s^0+\varepsilon\xi_s^1+\varepsilon^2\xi_s^2+\cdots \tag{20}$$

The infinitesimal generator vector and its first extension are

$$X^{(0)}=\xi_s\frac{\partial}{\partial q_s}, \tag{21}$$

$$X^{(1)}=\xi_s\frac{\partial}{\partial q_s}+\frac{\bar{\mathrm{d}}}{\mathrm{d}}\xi_s\frac{\partial}{\partial\dot{q}_s}. \tag{22}$$

Substituting Eq.(20) into Eq.(22), we have

$$X_m^{(1)}=\varepsilon^m\xi_s^m\frac{\partial}{\partial q_s}+\varepsilon^m\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s^m\frac{\partial}{\partial\dot{q}_s}, \tag{23}$$

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}=\frac{\partial}{\partial t}+\dot{q}_k\frac{\partial}{\partial q_k}+\Big(\alpha_k+\varepsilon\frac{M_{kj}}{D}W_j\Big)\frac{\partial}{\partial\dot{q}_k}. \tag{24}$$

Equation(24) leads to Eq.(13) based on the symmetrical transformations of the system without perturbation, after being perturbed, Eq.(14) becomes

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s=X^{(1)}(\alpha_s)+\varepsilon X^{(1)}\Big(\frac{M_{sk}}{D}W_k\Big). \tag{25}$$

Substituting Eq.(20) into Eq.(25), and making the coefficients of ε^m equal, we obtain

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s^m=X_m^{(1)}(\alpha_s)+X_{m-1}^{(1)}\Big(\frac{M_{sk}}{D}W_k\Big), \tag{26}$$

when $m=0$, the condition $\xi_s^{-1}=0$ holds.

Theorem 2 For nonholonomic controllable mechanical systems with non-Chetaev type constraints perturbed by small quantities εW_s, if the generators ξ_s^0 satisfy the determining Eq.(26), and there exists a gauge function $\mu=\mu(t,q_s,\dot{q}_s,\mu_r,\dot{\mu}_r)$ satisfying the following equation:

$$\frac{\partial\alpha_s}{\partial\dot{q}_s}+\varepsilon\frac{\partial}{\partial\dot{q}_s}\Big(\frac{M_{sk}}{D}W_k\Big)+\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\tilde{\mu}=0, \tag{27}$$

then

$$\begin{aligned}I_z=&\sum_{m=0}^{z}\varepsilon^m\Big[\frac{1}{\tilde{\mu}}\frac{\partial}{\partial q_s}(\tilde{\mu}\xi_s^m)+\frac{1}{\tilde{\mu}}\frac{\partial}{\partial\dot{q}_s}\Big(\tilde{\mu}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s^m\Big)\Big]\\ &=\text{const.}\end{aligned} \tag{28}$$

is a zth order Hojman adiabatic invariant of the mechanical system, where $m=0$, $\tilde{\mu}=\tilde{\mu}^0$.

Proof

$$\begin{aligned}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}I_z=&\sum_{m=0}^{z}\varepsilon^m\Big[\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial\xi_s^m}{\partial q_s}+\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial}{\partial\dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s^m\\ &+\frac{\bar{\mathrm{d}}}{\mathrm{d}t}X_m^{(1)}(\ln\tilde{\mu})\Big],\end{aligned} \tag{29}$$

the generators ξ_s^m under the special infinitesimal transformation are Lie symmetrical, we have[16,25]

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}X_m^{(1)}(\ln\tilde{\mu})=X_m^{(1)}\Big(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\tilde{\mu}\Big). \tag{30}$$

After calculation, we obtain

$$\begin{aligned}&\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\Big(\frac{\partial\xi_s}{\partial q_s}+\frac{\partial}{\partial\dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\Big)-\varepsilon^m X_m^{(1)}\Big(\frac{\partial\alpha_s}{\partial\dot{q}_s}\Big)\\&-\varepsilon X^{(1)}\Big[\frac{\partial}{\partial\dot{q}_s}\Big(\frac{M_{sk}}{D}W_k\Big)\Big]\\&=\frac{\partial}{\partial\dot{q}_s}\Big[\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s-X^{(1)}(\alpha_s)-\varepsilon X^{(1)}\Big(\frac{M_{sk}}{D}W_k\Big)\Big].\quad(31)\end{aligned}$$

