Vol 16 No 5, May 2007
1009-1963/2007/16(05)/1263-04

Chinese Physics

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# Chaotic synchronization for a class of fractional-order chaotic systems

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(Received 24 August 2006; revised manuscript received 22 September 2006)

In this paper, a very simple synchronization method is presented for a class of fractional-order chaotic systems only via feedback control. The synchronization technique, based on the stability theory of fractional-order systems, is simple and theoretically rigorous.

**Keywords:** synchronization, fractional-order chaotic systems, stability theory
**PACC:** 0545

## 1. Introduction

Derivatives and integrals of fractional-order have found many applications in recent studies in physics and engineering. Many systems[1,2] are known to display fractional order dynamics, such as viscoelastic systems, dielectric polarization, electrode-electrolyte polarization and electromagnetic waves. But, the dynamics of fractional-order systems has not yet been fully studied. Recently, many authors have investigated the chaotic dynamics of fractional-order dynamical systems. It was shown that the fractional-order system can produce a chaotic attractor.[2−6] The chaotic behaviour and its control in fractional-order chaotic system were studied in Refs.[4,6]. Chaos and hyperchaos in the fractional-order Rŏssler equations were studied in Ref.[5], in which they showed that chaos can exist in the fractional-order Rŏssler equation with order as low as 2.4, and hyperchaos can exist in the fractional-order Rŏssler hyperchaos equation with order as low as 3.8.

On the other hand, chaotic synchronization and control have held the interest of many authors in the past few decades. Recently, the synchronization and control of fractional-order chaotic systems has attracted much attentions,[7−10] and many results have obtained. In this paper, a very simple synchronization method is presented for a class of fractional-order chaotic systems via feedback control. The synchronization technique, based on stability theory of fractional-order systems, is simple and theoretically rigorous.

## 2. A class of fractional-order chaotic systems

First, we review the fractional derivatives. The Riemann–Liouville definition is

$$\frac{\mathrm{d}^{\alpha} f(t)}{\mathrm{d}t^{\alpha}} = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^{n}}{\mathrm{d}t^{n}} \int_{\alpha}^{t} \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} \mathrm{d}\tau \quad (1)$$

for $n-1 \leq \alpha < n$, where $\Gamma(n-\alpha)$ is the Gamma function.

Now, we consider a class of fractional derivative chaotic systems,

$$\begin{aligned} \frac{\mathrm{d}^{q} x_1}{\mathrm{d}t^{q}} &= f_1(x_1, x_2, x_3), \\ \frac{\mathrm{d}^{q} x_2}{\mathrm{d}t^{q}} &= f_2(x_1, x_2, x_3), \\ \frac{\mathrm{d}^{q} x_3}{\mathrm{d}t^{q}} &= f_3(x_1, x_2, x_3), \end{aligned} \quad (2)$$

where $0 < q \leq 1$ are the fractional orders, $x = (x_1, x_2, x_3)^{\mathrm{T}}$ are state variables and real numbers; $f_i (i = 1, 2, 3)$ are continuous functions and only one is nonlinear function. Without loss of generality, one can take $f_3(x_1, x_2, x_3)$ as the nonlinear function. (If $f_1(x_1, x_2, x_3)$ is nonlinear function, $x_1$ and $x_3$ can be interchanged, so $f_3(x_1, x_2, x_3)$ is a nonlinear function. If $f_2(x_1, x_2, x_3)$ is nonlinear function, $x_2$ and $x_3$ can be interchanged, so $f_3(x_1, x_2, x_3)$ is a nonlinear function.) There are many practical chaotic systems like system (2): for example, the fractional-order Chua chaotic circuit and fractional-order Rŏssler chaotic system, etc.

The fractional-order Rŏssler chaotic system[11] is system (3),

$$\begin{aligned}\frac{\mathrm{d}^q x_1}{\mathrm{d}t^q} &= -(x_2 + x_3),\\ \frac{\mathrm{d}^q x_2}{\mathrm{d}t^q} &= x_1 + \alpha x_2,\\ \frac{\mathrm{d}^q x_3}{\mathrm{d}t^q} &= 0.2 + x_3(x_1 - 10),\end{aligned} \tag{3}$$

where $q = 0.9$, $a = 0.4$, the chaotic attractors is shown in Fig.1.

