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Pfaffianization of the variable-coefficient Kadomtsev–Petviashvili equation*

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This paper constructs more general exact solutions than N-soliton solution and Wronskian solution for variable-coefficient Kadomtsev–Petviashvili (KP) equation. By using the Hirota method and Pfaffian technique, it finds the Grammian determinant-type solution for the variable-coefficient KP equation (VCKP), the Wronski-type Pfaffian solution and the Gram-type Pfaffian solutions for the Pfaffianized VCKP equation.

Keywords: variable-coefficient KP equation, Pfaffian technique, Pfaffian solution
PACC: 0220, 0290

1. Introduction

There has been rising concerns with the nonlinear science. As an important part of nonlinear science, the soliton theory is widely used in many correlated subjects . One of its important application is to search for the solutions of the nonlinear partial differential equations. In the past decades, there has been significant progress in the development of various methods, including the inverse scattering transformation,[1,2] the Bäcklund transformation,[3−6] the algebra-geometric method,[7−13] the Hirota method,[14−16] the homogeneous balance method,[17] the Darboux transformation method.[18] Among them, Hirota method is regarded as one of the most efficient methods to construct the N-soliton solution for the soliton equations. In recent years, this method has been extended to successfully obtain other more general kinds of exact solutions, such as Wronskian solutions and Pfaffian solutions.[19−22]

This paper is interested in the study of variable-coefficient Kadomtsev–Petviashvili (KP) (VCKP) equation[23]

$$\begin{aligned} &u_t - h(u_{xxx} + 6uu_x + 3\alpha^2\partial^{-1}u_{yy}) \\ &- b_1u_x + k(xu_x + 2u + 2yu_y) \\ &+ \alpha b_1xu_y + 2\alpha b_1\partial^{-1}u_y = 0. \end{aligned} \tag{1}$$

Deng *et al* have discussed the Wronskian solution and N-soliton solution.[23] Here by using the Hirota method and Pfaffian technique, we find the Grammian determinant solution, the Wronski-type solution and the Gram-type Pfaffian solutions for the VCKP equation (1).

Through the transformation

$$u = 2(\ln\tau)_{xx}, \tag{2}$$

equation (1) can be transformed into the following bilinear form

$$\begin{aligned} &D_xD_t\tau\cdot\tau - h(D_x^4\tau\cdot\tau + 3\alpha^2D_y^2\tau\cdot\tau) - b_1D_x^2\tau\cdot\tau \\ &+ k(xD_x^2\tau\cdot\tau + 2\tau_x\tau + 2yD_xD_y\tau\cdot\tau) \\ &+ \alpha b_1xD_xD_y\tau\cdot\tau + 2\alpha b_1\tau_y\tau = 0, \end{aligned} \tag{3}$$

where D is the well known Hirota bilinear operator:

$$\begin{aligned} D_x^lD_y^mD_t^na\cdot b = &(\partial_x - \partial_{x'})(\partial_y - \partial_{y'})(\partial_t - \partial_{t'}) \\ &\times a(x,y,t)b(x',y',t')|_{x'=x,y'=y,t'=t}. \end{aligned} \tag{4}$$

Expanding Eq.(3), we have

$$\begin{aligned} &2(\tau_{xt}\tau - \tau_x\tau_t) - 2h(\tau_{xxxx}\tau - 4\tau_x\tau_{xxx} \\ &+ 3\tau_{xx}^2 + 3\alpha^2\tau_{yy}\tau - 3\alpha^2\tau_y^2) - 2b_1(\tau_{xx}\tau - \tau_x^2) \\ &+ 2xk(\tau_{xx}\tau - \tau_x^2) + 2k\tau_x\tau + 4yk(\tau_{xy}\tau - \tau_x\tau_y) \\ &+ 2x\alpha b_1(\tau_{xy}\tau - \tau_y\tau_x) + 2\alpha b_1\tau_y\tau = 0. \end{aligned} \tag{5}$$

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2. Grammian expression of the VCKP equation