Substituting Eq.(20) into Eq.(31), and making the coefficients of ε^m equal, we obtain

$$\begin{aligned}&\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\Big(\frac{\partial\xi_s^m}{\partial q_s}+\frac{\partial}{\partial\dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s^m\Big)-X_m^{(1)}\Big(\frac{\partial\alpha_s}{\partial\dot{q}_s}\Big)\\&-X_{m-1}^{(1)}\Big[\frac{\partial}{\partial\dot{q}_s}\Big(\frac{M_{sk}}{D}W_k\Big)\Big]\\&=\frac{\partial}{\partial\dot{q}_s}\Big[\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s^m-X_m^{(1)}(\alpha_s)-X_{m-1}^{(1)}\Big(\frac{M_{sk}}{D}W_k\Big)\Big].\quad(32)\end{aligned}$$

Substituting Eqs.(32) and (30) into Eq.(29), using Eqs.(27) and (28), we have

$$\begin{aligned}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}I_z=&\sum_{m=0}^{z}\varepsilon^m\Big\{\frac{\partial}{\partial\dot{q}_s}\Big[\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s^m-X_m^{(1)}(\alpha_s)-X_{m-1}^{(1)}\Big(\frac{M_{sk}}{D}W_k\Big)\Big]+X_m^{(1)}\Big(\frac{\partial\alpha_s}{\partial\dot{q}_s}\Big)\\&+X_{m-1}^{(1)}\Big[\frac{\partial}{\partial\dot{q}_s}\Big(\frac{M_{sk}}{D}W_k\Big)\Big]+X_m^{(1}\Big(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\mu\Big)\Big\}\\=&\sum_{m=0}^{z}\varepsilon^m\Big\{X_{m-1}^{(1)}\Big[\frac{\partial}{\partial\dot{q}_s}\Big(\frac{M_{sk}}{D}W_k\Big)\Big]-\varepsilon X_m^{(1)}\Big[\frac{\partial}{\partial\dot{q}_s}\Big(\frac{M_{sk}}{D}W_k\Big)\Big]\Big\}\\=&\sum_{m=0}^{z}\varepsilon^m X_{m-1}^{(1)}\Big[\frac{\partial}{\partial\dot{q}_s}\Big(\frac{M_{sk}}{D}W_k\Big)\Big]-\sum_{m=0}^{z}\varepsilon^{m+1}X_m^{(1)}\Big[\frac{\partial}{\partial\dot{q}_s}\Big(\frac{M_{sk}}{D}W_k\Big)\Big]\\=&\varepsilon^0X_{-1}^{(1)}\Big[\frac{\partial}{\partial\dot{q}_s}\Big(\frac{M_{sk}}{D}W_k\Big)\Big]+\varepsilon X_0^{(1)}\Big[\frac{\partial}{\partial\dot{q}_s}\Big(\frac{M_{sk}}{D}W_k\Big)\Big]+\varepsilon^2X_1^{(1)}\Big[\frac{\partial}{\partial\dot{q}_s}\Big(\frac{M_{sk}}{D}W_k\Big)\Big]\\&+\varepsilon^3X_2^{(1)}\Big[\frac{\partial}{\partial\dot{q}_s}\Big(\frac{M_{sk}}{D}W_k\Big)\Big]+\cdots+\varepsilon^{z-1}X_{z-2}^{(1)}\Big[\frac{\partial}{\partial\dot{q}_s}\Big(\frac{M_{sk}}{D}W_k\Big)\Big]+\varepsilon^zX_{z-1}^{(1)}\Big[\frac{\partial}{\partial\dot{q}_s}\Big(\frac{M_{sk}}{D}W_k\Big)\Big]\\&-\varepsilon X_0^{(1)}\Big[\frac{\partial}{\partial\dot{q}_s}\Big(\frac{M_{sk}}{D}W_k\Big)\Big]-\varepsilon^2X_1^{(1)}\Big[\frac{\partial}{\partial\dot{q}_s}\Big(\frac{M_{sk}}{D}W_k\Big)\Big]\\&-\varepsilon^3X_2^{(1)}\Big[\frac{\partial}{\partial\dot{q}_s}\Big(\frac{M_{sk}}{D}W_k\Big)\Big]-\cdots-\varepsilon^{z-1}X_{z-2}^{(1)}\Big[\frac{\partial}{\partial\dot{q}_s}\Big(\frac{M_{sk}}{D}W_k\Big)\Big]\\&-\varepsilon^zX_{z-1}^{(1)}\Big[\frac{\partial}{\partial\dot{q}_s}\Big(\frac{M_{sk}}{D}W_k\Big)\Big]-\varepsilon^{z+1}X_z^{(1)}\Big[\frac{\partial}{\partial\dot{q}_s}\Big(\frac{M_{sk}}{D}W_k\Big)\Big]\\=&-\varepsilon^{z+1}X_z^{(1)}\Big[\frac{\partial}{\partial\dot{q}_s}\Big(\frac{M_{sk}}{D}W_k\Big)\Big]+\varepsilon^0X_{-1}^{(1)}\Big[\frac{\partial}{\partial\dot{q}_s}\Big(\frac{M_{sk}}{D}W_k\Big)\Big]=-\varepsilon^{z+1}X_z^{(1)}\Big[\frac{\partial}{\partial\dot{q}_s}\Big(\frac{M_{sk}}{D}W_k\Big)\Big],\end{aligned}$$

