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# Ground state of a superconducting $\pi$ ring array under external magnetic fields*

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The ground state of a two-dimensional square superconducting $\pi$ ring array has been investigated. The circulating currents of the $\pi$ ring array will spontaneously magnetize to the 'antiferromagnetic' arrangement with directions of the nearest-neighbouring currents circulating oppositely in the absence of an external magnetic field. It is found that the external magnetic field could destroy the anti-parallel configuration effectively. The external magnetic field needed to destroy the anti-parallel configuration is related to the superconducting $\pi$ ring's inductance parameter $\beta = 2\pi LI_c/\phi_0$. For a small $\beta$ the anti-parallel configuration, which is the lowest-energy ground state of the system, will be fully destroyed and changed to the configuration that the circulating currents have the same direction and parallel to the external magnetic field when the magnetic flux reaches $\phi_0/4$ in each ring. Moreover, the magnetic field needed to destroy the anti-parallel configuration will be very small when $\beta$ is large enough.

**Keywords:** $\pi$ ring array, Josephson junction, spontaneous magnetization
**PACC:** 7450, 7510H

## 1. Introduction

An important issue for the d wave symmetry of high-temperature superconductivity (HTS) is that the HTS grain boundary Josephson junction may have a negative critical current, which could be written as $I = -I_c \sin\varphi$, where $I_c > 0$, $\varphi$ is the quantum phase difference of the two electrodes' wavefunction of the junction. The relation can also be written as $I = I_c \sin(\varphi - \pi)$, therefore this kind of junction is also named a $\pi$ junction. Sigrist and Rice[1] indicated that a superconducting ring consisting of only one $\pi$ junction ($\pi$ ring) will spontaneously magnetize in the absence of external magnetic field when its inductance parameter $\beta = 2\pi LI_c/\phi_0 > 1$ ($\phi_0$ is the flux quantum). The spontaneous magnetic flux $LI$ is equal to $\phi_0/2$ when $\beta$ is large enough. In light of this knowledge Kirtley and Tsuei designed and confirmed the phase-sensitive d-wave experiment of HTS.[2]

The research of the superconducting $\pi$ ring not only verifies the d wave symmetry of HTS, but also has abundant content itself. In a previous work,[3] we have investigated the spontaneous magnetization of a coupled $2 \times 2$ $\pi$ ring array and found that the coupling between the neighbouring $\pi$ rings favours their spontaneous magnetization directions being opposite. Moreover we indicated that a one-dimensional (1D) $\pi$ ring chain and 2D $\pi$ ring arrays will spontaneously magnetize to the 'antiferromagnetic' arrangement in the absence of external magnetic field. Tian *et al*[4,5] gave the same result after computing large dimensional model of 1D $\pi$ ring chains and 2D $\pi$ ring arrays. This is in accordance with the scanning SQUID observations of the spontaneous magnetization with YBCO-Nb edge junction 1D $\pi$ ring chains and 2D $\pi$ ring arrays by Hilgenkemp *et al*,[6] in which they definitely exhibit the 'antiferromagnetic' ordering.

Recently, more and more interest has been attracted to the research of the superconducting $\pi$ ring for it not only has academic importance, but also has potential applications in many aspects such as the superconducting $\pi$ ring SQUID, superconducting single flux quantum circuit and the superconducting quantum computer.[7]

To the best of our knowledge, there is currently no research on the behaviour of the 2D $\pi$ ring arrays

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under external magnetic fields. In this paper we investigate the static behaviour of a 2D square $\pi$ ring array under an external magnetic field by calculating the system's free energy. It is found that the external magnetic field has a strong influence on the magnetization state of the 2D $\pi$ ring array, which could induce phase transition in the system.

## 2. Experiment

We focus on a 2D square superconducting $\pi$ ring array. As schematically shown in Fig.1, each superconducting ring consists of four Josephson junctions, namely junction 1, 2, 3 and 4, in which junction 4 is a $\pi$ junction and the other three are normal junction (zero junction). For simplicity of discussion, it is assumed that all superconducting rings have the same ring inductance $L$. Moreover, we assume that all the rings have the same effective area $S$ for the external magnetic field, which means that the uniform external magnetic field $B$ generates equal magnetic flux $\Phi_{\rm e} = S \cdot B$ in each ring. It is assumed further that every junction has the same absolute value of critical current, namely $I_{\rm c1} = I_{\rm c2} = I_{\rm c3} = I_{\rm c}$, and $I_{\rm c4} = -I_{\rm c}$; the inductance parameter of every superconducting ring is $\beta = 2\pi L I_{\rm c}/\phi_0$.