In this section, we will show that the VCKP equation (1) has a solution in the following Grammian form:

$$\tau = \det|a_{ij}|_{1\le i,j\le N} = (1,2,\ldots,N,N^*,\ldots,2^*,1^*),$$
$$a_{ij} = (i,j^*) \equiv c_{ij} + \int^x f_i g_j \,\mathrm{d}s,$$
$$(i,j) = (i^*,j^*) = 0, \quad c_{ij} = \text{constant}, \tag{6}$$

where

$$\frac{\partial f_i}{\partial y} = -\frac{1}{\alpha}\frac{\partial^2 f_i}{\partial x^2}, \quad (i = 1,2,\ldots,N),$$
$$\frac{\partial f_i}{\partial t} = 4h\frac{\partial^3 f_i}{\partial x^3} + \frac{2k}{\alpha}y\frac{\partial^2 f_i}{\partial x^2} + xb_1\frac{\partial^2 f_i}{\partial x^2} - xk\frac{\partial f_i}{\partial x} + 2b_1\frac{\partial f_i}{\partial x}\ , \tag{7}$$

$$\frac{\partial g_j}{\partial y} = \frac{1}{\alpha}\frac{\partial^2 g_j}{\partial x^2}, \quad (j = 1,2,\ldots,N),$$
$$\frac{\partial g_j}{\partial t} = 4h\frac{\partial^3 g_j}{\partial x^3} - \frac{2k}{\alpha}y\frac{\partial^2 g_j}{\partial x^2} - xb_1\frac{\partial^2 g_j}{\partial x^2} - xk\frac{\partial g_j}{\partial x}\ . \tag{8}$$

It is easy for us to get the following differential rules:

$$\frac{\partial}{\partial x}(i,j^*) = (d_0,d_0^*,i,j^*),$$
$$\frac{\partial^2}{\partial x^2}(i,j^*) = (d_1,d_0^*,i,j^*) + (d_0,d_1^*,i,j^*), \tag{9}$$

where

$$(d_n,j^*) = \frac{\partial^n}{\partial x^n}g_j, \quad (d_n^*,i) = \frac{\partial^n}{\partial x^n}f_i,$$
$$(d_m,d_n) = (d_m^*,d_n^*) = (d_m,d_n^*) = (d_n,i) = (d_m^*,i^*) = 0.$$

Through some calculations, we also have

$$\frac{\partial}{\partial y}(i,j^*) = -\frac{1}{\alpha}[-(d_1,d_0^*,i,j^*) + (d_0,d_1^*,i,j^*)], \tag{10}$$
$$\frac{\partial}{\partial t}(i,j^*) = 4h[(d_2,d_0^*,i,j^*) - (d_1,d_1^*,i,j^*) + (d_0,d_2^*,i,j^*)] + \frac{2k}{\alpha}y[-(d_1,d_0^*,i,j^*) + (d_0,d_1^*,i,j^*)]$$
$$+ xb_1[-(d_1,d_0^*,i,j^*) + (d_0,d_1^*,i,j^*)] + b_1(d_0,d_0^*,i,j^*) - xk(d_0,d_0^*,i,j^*) + k(i,j^*). \tag{11}$$

Using Eqs.(9)–(11), we obtain

$$\tau = (\bullet),\ \ \tau_x = (d_0,d_0^*,\bullet),\ \ \tau_{xx} = (d_1,d_0^*,\bullet) + (d_0,d_1^*,\bullet),$$
$$\tau_{xxx} = (d_2,d_0^*,\bullet) + 2(d_1,d_1^*,\bullet) + (d_0,d_2^*,\bullet),$$
$$\tau_{xxxx} = (d_3,d_0^*,\bullet) + 3(d_2,d_1^*,\bullet) + 2(d_0,d_0^*,d_1,d_1^*,\bullet) + 3(d_1,d_2^*,\bullet) + (d_0,d_3^*,\bullet); \tag{12}$$
$$\tau_y = -\frac{1}{\alpha}[-(d_1,d_0^*,\bullet) + (d_0,d_1^*,\bullet)]; \tag{13}$$
$$\tau_t = 4h[(d_2,d_0^*,\bullet) - (d_1,d_1^*,\bullet) + (d_0,d_2^*,\bullet)] + \frac{2k}{\alpha}y[-(d_1,d_0^*,\bullet) + (d_0,d_1^*,\bullet)]$$
$$+ xb_1[-(d_1,d_0^*,\bullet) + (d_0,d_1^*,\bullet)] + b_1(d_0,d_0^*,\bullet) - xk(d_0,d_0^*,\bullet) + k(\bullet), \tag{14}$$

where $(\bullet)$ denotes the character $(1,2,\ldots,N,N^*,\ldots,2^*,1^*)$ common to each Pfaffian.

