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Tokamak resistive magnetohydrodynamic ballooning instability in the negative shear regime*

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Improved confinement of tokamak plasma with central negative shear is checked against the resistive ballooning mode. In the negative shear regime, the plasma is always unstable for purely growing resistive ballooning mode. For a simplest tokamak equilibrium model, the s–α model, characteristics of this kind of instability are fully clarified by numerically solving the high n resistive magnetohydrodynamic ballooning eigen-equation. Dependences of the growth rate on the resistivity, the absolute shear value, the pressure gradient are scanned in detail. It is found that the growth rate is a monotonically increasing function of α while it is not sensitive to the changes of the shear s, the initial phase θ_0 and the resistivity parameter $\varepsilon_{\rm R}$.

Keywords: negative shear, resistive ballooning instability
PACC: 5235P, 5255G

1. Introduction

Improved confinement of tokamak plasma is thought to be necessary for an economic magnetic fusion reactor. Encouraging improved confinements realized in some large tokamaks, such as the TFTR,[1] DIII-D,[2] JET,[3] JT-60U,[4] are characterized by negative magnetic shear ($s < 0, s = \dfrac{r}{q}\dfrac{\mathrm{d}q}{\mathrm{d}r}$, with r being the minor radius, q the safety factor) in the central region of the toroidal plasma. In this region, the plasma is known to be stable against the high n (the toroidal mode number) ideal magnetohydrodymanic (MHD) mode (hereafter the IBM).[5] However, the plasma in this region can be unstable against non-ideal MHD modes[6] and kinetic modes.[7] The high n resistive MHD ballooning mode (hereafter the RBM) is the counterpart of the high n IBM. Basic theoretic treatments of the RBM can be cast into two types: the first is based on the asymptotic matching, i.e. to solve the marginal IBM equation in the large θ limit (θ is the poloidal angle) then to determine a parameter $\Delta_{\rm B}$ (a measure of the MHD energy available to drive instability, $\Delta_{\rm B} < 0$ corresponds to unstable);[8–11] the second is to directly solve the RBM eigen-mode equation.[12,13] Most of the studies belong to the first type. Fewer contributions have been devoted to the RBM in the negative shear case and only the first method has been used. In this paper, we start the RBM study on the negative shear plasma by using the second method.

2. Unified ideal and resistive ballooning mode eigen-equation[12]

The simplest theoretical mode for describing the plasma macroscopic instabilities is the single fluid incompressible MHD model. A unified ideal and resistive ballooning mode eigen-equation was derived by Bateman and Nelson[12] by using this model in the so-called ballooning representation.[14] In an orthogonal flux coordinates ψ, χ and ζ with ψ being the magnetic flux, χ the poloidal angle variable and ζ the toroidal angle variable, the perturbed quantity is expressed as

$$A(\psi, \chi, \zeta, t) = \hat{A}(\psi, \chi) \exp[\Gamma t + \mathrm{i}nS(\psi, \chi, \zeta)]. \quad (1)$$

Here

$$S(\psi, \chi, \zeta) = \zeta - \int^{\chi} v(\psi, \chi')\mathrm{d}\chi' \quad (2)$$

is called the eikonal, and $S =$ const represents a magnetic force line, Γ is the growth rate of the unstable mode, $v = FJ/R^2 = \dfrac{B_\zeta J}{R}$ is the local safety factor, $J = (\nabla\psi \times \nabla\chi \cdot \nabla\zeta)^{-1}$ is the Jacobian of the coordinates. If we define $X = RB_\chi \xi_\psi$, this equation (ξ is

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the displacement vector of the plasma) reads

$$\boldsymbol{B}\cdot\nabla\left\{\frac{|\nabla S|^2}{B^2G}\boldsymbol{B}\cdot\nabla X\right\}+\left(D_\kappa-\frac{\Gamma^2|\nabla S|^2}{B^2}\right)X=0, \tag{3}$$

where

$$G=1+\frac{n^2|\nabla S|^2}{\Gamma}\eta, \tag{4}$$

$$D_\kappa=\frac{\mathrm{d}p/\mathrm{d}\psi}{B^2}\left[2\frac{\mathrm{d}}{\mathrm{d}\psi}\left(p+\frac{B^2}{2}\right)-\frac{\partial S}{\partial\psi}\frac{RB_\zeta}{B^2}\boldsymbol{B}\cdot\nabla B^2\right]. \tag{5}$$