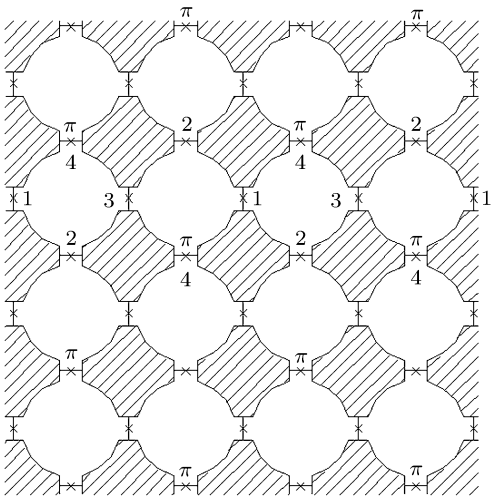


**Fig.1.** The schematic diagram of a $\pi$ ring array. The shadowed region is HTS film. The product sign with a $\pi$ mark denotes a $\pi$ junction, and that without a $\pi$ mark expresses a normal Josephson junction. Four neighbouring junctions construct a $\pi$ ring.

When investigating the spontaneous magnetization of a $2\times2$ $\pi$ ring array in the absence of an external magnetic field, we indicate that although the magnetic interaction among the neighbouring rings favours the directions of the spontaneously magnetized circulating currents to be opposite to each other, they are inclined to magnetize oppositely when the inductance of the rings is neglected. This phenomenon is essentially a phase coherence effect of the superconducting quantum wavefunctions in the rings through the shared Josephson junctions.[3] In this paper we will discuss the influence of the external magnetic field on the phase coherence of the superconducting wavefunctions at the junctions. For simplicity, we ignore the mutual inductance among the rings and consider only the influence of the external magnetic field on the circulating currents, the phase difference of superconducting wavefunctions at the junctions, and the free energy of the system.

### 2.1. The directions of the nearest-neighbouring circulating currents being opposite to each other (fully antiparallel configuration)

This configuration is schematically shown in Fig.2 and we name it State 1. Due to the existence of external magnetic field, the forward and reverse circulating currents are not symmetric. They may have different absolute circulating currents.

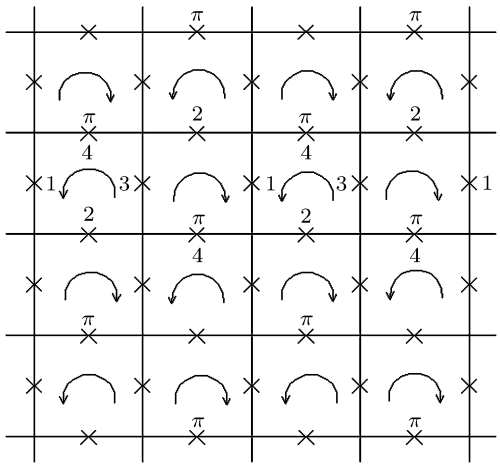


**Fig.2.** The schematic diagram of the full anti-parallel configuration, in which the semicircle sign represents the circulating current of the corresponding $\pi$ ring. The direction of a circulating current is determined by the right-hand rule.

We assume that the circulating current having the same direction as the external magnetic field is $I_{\rm u}$ and that having the opposite direction is $I_{\rm d}$, and the magnetic flux generated by the external magnetic field

is $\Phi_e$. The free energy of a couple of superconducting rings with forward and reverse directions could be written as

$$F = \frac{1}{2}LI_u^2 + \frac{1}{2}LI_d^2 - \frac{I_c\phi_0}{2\pi}\cos\theta_1 - \frac{I_c\phi_0}{2\pi}\cos\theta_2 - \frac{I_c\phi_0}{2\pi}\cos\theta_3 + \frac{I_c\phi_0}{2\pi}\cos\theta_4. \quad (1)$$

The first two terms are the magnetic energies of the two rings, and the last four terms are the Josephson coupling energies of the four junctions (each ring consists of four junctions and each junction is shared by two rings, and therefore each ring has two junctions and the two coupled rings have four junctions). There are phase relations among the physical quantities $I_u$, $I_d$, $\theta_1$, $\theta_2$, $\theta_3$ and $\theta_4$. As can be seen in Fig.2, for a superconducting ring whose circulating current is parallel to the external magnetic field, the relation can be written as

$$\theta_1 + \theta_2 + \theta_3 + \theta_4 + 2\pi(\Phi_e + LI_u)/\phi_0 = 2n\pi. \quad (2)$$

For a superconducting ring whose circulating current is antiparallel to the external magnetic field, we have

$$\theta_1 + \theta_2 + \theta_3 + \theta_4 + 2\pi(-\Phi_e + LI_d)/\phi_0 = 2n\pi. \quad (3)$$