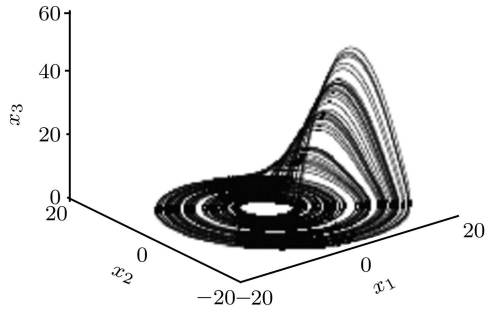



**Fig.1.** Chaotic attractor of the fractional-order Rŏssler system.

The fractional-order Chua chaotic circuit[11] is system (4),

$$\begin{aligned}\frac{\mathrm{d}^q x_1}{\mathrm{d}t^q} &= a\left(x_2 + \frac{x_1 - 2x_1^3}{7}\right),\\ \frac{\mathrm{d}^q x_2}{\mathrm{d}t^q} &= x_1 - x_2 + x_3,\\ \frac{\mathrm{d}^q x_3}{\mathrm{d}t^q} &= -\frac{100}{7}x_2.\end{aligned} \tag{4}$$

Where $q = 0.9$, $a = 12.75$, the chaotic attractors is shown in Fig.2.

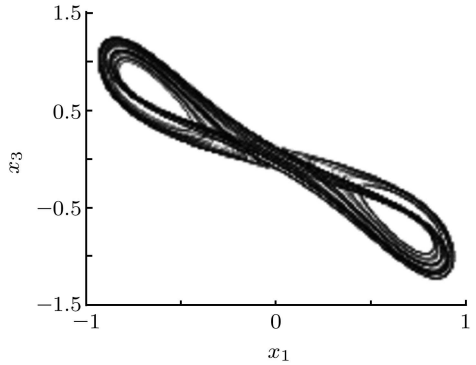



**Fig.2.** Chaotic attractor of the fractional-order Chua circuit.

Now, $x_1$ and $x_3$ can be interchanged, so chaotic system (4) can be changed to chaotic system (5),

$$\begin{aligned}\frac{\mathrm{d}^q x_1}{\mathrm{d}t^q} &= -\frac{100}{7}x_2,\\ \frac{\mathrm{d}^q x_2}{\mathrm{d}t^q} &= x_1 - x_2 + x_3,\\ \frac{\mathrm{d}^q x_3}{\mathrm{d}t^q} &= a\left(x_2 + \frac{x_3 - 2x_3^3}{7}\right).\end{aligned} \tag{5}$$

Chaotic system (2) can be changed to chaotic system (6) by the reversible linear transform $X = AY$,[12]

$$\begin{aligned}\frac{\mathrm{d}^q y_1}{\mathrm{d}t^q} &= y_2,\\ \frac{\mathrm{d}^q y_2}{\mathrm{d}t^q} &= y_3,\\ \frac{\mathrm{d}^q y_3}{\mathrm{d}t^q} &= g(y_1, y_2, y_3).\end{aligned} \tag{6}$$

Where $Y = (y_1, y_2, y_3)^{\mathrm{T}}$ are state variables and real numbers of system (6), and $A$ is constant matrix and is reversible, $g(y_1, y_2, y_3)$ is one nonlinear function determined by transform $X = AY$. For example, use the following reversible linear transform: $X = AY$,

$$\begin{cases} x_1 = -0.4y_1 + y_2,\\ x_2 = y_1,\\ x_3 = -y_1 + 0.4y_2 - y_3.\end{cases}$$

The fractional-order Rŏssler chaotic system (3) can be changed to the following system:

$$\begin{aligned}\frac{\mathrm{d}^q y_1}{\mathrm{d}t^q} &= y_2,\\ \frac{\mathrm{d}^q y_2}{\mathrm{d}t^q} &= y_3,\\ \frac{\mathrm{d}^q y_3}{\mathrm{d}t^q} &= -0.2 - y_2 + 0.4y_3\\ &\quad -(0.4y_2 - y_1 - y_3)(y_2 - 0.4y_1 - 1).\end{aligned}$$