In the above equations, $F=B_\zeta R$, R is the major radius of the plasma torus, $\boldsymbol{B}=B_\zeta\boldsymbol{e}_\zeta+B_\chi\boldsymbol{e}_\chi=\nabla\psi\times\nabla\zeta+F\nabla\zeta$ is the total magnetic field, η is the plasma resistivity. Parameter D_κ represents the pressure gradient driving force that is proportional to the curvature of the magnetic force line while the first term on the right-hand side of Eq.(5) is due to the normal curvature, and the second term due to the geodesic curvature. For a general toroidally axisymmetric plasma equilibrium, we have

$$\nabla S=\frac{1}{R}\boldsymbol{e}_\zeta-\frac{B_\zeta}{RB_\chi}\boldsymbol{e}_\chi-(RB_\chi)\left(\int^\chi v'_\psi\mathrm{d}\chi'\right)\boldsymbol{e}_\psi. \tag{6}$$

Substituting it into Eq.(3) and making use of Eq.(5), we obtain the resistive ballooning eigen-equation as

$$\frac{1}{J}\frac{\partial}{\partial\theta}\left\{\frac{1}{JR^2B_\chi^2G}\left[1+\left(\frac{R^2B_\chi^2}{B}\int^\theta v'\mathrm{d}\theta'\right)^2\right]\frac{\partial X}{\partial\theta}\right\}+\frac{2p'}{B^2}\left[\left(p+\frac{B^2}{2}\right)'-\frac{F}{B^2}\left(\int^\theta v'\mathrm{d}\theta'\right)\frac{1}{J}\frac{\partial}{\partial\theta}\left(\frac{B^2}{2}\right)\right]X$$
$$-\frac{\Gamma^2}{R^2B_\chi^2}\left[1+\left(\frac{R^2B_\chi^2}{B}\int^\theta v'\mathrm{d}\theta'\right)^2\right]X=0. \tag{7}$$

For the ideal MHD ballooning mode, $G=1$, and this equation is completely the same as that derived by Connor *et al* in Ref.[14]. In the following, we use the simplest tokamak equilibrium model, i.e. the (s,α) model, which describes the tokamak magnetic surfaces as concentric circular toroids, then the unified ideal and resistive ballooning eigenequation (7) becomes[13]

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left[\frac{f}{1+Gf}\frac{\mathrm{d}F}{\mathrm{d}\theta}\right]+KF-\lambda(\Gamma/\omega_\mathrm{A})^2fF=0, \tag{8}$$

where

$$f=1+[s(\theta-\theta_0)-\alpha\sin\theta]^2, \tag{9}$$

$$G=1=\frac{n^2\eta q^2}{\mu_0r^2\Gamma}f, \tag{10}$$

$$K=\alpha[\cos\theta-\alpha\sin\theta(s(\theta-\theta_0)-\alpha\sin\theta)], \tag{11}$$

$$\alpha=-2\mu_0\frac{R_0q^2}{B^2}\frac{\mathrm{d}p}{\mathrm{d}r}. \tag{12}$$

In the above equations, q is the safety factor, $\omega_\mathrm{A}=V_\mathrm{A}/qR$ is the Alfven frequency, $V_\mathrm{A}=B/\sqrt{\mu_0\rho_0}$ is the Alfven velocity. For the resistive ballooning mode, we have $\lambda=\varepsilon_\mathrm{R}$ while for the ideal ballooning mode, $\varepsilon_\mathrm{R}=0,\lambda=-1$. As in ideal ballooning mode case, the poloidal angle now is extended to $-\infty<\theta<\infty$.