Considering the symmetry of the system's structure, every junction has the same current $I_J$, namely

$$I_J = I_u + I_d = I_c\sin\theta_1 = I_c\sin\theta_2 = I_c\sin\theta_3 = -I_c\sin\theta_4. \quad (4)$$

To obtain the lowest free energy of the system, namely to reach the minimum value of Eq.(1), the value of the phase difference should be set to[5]

$$\theta_1 = \theta_2 = \theta_3, \quad \theta_4 = \theta_1 - \pi. \quad (5)$$

Consequently, Eqs.(1)–(3) can be respectively rewritten as

$$F = \frac{1}{2}LI_u^2 + \frac{1}{2}LI_d^2 - 4\frac{I_c\phi_0}{2\pi}\cos\theta_1, \quad (6)$$

$$4\theta_1 - \pi + 2\pi(\Phi_e + LI_u)/\phi_0 = 2n\pi, \quad (7)$$

$$4\theta_1 - \pi + 2\pi(-\Phi_e + LI_d)/\phi_0 = 2n\pi. \quad (8)$$

When $n = 0$, we obtain

$$2(\pi - 4\theta_1) = 2\pi L(I_u + I_d)/\phi_0 = \beta\sin\theta_1, \quad (9)$$

and thus

$$\beta = 2(\pi - 4\theta_1)/\sin\theta_1, \quad (10)$$

which means that the phase difference is determined by the inductance parameter $\beta$ only and independent of the external magnetic field $\Phi_e$. The current flowing through the junction is determined by Eq.(4) and also independent of the external magnetic flux $\Phi_e$. Let $n = 0$, from Eqs.(7) and (8) we can obtain

$$LI_u = LI_c\sin\theta_1/2 - \Phi_e, \quad (11)$$

$$LI_d = LI_c\sin\theta_1/2 + \Phi_e. \quad (12)$$

Substituting these results into the free energy relation (1), we have

$$F = F_0[2(\pi - 4\theta_1)^2 + 8\pi^2\phi_e^2 - 16(\pi - 4\theta_1)\cot\theta_1], \quad (13)$$

where $F_0 = \frac{1}{2L}\left(\frac{\phi_0}{2\pi}\right)^2$, and $\phi_e = \frac{\Phi_e}{\phi_0}$ is the normalized external magnetic flux; $\theta_1$ is determined by the inductance parameter $\beta$ and independent of the external magnetic flux.

## 2.2. Half of the nearest-neighbouring circulating currents being equidirectional, half of them in opposite directions

As schematically shown in Fig.3, the circulating currents in each column of a superconducting $\pi$ ring array have the same direction, and the directions of the circulating currents in every two neighbouring columns are opposite.

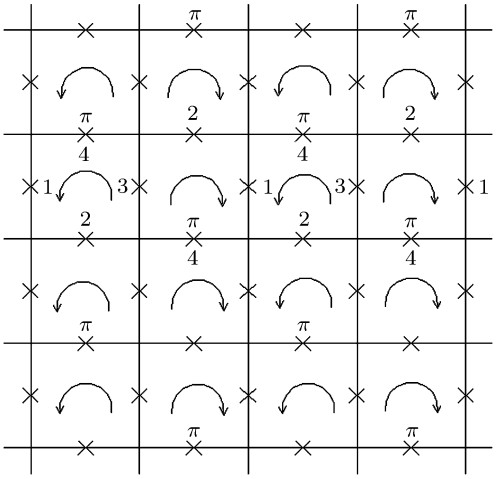



**Fig.3.** Schematic diagram of the state with half of the nearest-neighbouring circulating currents being in same directions and half of them in the opposite.

For each $\pi$ ring, its two neighbouring rings in the same column have the same circulating currents, and the two neighbouring rings in the same row have opposite circulating currents. This configuration is named

State 2. Because of translation invariance, all of the $\pi$ rings anti-parallel to the external magnetic field have the same circulating current, which is marked as $I_{\rm u}$. For the same reason all of the $\pi$ rings that have the same direction as the external magnetic field have the same circulating current also, and we mark it as $I_{\rm d}$. Consequently, in the circulating current structure shown in Fig.3, the current flowing through junction 2 and 4 will be zero, namely there exists the relation $I_{\rm c}\sin\theta_2 = -I_{\rm c}\sin\theta_4 = 0$. For the purpose of obtaining the lowest free energy of the system, the value of the phase difference should be set to $\theta_2 = 0$ and $\theta_4 = \pi$. However, the currents flowing through junctions 1 and 3 are equal and could be written as

$$I_{\rm J} = I_{\rm u} + I_{\rm d} = I_{\rm c}\sin\theta_1 = I_{\rm c}\sin\theta_3. \tag{14}$$