Substituting Eqs.(12)–(14) into Eq.(5), we get

$$-24h[(d_0,d_0^*,d_1,d_1^*,\bullet)(\bullet) - (d_1,d_1^*,\bullet)(d_0,d_0^*,\bullet) + (d_1,d_0^*,\bullet)(d_0,d_1^*,\bullet))] = 0. \tag{15}$$

It is easy for us to see that Eq.(15) is nothing but the Jacobi identity for determinants and also a special case of the Pfaffian identity

$$(d_0,d_0^*,d_1,d_1^*,\bullet)(\bullet) = (d_1,d_1^*,\bullet)(d_0,d_0^*,\bullet) - (d_1,d_0^*,\bullet)(d_0,d_1^*,\bullet) - (d_0,d_1,\bullet)(d_0^*,d_1^*,\bullet), \tag{16}$$

since

$$(d_0,d_1,\bullet)(d_0^*,d_1^*,\bullet) \equiv 0. \tag{17}$$

Therefore we have proven that the Pfaffian (6) solves the bilinear equation (3).

3. Wronski-type Pfaffian solution

Here we introduce the coupled system for VCKP:

$$D_xD_t\tau\cdot\tau - h(D_x^4\tau\cdot\tau + 3\alpha^2D_y^2\tau\cdot\tau) - b_1D_x^2\tau\cdot\tau$$
$$+ k(xD_x^2\tau\cdot\tau + 2\tau_x\tau + 2yD_xD_y\tau\cdot\tau) + \alpha b_1xD_xD_y\tau\cdot\tau + 2\alpha b_1\tau_y\tau = -24h\tilde{\sigma}\sigma, \tag{18}$$

$$[2hD_x^3 + D_t - 6\alpha hD_xD_y - (2ky - x\alpha b_1)D_y + (xk - b_1)D_x]\sigma\cdot\tau = 0, \tag{19}$$

$$[2hD_x^3 + D_t + 6\alpha hD_xD_y - (2ky - x\alpha b_1)D_y + (xk - b_1)D_x]\tilde{\sigma}\cdot\tau = 0. \tag{20}$$

Later we will easily see that Eqs.(18), (19) and (20) are the Pfaffianization of the VCKP. Through the dependent variable transformations,

$$u = 2(\log\tau)_{xx}, \quad v = \sigma/\tau, \quad v = \tilde{\sigma}/\tau,$$

the equations above can be written as

$$u_t - h(u_{xxx} + 6uu_x + 3\alpha^2\partial^{-1}u_{yy}) - b_1u_x + k(xu_x + 2u + 2yu_y) + \alpha b_1xu_y + 2\alpha b_1\partial^{-1}u_y + 24v\tilde{v} = 0,$$
$$v_t + 2hv_{xxx} - (2ky - x\alpha b_1)v_y + (xk - b_1 + 6uh)v_x - 6\alpha h\Big(v_{xy} + v\int^x u_y\mathrm{d}x\Big) = 0,$$
$$v_t + 2hv_{xxx} - (2ky - x\alpha b_1)v_y + (xk - b_1 + 6uh)v_x + 6\alpha h\Big(v_{xy} + v\int^x u_y\mathrm{d}x\Big) = 0, \tag{21}$$

which reduces to the VCKP equation when $v = \tilde{v} = 0$.

Here we will propose Wronski-type Pfaffian solution for the Pfaffianized VCKP.

The Wronski-type Pfaffian solution for the Pfaffianized VCKP (18) and (19) is given by

$$\tau^W = (0, 1, \ldots, 2N-1), \tag{22}$$

where

$$\sigma^W = (0, 1, \ldots, 2N-3),$$
$$\tilde{\sigma}^W = (0, 1, \ldots, 2N+1), \tag{23}$$
$$\frac{\partial}{\partial x_n}(i,j) = (i+n, j) + (i, j+n),$$
$$(i, j = 0, 1, \ldots, 2N+1; \quad n = 0, 1, 2, \ldots). \tag{24}$$

The entries of one special kind of the Wronski-type Pfaffian solution are expressed as:

$$(i,j) = \sum_{k=1}^{M}[\Phi_k^{(i)}\Psi_k^{(j)} - \Phi_k^{(j)}\Psi_k^{(i)}], \tag{25}$$

where

$$\frac{\partial}{\partial x_n}\Phi_k = \Phi_k^{(n)}, \quad \frac{\partial}{\partial x_n}\Psi_k = \Psi_k^{(n)}, \tag{26}$$

so we will obtain a differential rule for the Wronski-type Pfaffian

$$\frac{\partial}{\partial x_n}(i_0, i_1, \ldots, i_{2N-1}) = \sum_{k=0}^{2N-1}(i_0, \ldots, i_k + n, \ldots, i_{2N-1}). \tag{27}$$