As in Ref.[14], we introduce $\hat{\Gamma}=\Gamma/\Gamma_\mathrm{R}$, where $\Gamma_\mathrm{R}=(n^2q^2/\tau_\mathrm{R}\tau_\mathrm{A}^2)^{1/3}$, $\tau_\mathrm{R}=\mu_0r^2/\eta$ (the classical diffusion time), and $\varepsilon_\mathrm{R}=(n^2q^2/S_\mathrm{M})^{2/3}$, with $S_\mathrm{M}=\tau_\mathrm{R}/\tau_\mathrm{A}$ (the magnetic Reynolds number), hence we obtain the final eigen-equation of the resistive ballooning mode for s–α equilibrium,

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left[\frac{\hat{\Gamma}f}{\hat{\Gamma}+\varepsilon_\mathrm{R}f}\frac{\mathrm{d}F}{\mathrm{d}\theta}\right]+KF-\varepsilon_\mathrm{R}\hat{\Gamma}^2fF=0. \tag{13}$$

Equation (13) is the same as that in Ref.[13] where this equation was used to calculate the growth rate of the unstable mode in the second stability region of an ideal MHD ballooning mode. In this paper, we utilize an eighth-order Runge–Kutta scheme to numerically solve this equation for a negative shear tokamak plasma. Basic parameters encountered are $s,\alpha,\varepsilon_\mathrm{R},\theta_0$. Different from Ref.[13] where the initial phase $\theta_0=0$ was assumed, we consider $0\le\theta_0\le\pi$ so that the boundary condition is $X\to0$ for $\theta\to\infty$. For $\theta_0=0$, this condition is identical to the even mode parity: $\partial X/\partial\theta=0$ at $\theta=0$. For $\theta_0\neq0$, however, the mode is an even-odd mixed parity one.

3. s–α equilibrium model for a negative shear tokamak plasma

For improved confinement discovered in Refs.[1]–[4], the negative shear region is in the central shear region. We neglect the toroidal effect and the Shafranov shift effect because they are very small in the central region and the plasma is already stable by only the negative shear effect against the ideal MHD ballooning mode.[15] Now assuming the pressure profile to be $P = P_0(1 - x^2)^\lambda$, with λ a parameter to measure the pressure profile peakedness , we have

$$\alpha = -2\lambda\beta_0 q^2 (R_0/a)(r/a)(1 - x^2)^{\lambda-1}. \qquad (14)$$

Taking $\beta_0 \leq 0.1, q = 2, R_0/a = 4, r/a \leq 0.3$, we have $\alpha \leq 1.0$ for $\lambda = 1$ (parabolic profile); $\alpha < 2$ for $\lambda = 2$ (peaked profile). The appropriate negative shear value should be in the range $-0.1 \leq s \leq 0$.

4. Characteristic of the most unstable modes

Usually, we can obtain several unstable modes, i.e. for any fixed $s, \alpha, \varepsilon_{\rm R}, \theta_0$, there are several eigenvalues $\hat{\Gamma}$ to satisfy the boundary condition. Physically, the largest eigen value $\hat{\Gamma}_1$ corresponds to the most unstable mode. It is found that the eigenfunction of the most unstable mode has the most reasonable structure while the eigenfunction of the less unstable mode has a much complex structure. Because their eigenfunctions are very sensitive to the numerical errors of their eigenvalues, it requires careful treatment to obtain the correct solution. We need to check carefully every solution by simultaneously determining the eigenvalue and the corresponding eigenfunction. Figures 1(a) and 1(b) are examples of eigenfunctions of a most unstable mode and a less unstable mode with the same external parameters: $s = -0.5, \alpha = 0.5, \theta_0 = 0$, and $\varepsilon_{\rm R} = 0.01$ and eigenvalues $\hat{\Gamma}_1 = 0.49215, \hat{\Gamma}_2 = 0.48311$. If not particularly noted, in the following, we are mainly concerned with the most unstable mode.