And thus we have $\theta_1 = \theta_3$. The relation connecting the circulating currents $I_{\rm u}$, $I_{\rm d}$ with the phase difference satisfies the following expression: for a ring in which the current circulating direction is the same as. the external magnetic field, we have

$$\theta_1 + \theta_2 + \theta_3 + \theta_4 + 2\pi(\Phi_{\rm e} + LI_{\rm u})/\phi_0 = 2n\pi.$$

This can be simplified to

$$2\theta_1 + \pi + 2\pi(\Phi_{\rm e} + LI_{\rm u})/\phi_0 = 2n\pi. \tag{15}$$

For a ring in which the current circulating direction is opposite to that of the external magnetic field, we obtain

$$2\theta_1 + \pi + 2\pi(-\Phi_{\rm e} + LI_{\rm d})/\phi_0 = 2n\pi. \tag{16}$$

From Eqs.(14)–(16) we can obtain

$$\beta = 2(\pi - 2\theta_1)/\sin\theta_1, \tag{17}$$

$$LI_{\rm u} = LI_{\rm c}\sin\theta_1/2 - \Phi_{\rm e}, \tag{18}$$

$$LI_{\rm d} = LI_{\rm c}\sin\theta_1/2 + \Phi_{\rm e}. \tag{19}$$

The free energy of the two coupled superconducting $\pi$ rings in which the currents circulating in opposite directions could be written as

$$\begin{aligned} F = & \frac{1}{2}LI_{\rm u}^2 + \frac{1}{2}LI_{\rm d}^2 - \frac{I_{\rm c}\phi_0}{2\pi}\cos\theta_1 - \frac{I_{\rm c}\phi_0}{2\pi}\cos\theta_2 - \frac{I_{\rm c}\phi_0}{2\pi}\cos\theta_3 + \frac{I_{\rm c}\phi_0}{2\pi}\cos\theta_4 \\ = & F_0[2(\pi - 2\theta_1)^2 + 8\pi^2\phi_{\rm e}^2 - 8\frac{\pi - 2\theta_1}{\sin\theta_1} - 8(\pi - 2\theta_1)\cot\theta_1]. \end{aligned} \tag{20}$$

## 2.3. The circulating currents flowing through all of the rings being in the same direction (full-parallel configuration)

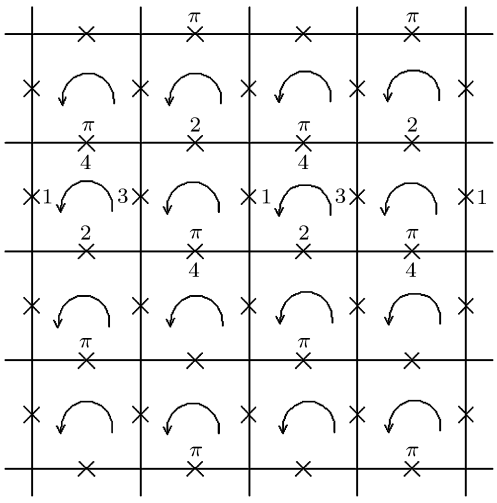



**Fig.4.** The schematic diagram of the fully parallel configuration.

This configuration is schematically shown in Fig.4. We can only investigate the characteristic of a couple of the nearest-neighbouring rings for the translation invariance. On the other hand, because of the geometric symmetry, the junctions belonging to the two rings having the same label will have the same phase difference of the superconducting wavefunctions. This configuration can be divided into two types, described as follows.

### 2.3.1. The direction of the circulating currents parallel to the external magnetic field

This configuration is named State 3. It is assumed that the circulating current of one ring is $I_{\rm u}$ and that of another one of the coupled rings is $\alpha I_{\rm u}$ where $\alpha$ is the ratio of the two currents and $0 \le \alpha \le 1$. Evidently, the circulating currents of the four nearest-neighbouring rings of $I_{\rm u}$ are all $\alpha I_{\rm u}$ and those of the four rings around $\alpha I_{\rm u}$ are $I_{\rm u}$ due to the symmetry of

the configuration. Therefore, the phase equation of a ring with circulating current $I_{\rm u}$ can be written as

$$\theta_1+\theta_2+\theta_3+\theta_4+2\pi(\Phi_{\rm e}+LI_{\rm u})/\phi_0=2n\pi. \quad (21)$$