We also assume that

$$\frac{\partial}{\partial y}(i,j) = -\frac{1}{\alpha}[(i+2, j) + (i, j+2)],$$
$$\frac{\partial}{\partial t}(i,j) = 4h[(i+3, j) + (i, j+3)] + \Big(\frac{2k}{\alpha}y + xb_1\Big)[(i+2, j) + (i, j+2)] + (-xk + b_1)$$
$$\times[(i+1, j) + (i, j+1)] - (2N - i)b_1(i+1, j) + (2N - j)b_1(i, j+1). \tag{28}$$

From Eq.(28), we come up with the following differential rules

$$\tau_t^W = 4h[(\widehat{2N-4}, 2N-2, 2N-1, 2N) - (\widehat{2N-3}, 2N-1, 2N+1) + (\widehat{2N-2}, 2N+2)]$$

$$-\frac{2k}{\alpha}y[(\widehat{2N-3},2N-1,2N)-(\widehat{2N-2},2N+1)]-xk(\widehat{2N-2},2N)-N(2N-1)k(\widehat{2N-1})$$
$$-xb_1[(\widehat{2N-3},2N-1,2N)-(\widehat{2N-2},2N+1)]+b_1(\widehat{2N-2},2N), \tag{29}$$

$$\tau_{tx}^W=4h[(\widehat{2N-5},2N-3,2N-2,2N-1,2N)-(\widehat{2N-3},2N,2N+1)+(\widehat{2N-2},2N+3)]$$
$$-\frac{2k}{\alpha}y[(\widehat{2N-4},2N-2,2N-1,2N)-(\widehat{2N-2},2N+2)]-xk[(\widehat{2N-3},$$
$$2N-1,2N)+(\widehat{2N-2},2N+1)]-k(\widehat{2N-2},2N)-N(2N-1)k(\widehat{2N-2},2N)$$
$$-xb_1[(\widehat{2N-4},2N-2,2N-1,2N)-(\widehat{2N-2},2N+2)]-b_1[(\widehat{2N-3},2N-1,2N)$$
$$-(\widehat{2N-2},2N+1)]+b_1[(\widehat{2N-3},2N-1,2N)+(\widehat{2N-2},2N+1)]. \tag{30}$$

Substituting Eqs.(29) and (30) into Eq.(18), we find that Eq.(18) is equivalent to

$$-24h[(\widehat{2N-1})(\widehat{2N-3},2N,2N+1)-(\widehat{2N-2},2N)(\widehat{2N-3},2N-1,2N+1)$$
$$+(\widehat{2N-3},2N-1,2N)(\widehat{2N-2},2N+1)]=-24h(\widehat{2N-3})(\widehat{2N+1}). \tag{31}$$

Equation (31) is nothing but the Pfaffian identity.

Substituting Eqs.(29) and (30) into Eq.(19), we find that Eq.(19) can be written as

$$(\widehat{2N-4},2N)(\widehat{2N-1})-(\widehat{2N-3})(\widehat{2N-4},2N-2,2N-1,2N)$$
$$-(\widehat{2N-4},2N-1)(\widehat{2N-2},2N)+(\widehat{2N-4},2N-2)(\widehat{2N-3},2N-1,2N)=0, \tag{32}$$

which is nothing but the Pfaffian identity. In the same way, we can verify that the Wronski-type Pfaffian (22) and (23) satisfy Eq.(20).

Thus we have proved that the Pfaffianized VCKP (18)–(20) has the Wronski-type Pfaffian solution.

4. Gram-type Pfaffian solution

In this section, we discuss another Pfaffian solution, which is the Gram-type Pfaffian solution for the Pfaffianized VCKP Eqs.(18)–(20).

The Gram-type Pfaffian solution is proposed as

$$\tau^{\mathrm{G}}=(1,2,\ldots,2N),$$
$$(i,j)=c_{ij}+\int^x(f_ig_j-f_jg_i)\mathrm{d}s, \tag{33}$$

where

$$\frac{\partial f_i}{\partial y}=-\frac{1}{\alpha}\frac{\partial^2 f_i}{\partial x^2},\quad \frac{\partial g_j}{\partial y}=\frac{1}{\alpha}\frac{\partial^2 g_j}{\partial x^2},$$
$$\frac{\partial f_i}{\partial t}=4h\frac{\partial^3 f_i}{\partial x^3}+\frac{2k}{\alpha}y\frac{\partial^2 f_i}{\partial x^2}+xb_1\frac{\partial^2 f_i}{\partial x^2}$$
$$-xk\frac{\partial f_i}{\partial x}+2b_1\frac{\partial f_i}{\partial x},$$
$$\frac{\partial g_j}{\partial t}=4h\frac{\partial^3 g_j}{\partial x^3}-\frac{2k}{\alpha}y\frac{\partial^2 g_j}{\partial x^2}-xb_1\frac{\partial^2 g_j}{\partial x^2}-xk\frac{\partial g_j}{\partial x},$$
$$(i,j=1,2,\ldots,2N). \tag{34}$$