In a positive shear case considered in previous papers, the authors usually deal with the $\theta_0 = 0$ case.[13] However, some authors indicated that the most unstable mode does not definitely correspond to the $\theta_0 = 0$ case. We need check this point for the negative shear case. However, we do not have a conclusion about whether the $\theta_0 = 0$ or the $\theta_0 = \pi$ case corresponds to the most unstable mode. It seems that this depends also on other parameters like the s, α and $\varepsilon_{\rm R}$. It is observed that the eigenvalue does not change much for different θ_0 though the eigenfunction changes significantly.

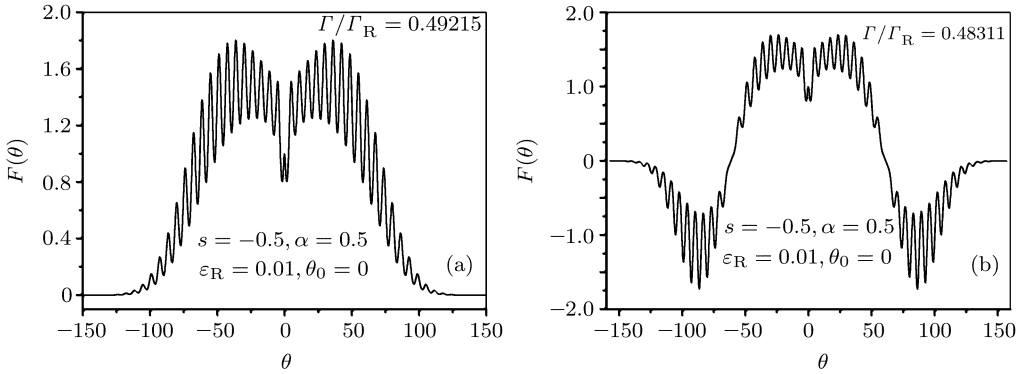


Fig.1. Eigenfunctions for the unstable modes. (a) Eigenfunction of the most unstable mode for $s = -0.5, \alpha = 0.5,\ \varepsilon_{\rm R} = 0.01, \theta_0 = 0$. (b) Eigenfunction of the less unstable mode for the same parameters cited in (a).

5. Dependence of growth rate on the absolute shear value

Our numerical study scanned over the absolute values of the magnetic shear for the normalized growth rate of the most unstable mode. It is shown in Fig.2 that it is rather flat against the change of the absolute value of the magnetic shear s. The difference between the two curves is mainly due to the different α values. The corresponding eigenfunctions, however, are significantly different for different s while other

parameters keep constant, as shown in Figs.3(a) and 3(b). Another comparable case with an intermediate value $s = -0.5$ is found in Fig.1(a).

As can be seen from Fig.2, the normalized growth rate of the most unstable mode decreases only slowly with the increase of the absolute value of the negative shear. With the decrease of the absolute value of the negative shear, the eigenfunction extends its profile over the extended poloidal angle. Therefore, we obtain a basic result that the growth rate of the most unstable mode is written as

$$\Gamma = C(\alpha, s, \varepsilon_{\rm R}, \theta_0)\Gamma_{\rm R}. \tag{15}$$

Now the coefficient $C(\alpha, s, \varepsilon_{\rm R}, \theta_0)$ is nearly unchanged against the change of the shear value. Dependence on other three parameters $\alpha, \varepsilon_{\rm R}, \theta_0$ needs to be further checked.

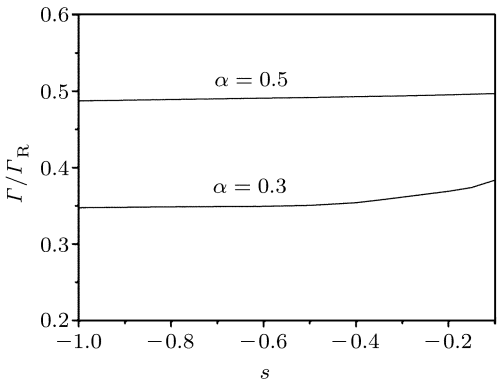


Fig.2. The growth rate of the most unstable mode versus the shear value for two selected α.