And the phase equation of the ring with circulating current $\alpha I_{\rm u}$ can be written as

$$-\theta_1-\theta_2-\theta_3-\theta_4+2\pi(\Phi_{\rm e}+\alpha LI_{\rm u})/\phi_0=2n'\pi. \quad (22)$$

The relation between the circulating current and the phase difference is as follows:

$$\begin{aligned} I_{\rm u}-\alpha I_{\rm u} &= I_{\rm c}\sin\theta_1=I_{\rm c}\sin\theta_2=I_{\rm c}\sin\theta_3 \\ &= -I_{\rm c}\sin\theta_4. \end{aligned} \quad (23)$$

So we have a relationship between the phase differences, given by

$$\theta_1=\theta_2=\theta_3, \quad \theta_4=\theta_1-\pi. \quad (24)$$

From Eqs.(21) and (22) we can obtain

$$2\Phi_{\rm e}+LI_{\rm u}(1+\alpha)=m\phi_0, \quad I_{\rm u}=\frac{\phi_0-2\Phi_{\rm e}}{(1+\alpha)L}. \quad (25)$$

Using Eqs.(23)–(25), we can write the free energy of a couple of rings as

$$\begin{aligned} F &= \frac{1}{2}LI_{\rm u}^2+\frac{1}{2}L(\alpha I_{\rm u})^2-\frac{I_{\rm c}\phi_0}{2\pi}\cos\theta_1-\frac{I_{\rm c}\phi_0}{2\pi}\cos\theta_2-\frac{I_{\rm c}\phi_0}{2\pi}\cos\theta_3+\frac{I_{\rm c}\phi_0}{2\pi}\cos\theta_4 \\ &= \frac{(1+\alpha^2)(\phi_0-2\Phi_{\rm e})^2}{2(1+\alpha)^2L}-\frac{2\phi_0}{\pi}\sqrt{I_{\rm c}^2-\frac{(1-\alpha)^2(\phi_0-2\Phi_{\rm e})^2}{(1+\alpha)^2L^2}}, \end{aligned}$$

and we have

$$\frac{\partial F}{\partial\alpha}=\frac{(\alpha-1)(\phi_0-2\Phi_{\rm e})^2\left[4L(1+\alpha)\phi_0+\pi L\sqrt{(1+\alpha)^2L^2I_{\rm c}^2-(\alpha-1)^2(\phi_0-2\Phi_{\rm e})^2}\right]}{\pi(1+\alpha)^3L^2\sqrt{(1+\alpha)^2L^2I_{\rm c}^2-(\alpha-1)^2(\phi_0-2\Phi_{\rm e})^2}}. \quad (26)$$

The partial differential $\dfrac{\partial F}{\partial\alpha}$ will be negative when $0\leq\alpha<1$, indicating that the free energy $F$ is a monotonically decreasing function in this interval. To obtain the lowest free energy of the system, we have $\alpha=1$. Consequently, Eq.(25) can be re-written as

$$LI_{\rm u}=\phi_0/2-\Phi_{\rm e}. \quad (27)$$

We thus have $\sin\theta_1=0$, from which the phase difference $\theta_1$ should be zero, and the free energy of the two coupled rings is expressed as

$$\begin{aligned} F &= \frac{(\phi_0-2\Phi_{\rm e})^2}{4L}-\frac{2I_{\rm c}\phi_0}{\pi} \\ &= F_0\left[8\pi^2\left(\frac{1}{2}-\phi_{\rm e}\right)^2-8\beta\right]. \end{aligned} \quad (28)$$

#### 2.3.2. The direction of the circulating currents antiparallel to the external magnetic field

This configuration is named State 4. Using the same procedure as in Section 2.3.1, we can deduce that a couple of the nearest-neighbouring rings will have the lowest free energy when their circulating currents have the same value, which can be written as

$$\begin{aligned} F &= \frac{(\phi_0+2\Phi_{\rm e})^2}{4L}-\frac{2I_{\rm c}\phi_0}{\pi} \\ &= F_0\left[8\pi^2\left(\frac{1}{2}+\phi_{\rm e}\right)^2-8\beta\right], \end{aligned} \quad (29)$$

and the circulating current is

$$LI_{\rm u}=\phi_0/2+\Phi_{\rm e}. \quad (30)$$

When the external magnetic field is equal to zero, there is no difference between the above two categories. The free energy is just the same as the above, i.e.

$$F=F_0(2\pi^2-8\beta).$$

### 2.4. Comparison of the free energies of the system under different external magnetic fields

Using the above equations, we can calculate the free energy of each state under an arbitrary external magnetic field. The variation of the free energy with the external magnetic field for the above circulating

current states is shown in Fig.5, at an inductance parameter $\beta = 10$. Considering the periodic response of the superconducting ring consisting of Josephson junctions to the external magnetic flux, we give only the status at which the external magnetic field increases from zero to $\phi_0/2$.