when $m=0$, the condition $W_s=0$ holds.

This shows that $(\bar{\mathrm{d}}/\mathrm{d}t)I_z$ is directly proportional to ε^{z+1}, so I_z is a zth order Hojman adiabatic invariant of the mechanical system.

5. Illustrative example

The Lagrangian of nonholonomic system is

$$L=\frac{1}{2}(\dot{q}_1^2+\dot{q}_2^2).\quad(33)$$

The nonpotential generalized forces are

$$Q_1=Q_2=0.\quad(34)$$

The nonholonomic constraint is

$$F_\beta=\dot{q}_2-\mu(t)\dot{q}_1=0,\quad(35)$$

where $\mu(t)$ is the controlling parameters.

Suppose that the constraint Eq.(35) is non-Chetaev type, the virtual displacement equation of the velocity space is

$$\delta q_1-\delta q_2=0.\quad(36)$$

Let us study the perturbation to symmetries and adiabatic invariant of this system.

Equation (3) give the equations of motion as follows

$$\ddot{q}_1 = \lambda, \quad \ddot{q}_2 = -\lambda. \tag{37}$$

From Eqs.(33) and (35), we obtain

$$\lambda = -\frac{\dot{\mu}}{1+\mu}\dot{q}_1. \tag{38}$$

Substituting Eq.(38) into Eq.(37), we obtain

$$\ddot{q}_1 = -\frac{\dot{\mu}}{1+\mu}\dot{q}_1, \quad \ddot{q}_2 = \frac{\dot{\mu}}{1+\mu}\dot{q}_1. \tag{39}$$

The determining Eq.(14) gives

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_1^0 = \Big(-\frac{\dot{\mu}}{1+\mu}\Big)\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_1^0, \tag{40}$$

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_2^0 = 0, \tag{41}$$

Eqs.(40) and (41) have solutions

$$\xi_1^0 = \xi_2^0 = \dot{q}_1 + \dot{q}_2. \tag{42}$$

Equation (15) gives

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\tilde{\mu}^0 = \frac{\dot{\mu}}{1+\mu}. \tag{43}$$

Equation (43) has solution

$$\tilde{\mu}^0 = \exp[\dot{q}_1 t + \dot{q}_2 t - q_1 - q_2 + \ln(1+\mu)]. \tag{44}$$

So we can obtain the following exact invariant by Theorem 1

$$I_0 = -2(\dot{q}_1 + \dot{q}_2). \tag{45}$$

Next we study the adiabatic invariants of the system. Suppose the system is perturbed by the small quantities

$$\varepsilon W_1 = -\varepsilon\dot{q}_1, \quad \varepsilon W_2 = \varepsilon\dot{q}_1. \tag{46}$$

Then Eq.(26) gives

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_1^m + \Big(\frac{\dot{\mu}}{1+\mu}\Big)\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_1^m + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_1^{m-1} = 0, \tag{47}$$

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_2^m = 0, \tag{48}$$

where $m = 0$, $\xi_1^{-1} = \xi_2^{-1} = 0$.

Equations (47) and (48) have solutions

$$\xi_1^1 = \xi_2^1 = \dot{q}_1 + \dot{q}_2. \tag{49}$$

Equation (27) gives

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\tilde{\mu} = \frac{\dot{\mu}}{1+\mu} + \varepsilon. \tag{50}$$

Equation (50) has solution

$$\tilde{\mu} = \exp[\dot{q}_1 t + \dot{q}_2 t - q_1 - q_2 + \varepsilon t + \ln(1+\mu)]. \tag{51}$$

So we can obtain the following exact invariant by Theorem 2

$$I_1 = -2(\dot{q}_1 + \dot{q}_2) - 2\varepsilon(\dot{q}_1 + \dot{q}_2). \tag{52}$$

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