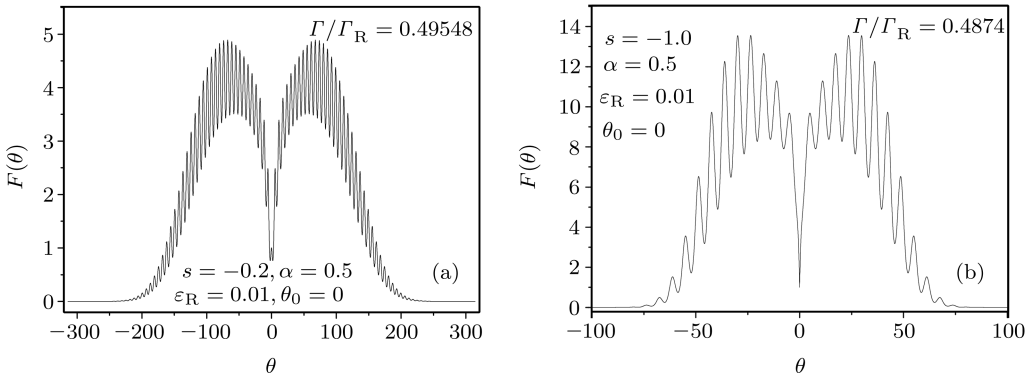


Fig.3. Eigenfunctions of the unstable modes. (a) The eigen-function of the most unstable mode for parameters shown in this figure with $s = -0.2$. (b) The same as in 3(a) with $s = -1.0$.

6. Dependence of growth rate on $\varepsilon_{\rm R}$

How the resistivity itself affects the growth rate is an interesting problem in the resistive ballooning mode. Because, for a very small value of $\varepsilon_{\rm R}$, it is difficult to keep the numerical error small, we restrict ourselves to the case of $\varepsilon_{\rm R} \geq 0.0001$. It is found that the normalized growth rate of the most unstable mode is only slowly changing over a very wide range of $\varepsilon_{\rm R}$ as shown in Fig.4. Here we purposely select the initial phase $\theta_0 = \pi$. It can be seen in next section that the normalized growth rate of the most unstable mode is also a slowly varying function of θ_0 so we do not show other curves for different θ_0. For the eigenfunctions that are quite different for different $\varepsilon_{\rm R}$, we still use the $\theta_0 = 0$ (Fig.5). Then we learn that the coefficient $C(\alpha, s, \varepsilon_{\rm R}, \theta_0)$ only slowly varies for quite large changes of $\varepsilon_{\rm R}$; while for the smaller value of $\varepsilon_{\rm R}$, this change seems to be a little larger.

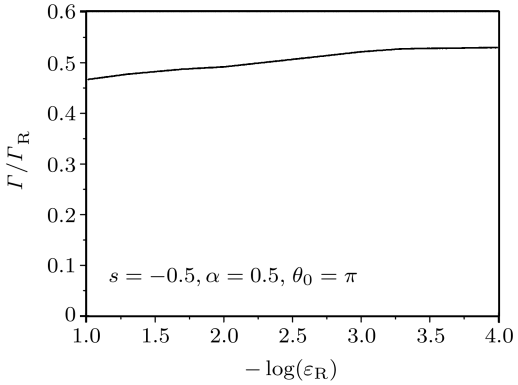


Fig.4. The normalized growth rate of the most unstable mode versus the change of $\varepsilon_{\rm R}$.