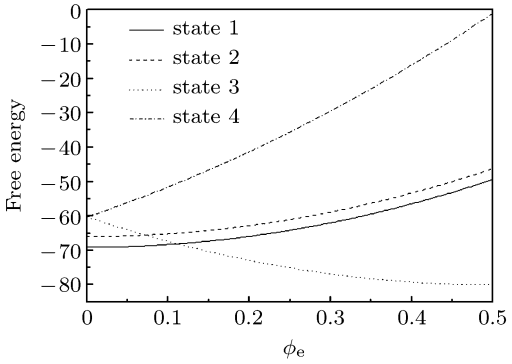



**Fig.5.** Relationship between the reduced free energy $f = F/F_0$ and the external magnetic field $\phi_e = \Phi_e/\phi_0$ for different configurations when $\beta = 10$.

The circulating current flowing though a $\pi$ ring reflects the spontaneous magnetization of the ring when the external magnetic field is equal to zero. As can be seen in Fig.5, spontaneous magnetization favours 'anti-ferromagnetic' arrangement with the nearest-neighbouring circulating currents being in opposite direction, which is just the spontaneous magnetization result of a $\pi$ ring array.[3,4] It can be seen also that the state with the nearest-neighbouring $\pi$ ring's circulating currents being in opposite directions has lower free energy when the external magnetic field is small. The free energy of the full anti-parallel configuration (State 1) increases with the increase of the external magnetic field. However, the free energy of State 3 decreases with the increase of the external magnetic field, in which state the circulating currents' directions are the same and parallel to the external magnetic field. The free energy of State 3 will be lower than that of State 1 when the external magnetic field increases to a certain value. This means that the ground state with the minimum free energy is the full parallel configuration in which all the circulating currents have the same direction and parallel to the external magnetic field. The external magnetic field needed to transfer the system from full anti-parallel to full parallel is related to the inductance parameter $\beta$. When the external magnetic field in each $\pi$ ring reaches $\phi_0/4$, the lowest-energy state is the full parallel configuration with all the circulating currents parallel to the external magnetic field.

## 3. Phase transition of $\pi$ ring array under an external magnetic field

It is indicated above that for a given regular $\pi$ ring array ($\beta$ fixed) to reach the lowest-energy state of the system, the circulating currents in all the $\pi$ rings should transfer from the state in which the currents in nearest-neighbouring rings are in opposite directions, to the state having the currents in the same direction when the external magnetic field increases from zero. The transition point is related to the inductance parameter $\beta$. It can be inferred that if a superconducting $\pi$ ring array is placed in a certain external magnetic field, the variation of the critical current of the Josephson junctions will induce the change of the inductance parameter when the temperature of the system changes. Subsequently, it may conduce to the transition of the $\pi$ ring array configuration from antiparallel to parallel. To calculate the phase transition quantitatively, the relation between the critical current $I_c$ and the ambient temperature $T$ are needed. For simplicity, a linear relation of $I_c$ and $T$ namely $I_c = I_{c0}(1 - T/T_c)$ is assumed. It is a good approximation for HTS when the temperature approaches $T_c$ despite that there will be a larger deviation for temperatures much lower than $T_c$. However, we investigate the question just qualitatively, thus the approximate expression extending to low temperatures is used.

Using the $I_c - T$ relation and the equations deduced above, we could calculate the variation of the free energy of the system with the temperature under a certain magnetic field. Figure 6 shows the variation of the difference of free energy between State 1 and State 3 with the temperature under different external magnetic fields. As can be seen in Fig.6, the system always remains in the antiparallel configuration at lower free energy at any temperature when the external magnetic field is small. The system keeps the parallel configuration when the external magnetic field is much larger and the external magnetic flux approaches $\phi_0/4$. The system holds the antiparallel configuration when the temperature is higher and the magnetic field is in a medium range. The system will transfer from the antiparallel configuration to the par-

allel one when the temperature decreases, hence $\beta$ increases. Subsequently, the system will turn into the parallel configuration at lower temperatures. Figure 7 is a numerically calculated phase diagram, showing the transition from the antiparallel to the parallel configuration of the circulating currents in a $\pi$ ring array. As can be seen in Fig.7(a), the circulating currents flowing in a $\pi$ ring array construct a full antiparallel configuration 'anti-ferromagnetic' state below the phase transition line, and form a parallel configuration 'ferromagnetic' state above the line. Figure 7(b) shows that the location and shape of the phase transition line are mostly determined by the inductance parameter of the $\pi$ ring array. The magnitude of the critical external magnetic flux of phase transition is equal to $\phi_0/4$ when $\beta$ approaches zero, while it approaches zero when $\beta$ approaches infinity.