By using the above differential rules of Gram-type Pfaffian, we get

$$\frac{\partial}{\partial x}(i,j)=(c_0,d_0,i,j),$$
$$\frac{\partial^2}{\partial x^2}(i,j)=(c_0,d_1,i,j)+(c_1,d_0,i,j), \tag{35}$$

where

$$\begin{cases}(c_n,j)=\dfrac{\partial^n}{\partial x^n}g_j,\quad (d_n,i)=\dfrac{\partial^n}{\partial x^n}f_i,\\ (d_m,d_n)=(c_m,c_n)=(d_m,c_n)=(d_n,i)=(d_m^*,i^*)=0,\end{cases}\qquad (m,n=0,1,\ldots).$$

We also get the expression of σ^{G} and $\tilde{\sigma}^{\mathrm{G}}$

$$\sigma^{\mathrm{G}}=-(c_0,c_1,1,2,\ldots,2N),\quad \tilde{\sigma}^{\mathrm{G}}=(d_0,d_1,1,2,\ldots,2N). \tag{36}$$

Through direct calculation, we also have

$$\frac{\partial}{\partial y}(i,j) = -\frac{1}{\alpha}[(c_0,d_1,i,j)-(c_1,d_0,i,j)], \tag{37}$$

$$\begin{aligned}\frac{\partial}{\partial t}(i,j) &= 4h[(c_0,d_2,i,j)-(c_1,d_1,i,j)+(c_2,d_0,i,j)]+\frac{2k}{\alpha}y[(c_0,d_1,i,j)-(c_1,d_0,i,j)]\\ &\quad + xb_1[(c_0,d_1,i,j)-(c_1,d_0,i,j)]+b_1(c_0,d_0,i,j)-xk(c_0,d_0,i,j)+k(i,j).\end{aligned} \tag{38}$$

Thus it is easy for us to have the following equation:

$$\begin{aligned}\tau_t{}^{\mathrm{G}} &= 4h[(c_0,d_2,1,\ldots,2N)-(c_1,d_1,1,\ldots,2N)+(c_2,d_0,1,\ldots,2N)]\\ &\quad+\frac{2k}{\alpha}y[(c_0,d_1,1,\ldots,2N)-(c_1,d_0,1,\ldots,2N)]+xb_1[(c_0,d_1,1,\ldots,2N)-(c_1,d_0,1,\ldots,2N)]\\ &\quad+b_1(c_0,d_0,1,\ldots,2N)-xk(c_0,d_0,1,\ldots,2N)+k(1,\ldots,2N).\end{aligned} \tag{39}$$

Using the same procedure as Hirota and Ohta[15] and substituting Eqs.(37) and (38) into Eqs.(18)–(20), then we find that Eqs. (18), (19) and (20) respectively equal to

$$-24h[(c_0,d_0,c_1,d_1,\bullet)(\bullet)-(c_1,d_1,\bullet)(c_0,d_0,\bullet)+(c_0,d_1,\bullet)(c_1,d_0,\bullet)] = 24h(d_0,d_1,\bullet)(c_0,c_1,\bullet), \tag{40}$$

$$6[-(c_0,c_1,c_2,d_0,\bullet)(\bullet)+(c_0,c_1,\bullet)(c_2,d_0,\bullet)+(c_1,c_2,\bullet)(c_0,d_0,\bullet)-(c_0,c_2,\bullet)(c_1,d_0,\bullet)] = 0, \tag{41}$$

and

$$6[(c_0,d_0,d_1,d_2,\bullet)(\bullet)+(d_0,d_1,\bullet)(c_0,d_2,\bullet)-(d_1,d_2,\bullet)(c_0,d_0,\bullet)+(d_0,d_2,\bullet)(c_0,d_1,\bullet)] = 0, \tag{42}$$

where $(\bullet)$ denotes the character $(1,2,\ldots,2N)$ common to each Pfaffian.

It is not hard for us to see that Eqs.(40), (41) and (42) are nothing but the Pfaffian identities.

Therefore, we have proven that the Pfaffianized VCKP Eqs.(18)–(20) has the Gram-type Pfaffian solution.

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