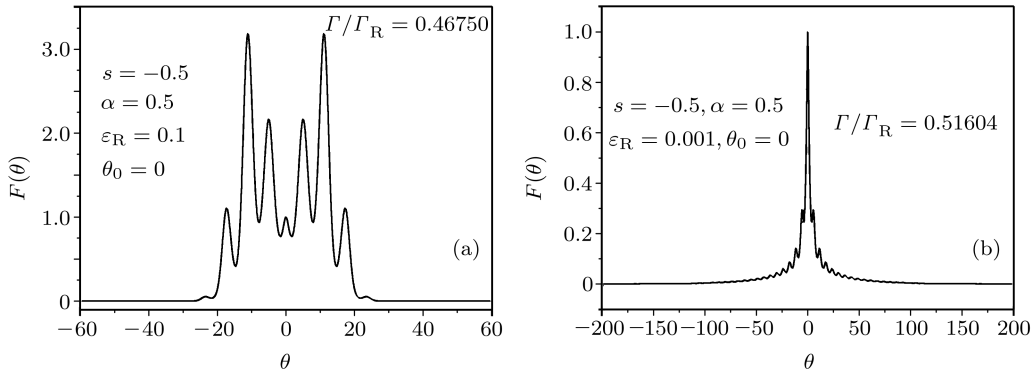


Fig.5. Eigenfunctions of the unstable modes. (a) The eigenfunction of the most unstable mode for $\varepsilon_R = 0.1$. (b) The eigenfunction for $\varepsilon_R = 0.001$. For an intermediate $\varepsilon_R = 0.01$, the eigenfunction is already shown in Fig.1(a).

7. Dependence of growth rate on the pressure gradient

As shown in Fig.6, the growth rate of the most unstable mode is a monotonically increasing function of α. This tendency remains when we change the parameters $s, \varepsilon_R, \theta_0$. The three curves shown in this figure are for $\varepsilon_R = 0.01, \theta_0 = 0$, and $s = -0.1, -0.5$ and -1.0, respectively. We find that the difference among these curves is too small to distinguish. It means that the coefficient C in Eq.(15) depends only very weakly on the negative shear value. The eigenfunction becomes much narrower for larger α as shown in Figs.7(a) and 7(b). The curves of an intermediate $\alpha = 0.5$ with other parameters being the same as those in Fig.7 are already shown in Fig.1(a). If neglecting its dependence on $s, \varepsilon_R, \theta_0$, the coefficient C in expression (15) can be fitted approximately by the following polynomial:

$$C(\alpha, s, \varepsilon_R, \theta_0) = \begin{array}{l} 0.040884 + 1.25279\alpha - 0.89314\alpha^2 \\ +0.47095\alpha^3 - 0.09449\alpha^4 \end{array}$$

$$\text{for} \quad 0.02 \le \alpha \le 2.0 \tag{16}$$

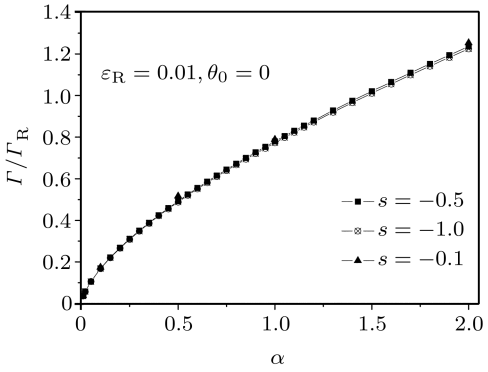


Fig.6. Dependence of the growth rate of the most unstable mode on α for fixed $\varepsilon_R = 0.01, \theta_0 = 0$ and three values of s:$s = -0.1, -0.5, -1.0$.