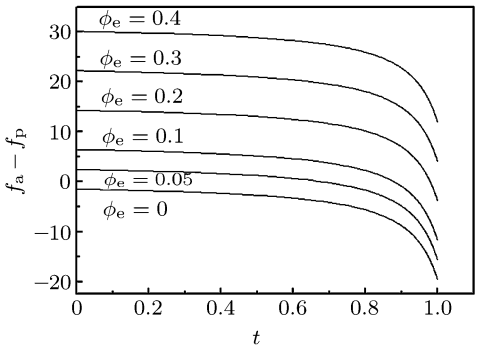



**Fig.6.** Variation of the difference of free energy between the antiparallel and the parallel configurations with the temperature for different external magnetic fields.

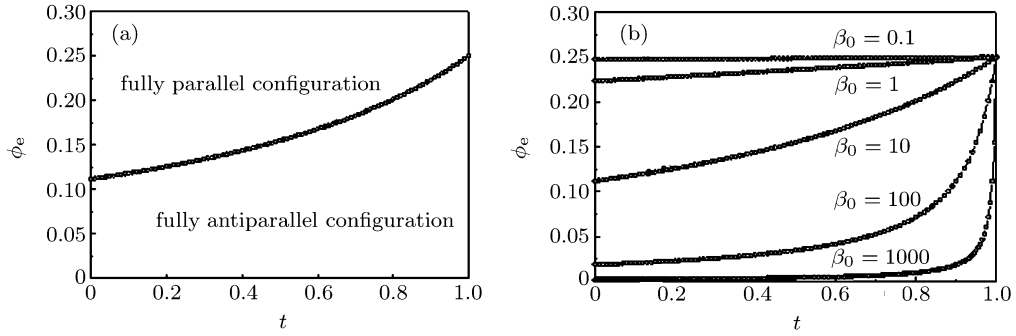



**Fig.7.** The calculated phase transition line between the full antiparallel and the parallel configurations for (a) $\beta_0 = 10$, and (b) different inductance parameters. There is a relation of $\beta = \beta_0(1 - t)$.

It should be pointed out that we only discuss the difference of free energy between the full parallel configuration and the full anti-parallel configuration of the circulating currents. Actually there are some transition states between these two states, for instance the configuration with part of the circulating currents parallel and part of them antiparallel. Therefore the real phase transition will be far more complex than we discussed above.

It should be noticed here that we neglect the mutual inductance among the rings, which means that the static magnetic coupling among the rings is ignored. From electromagnetics, the static magnetic interaction energy of the $\pi$ ring array reaches the minimum when the directions of the currents circulating in nearest-neighbouring rings are opposite. The antiparallel configuration of the circulating currents will be favoured if the mutual inductance is considered.[2] Consequently, the external magnetic flux $\Phi_e$ needed to transfer the anti-parallel configuration of the $\pi$ ring array's magnetic state to the parallel configuration will increase, and the specific value is related to the magnitude of the mutual inductance among the rings.

# 4. Discussion

To understand the phase transition discussed above from the viewpoint of physics, we investigate the relation between the circulating current flowing in each $\pi$ ring and the external magnetic field. Using Eqs.(11), (12), (27) and (30), we can calculate the circulating current in one $\pi$ ring under a given external magnetic field. Figure 8 shows the variation of the reduced circulating current $LI/\phi_0$ with the external magnetic flux when the $\pi$ ring array is in different state. It can be seen that for the configuration with the nearest-neighbouring circulating currents parallel to each other (State 3 and 4), the magnetic flux

generated from the circulating current is $LI = \phi_0/2$ when the external magnetic field is zero. For the antiparallel configuration (State 1) the reduced circulating current approaches $1/2$ when $\beta$ is large enough. In the parallel configuration, the circulating current for the state that has the same direction as the external magnetic flux (State 3) decreases gradually when the external magnetic field increases. The circulating current will reach zero when the external magnetic flux is equal to $\phi_0/2$. However, for the state that has opposite direction to the external magnetic flux (State 4), the circulating current increases gradually. For the antiparallel configuration, also the circulating current parallel to the external magnetic flux decreases from approximate $1/2$ to a small negative value, while the circulating current opposite to the external magnetic flux increases to about unity.