For the fractional-order Chua chaotic circuit system (5), we can use the following reversible linear transform: $X = AY$,

$$\begin{cases} x_1 = m^{2/3}y_1,\\ x_2 = -m^{-1/3}y_2,\\ x_3 = -m^{2/3}y_1 - m^{-1/3}(y_2 + y_3),\end{cases}$$

where $m = \dfrac{100}{7}$. So, the fractional-order Chua chaotic circuit system (5) can be changed to the following system:

$$\frac{\mathrm{d}^q y_1}{\mathrm{d}t^q} = y_2,$$

$$\begin{aligned}\frac{\mathrm{d}^q y_2}{\mathrm{d}t^q} &= y_3,\\ \frac{\mathrm{d}^q y_3}{\mathrm{d}t^q} &= -12.75m^{-1/3}y_2+\frac{12.75}{7}(-m^{2/3}y_1\\ &\quad -m^{-1/3}(y_2+y_3))\\ &\quad -\frac{25.5}{7}(-m^{2/3}y_1-m^{-1/3}(y_2+y_3))^3. \quad (7)\end{aligned}$$

So, there exists a class of fractional-order chaotic systems having the same form as system (2) or as system (6).

## 3. Synchronization for a class of fractional-order chaotic systems

Now, we discuss the chaotic synchronization for a class of fractional-order chaotic systems via feedback control.

System (6) can be changed as system (8),

$$\begin{aligned}\frac{\mathrm{d}^q y_1}{\mathrm{d}t^q} &= y_2,\\ \frac{\mathrm{d}^q y_2}{\mathrm{d}t^q} &= y_3,\\ \frac{\mathrm{d}^q y_3}{\mathrm{d}t^q} &= b_1y_1+b_2y_2+b_3y_3+h(y_1,y_2,y_3), \quad (8)\end{aligned}$$

where $g(y_1,y_2,y_3) = h(y_1,y_2,y_3)+\sum_{i=1}^{3}b_iy_i$, and $b_iy_i(i=1,2,3)$ are the linear parts of $g(y_1,y_2,y_3)$ and $b_i(i=1,2,3)$ are real numbers, $h(y_1,y_2,y_3)$ is the nonlinear parts of $g(y_1,y_2,y_3)$.

System (8) is called the drive system.

The response system with feedback control is constructed as follows:

$$\begin{aligned}\frac{\mathrm{d}^q z_1}{\mathrm{d}t^q} &= z_2+k_1(y_1-z_1),\\ \frac{\mathrm{d}^q z_2}{\mathrm{d}t^q} &= z_3+k_2(y_1-z_1),\\ \frac{\mathrm{d}^q z_3}{\mathrm{d}t^q} &= b_1z_1+b_2z_2+b_3z_3+h(y_1,y_2,y_3)\\ &\quad +k_3(y_1-z_1). \quad (9)\end{aligned}$$

Where $K=[k_1,k_2,k_3]^{\mathrm{T}}$ are control parameters, which will be determined later. Defining the synchronization error $e_i=z_i-y_i,(i=1,2,3)$ between system (8) and system (9), we can obtain the error of dynamical system as follows:

$$\begin{aligned}\frac{\mathrm{d}^q e_1}{\mathrm{d}t^q} &= e_2-k_1e_1,\\ \frac{\mathrm{d}^q e_2}{\mathrm{d}t^q} &= e_3-k_2e_1,\\ \frac{\mathrm{d}^q e_3}{\mathrm{d}t^q} &= b_1e_1+b_2e_2+b_3e_3-k_3e_1. \quad (10)\end{aligned}$$

Error of dynamical system (10) can be written as

$$\frac{\mathrm{d}^q e}{\mathrm{d}t^q} = (A-KC)e, \quad (11)$$

where $e=[e_1,e_2,e_3]^{\mathrm{T}}$, $C=[1,0,0]$ and $A=\begin{bmatrix}0&1&0\\0&0&1\\b_1&b_2&b_3\end{bmatrix}$. Certainly, the matrix $[C,CA,CA^2]^{\mathrm{T}}$ is of full rank.

According to the stability theory of fractional-order linear systems,[13] we can obtain the following synchronization theorem.