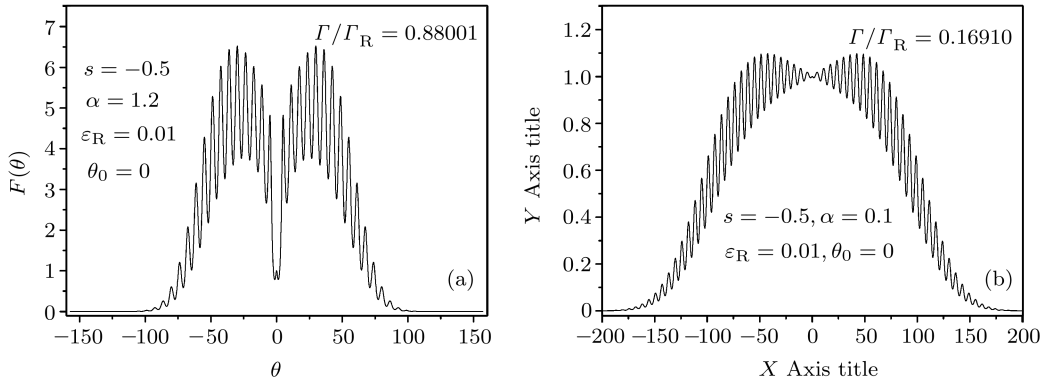


Fig.7. The eigenfunctions of the most unstable mode for different α. (a) The eigenfunction of the most unstable mode for $s = -0.5, \alpha = 1.2$ $\varepsilon_R = 0.01$ and $\theta_0 = 0$. (b) The eigenfunction of the most unstable mode for $s = -0.5, \alpha = 0.1$, $\varepsilon_R = 0.01$ and $\theta_0 = 0$.

8. Summary

Now we can conclude from the above investigations: (1) In the negative shear region of an improved confinement of tokamak plasma, the plasma is unstable against the resistive ballooning mode, (2) The growth rate of the most unstable mode can be expressed by expressions (15) and (16) quite accurately, and it basically is a monotonically increasing function of α, keeping almost unchanged against changes in the absolute value of the magnetic shear and the initial phase θ_0. The dependence of this growth rate on resistivity, toroidal mode number n, and the safety factor q is expressed through $\Gamma_{\rm R} = (n^2q^2\eta/\mu_0 r^2)^{1/3}\Gamma_{\rm A}^{2/3}$ with $\Gamma_{\rm A}$ being the typical Alfvenic time. This expression to some extent can provide a basis for understanding the resistive turbulence in the negative shear region.

References

[1] Levinton F M, Zarnstoff M C, Batha S H, Bell M, Bell R E, Bundny R V, Bush C, Zhang Z, Fridrickson E, Janos A, Manicham J, Ramsey A, Sabbagh S A, Schmidt G L, Synakowski E J and Taylor G 1995 *Phys. Rev. Lett.* **75** 4417

[2] Strait E J, Lao L L, Mauel M E, Rice B W, Taylor T S, Burrell K H, Chu M S, Lazarus E A, Osborne T H, Tompson S J and Tunbull A D 1995 *Phys. Rev. Lett.* **75** 4421

[3] JET Team and Gormezano C 1997 *Fusion Energy*: *Proc. 16th Int. Conf.* (Montreal 1996) Vol **1** (Vienna IAEA) p487

[4] Fujita T, Ide S, Shirai H, Kikuchi M, Naito O, Takeji S, Kubo H and Ishida S 1997 *Phys. Rev. Lett.* **78** 2377

[5] Connor J W 2004 *Plasma Phys. Control. Fusion* **45** R1

[6] Chu M S, Greene J M, Lao L L, Miller R L, Bonderson A, Sauter O, Rice B W, Strait E J, Taylor T S and Turnbull A D 1996 *Phys. Rev. Lett.* **77** 2710

[7] Hirose A and Elina M 1996 *Phys. Rev. Lett.* **76** 628

[8] Drake J F and Antonson T M 1985 *Phys. Fluids* **28** 544

[9] Connor J W, Hastie R J and Martin T J 1985 *Plasma Phys. Control. Fusion* **27** 1509

[10] Sykes A, Bishop C M and Hastie R J 1987 *Plasma Phys. Control. Fusion* **29** 719

[11] Fu G Y and Van Dam J W 1990 *Phys. Fluids* B **2** 2623

[12] Bateman G and Nelson D B 1978 *Phys. Rev. Lett.* **41**(26) 1804

[13] Kim J Y and Choi D I 1989 *Phys. Fluids* B **1** 1444

[14] Connor J W, Hastie R J and Taylor J B 1979 *Proc. Roy. Soc.* (London) A **365** 1

[15] Shi B R and Qu W X 2006 *Chin. Phys.* **15** 1532