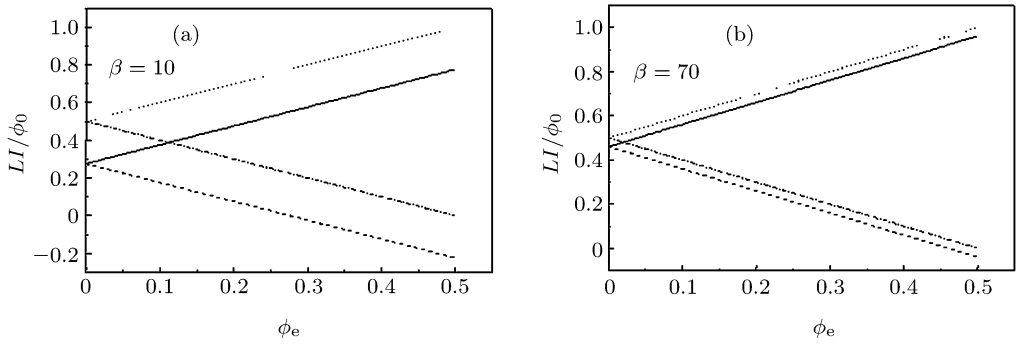



**Fig.8.** Relation between the circulating current and the external magnetic flux for a $\pi$ ring. (i) The circulating currents in the $\pi$ ring array are full antiparallel, and the current in a $\pi$ ring is opposite (solid line) or parallel (dashed line) to the external magnetic field. (ii) The circulating currents of the $\pi$ ring array are full parallel, and the current of a $\pi$ ring is opposite (dotted line) or parallel (dot-dashed line) to the external magnetic field. Different inductance parameters are used (a) $\beta = 10$ and (b) $\beta = 70$.

As can be seen in Figs.5 and 8, for a regular two-dimensional square $\pi$ ring array the ground state with the lowest-energy is the state with zero circulating current (the currents flowing through the Josephson junctions are also zero) when the external magnetic flux is equal to $\phi_0/2$, which is just the same as the normal Josephson junction array in the absence of external magnetic field. This is natural in physics. By comparing a $\pi$ ring with a zero ring with the same number of normal junctions, a $\pi$ phase difference will be generated when we encircle the $\pi$ ring, which corresponds to the decrease of half a fluxon. Therefore, when half a fluxon is added to a $\pi$ ring, the state of the ring should be equivalent to a zero ring in the absence of external magnetic field. This conclusion will hold true for the $\pi$ ring array, which can be proven from the phase equations in Section 2. Thus it is easy to understand that the anti-parallel configuration has the minimum free energy for a granular $\pi$ ring array when it spontaneously magnetizes in the absence of external magnetic field. The magnetization state will be maintained around the zero external magnetic field. However, when the external magnetic flux approaches $\phi_0/2$, it behaves just like a normal junction array under a small magnetic field with all the circulating currents parallel to the external magnetic flux.

## 5. Conclusions

In the absence of external magnetic field, the ground state of a $\pi$ ring array is in 'anti-ferromagnetic' ordering with the spontaneous magnetization circulating currents of the nearest-neighbouring $\pi$ rings fully anti-parallel, which is a quantum phase coherence result of the superconducting wavefunctions. However, the magnetic field has strong influence on the anti-parallel configuration. The external magnetic field needed to destroy the configuration is related to the inductance parameter $\beta$ of the rings. If $\beta$ is small enough, the configuration will be destroyed completely when the external magnetic flux reaches $\phi_0/4$ in each $\pi$ ring, and the system's ground state with the lowest-energy will change to the fully parallel configuration. Moreover, the external magnetic field needed to destroy the anti-parallel configuration will be very small when $\beta$ is large enough.

A point should be noted is that the discussion in this paper is only on the ground state of the system for different $\beta$ values under varied external magnetic fields. This, however, does not mean there is no metastable state with higher energy. Metastable states could appear in local areas of the sample, which subsequently causes the rise of the system's energy. The appearance of the metastable state is related to external factors, and it will result in the complexity of the experimental observation. Only the ground state phase transition of the $\pi$ ring array is considered in this paper. The topic of the metastable state exceeds the extent of this discussion.

Finally it should be noticed that all the $\pi$ ring arrays investigated in this paper do have regular periodic structures. Things will be complex when a $\pi$ ring array has different individual $\pi$ ring areas and Josephson critical currents in different regions. However, the conclusion that the external magnetic field could strongly destroy the antiparallel configuration of the circulating currents is correct qualitatively, although the external magnetic field needed to transfer the anti-parallel configuration to the parallel configuration varies at different regions of the sample. It is believed that our work will redound to understanding the model calculation of the $\pi$ ring arrays using other methods, and may give some advice to the design of further experiments for investigating the $\pi$ ring arrays.

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