**Theorem** If $|\arg(\lambda_i(A-KC))|>0.5\pi q, i=1,2,3$, by choosing control parameters $K$, then the fractional-order chaotic system (8) and the fractional-order chaotic system (9) can arrive to the global asymptotic chaotic synchronization. Where $\arg(\lambda_i(A-KC))$ denotes the argument of the eigenvalue $\lambda_i$ of $(A-KC)$ in Eq.(11).

**Proof** Because the matrix $[C,CA,CA^2]^{\mathrm{T}}$ is of full rank, according to the linear system theory,[14] we can obtain that the eigenvalue $\lambda_i$ of $(A-KC)$ in Eq.(11) can be placed anywhere by a suitable choice of $K=[k_1,k_2,k_3]^{\mathrm{T}}$, so there exists $K=[k_1,k_2,k_3]^{\mathrm{T}}$ such that $|\arg(\lambda_i(A-KC))|>0.5\pi q,(i=1,2,3)$. The solution of the problem to determine $K=[k_1,k_2,k_3]^{\mathrm{T}}$ such that the arguments of all the eigenvalues $\lambda_i$ of $(A-KC)$ in Eq.(11) satisfy $|\arg(\lambda_i(A-KC))|>0.5\pi q,(i=1,2,3)$. According to the stability theory of fractional-order linear systems,[13] we can obtain $\lim\limits_{t\to+\infty}e_i=0,(i=1,2,3)$, i.e., the fractional-order chaotic system (8) and fractional-order chaotic system (9) are in a state of global asymptotic chaotic synchronization. Namely, the fractional-order chaotic system (8) and the fractional-order chaotic system (9) can arrive at the global asymptotic chaotic synchronization by the 'pole placement technique' from control systems theory such as Ackermann's method.[14]

According to the above theorem, we can realize chaotic synchronization for a class of fractional-order chaotic systems.

In the following, we take a special case for example. According to the above theorem, we obtain that the fractional-order chaotic system (8) and the fractional-order chaotic system (9) are in a state of global asymptotic synchronization for $|\arg(\lambda_i(A-KC))|>0.5\pi q,(i=1,2,3)$. For example, we can choose $k_1=3+b_3$, $k_2=3+b_2+b_3(3+b_3)$, $k_3=1+b_1+b_2(3+b_3)+b_3(3+b_2+b_3(3+b_3))$ such

that the eigenvalues of $A - KC$ are those specified in vector $[-1, -1, -1]$. The synchronization simulation result for fractional-order Rŏssler chaotic system is shown in Fig.3. The synchronization simulation result for fractional-order Chua chaotic circuit system is shown in Fig.4, and $\varepsilon = \left(\sum_{i=1}^{3} e_i^2\right)^{1/2}$.

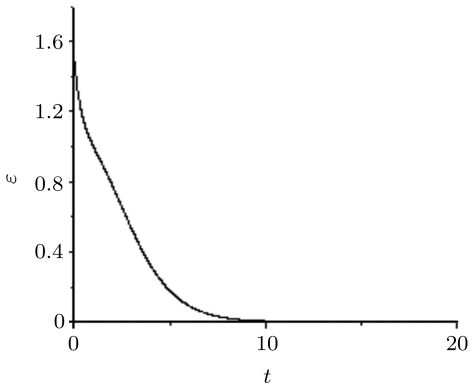



**Fig.3.** The synchronization result for fractional-order Rŏssler chaotic system.

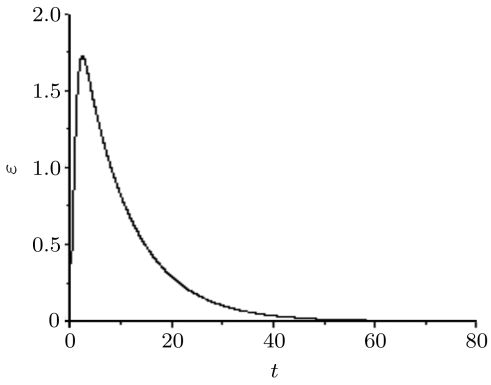



**Fig.4.** The synchronization result for fractional-order Chua chaotic circuit system.

# 4. Conclusion

According to the stability theory of fractional-order linear systems, a very simple chaotic synchronization method is presented for a class of fractional-order chaotic systems, and this method is theoretically rigorous.

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