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# Chaotifying a stable linear controllable system by single input state feedback

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In this paper, an approach for chaotifying a stable controllable linear system via single input state-feedback is presented. The overflow function of the system states is designed as the feedback controller, which can make the fixed point of the closed-loop system to be a snap-back repeller, thereby yields chaotic dynamics. Based on the Marotto theorem, it proves theoretically that the closed-loop system is chaotic in the sense of Li and Yorke. Finally, the simulation results are used to illustrate the effectiveness of the proposed method.

**Keywords:** chaotification, overflow function, Marotto theorem, state-feedback
**PACC:** 0545

## 1. Introduction

With the rapid development of chaos control[1−3] and synchronization[4−7] in the last decade, chaotification (or called 'anticontrol of chaos') by means of making an original non-chaotic dynamic system chaotic, or enhancing the existing chaos of a chaotic system, has attracted increasing attention in recent years due to the great potential of chaos in some novel time- and/or energy-critical applications.[8−11] Chaos under monitoring can provide a system designer with a variety of special properties, richness of flexibility, and a cornucopia of opportunities. For example, chaos is thought to be important in the human brain,[12] heartbeat regulations,[13] information processing, liquid mixing, and secure communication, etc.

In order to chaotify discrete dynamic systems, a simple mathematically rigorous chaotification method was developed by Chen and Lai with the state feedback control approach.[14−16] Afterwards, Wang and Chen[17,18] further showed that the Chen–Lai algorithm leads to chaos not only in the sense of Devany[19] but also in the sense of Li–Yorke.[20] However, most existing chaotification methods assume that the gain matrix of the chaotification controller is nonsingular. In fact, it is very difficult to let the controller gain matrix to be a nonsingular one in some practical applications. Therefore, a natural yet nontrivial question is whether one can chaotify the controlled system when the controller gain matrix is singular. The present paper attempts to provide a positive answer to this question. More precisely, for a controllable linear asymptotic stable system only with single input in which the control gain only has one column, we design a feedback controller which is a overflow function of the system states to chaotify the stable system. We apply the Marotto theorem to prove theoretically that the closed-loop system is indeed chaotic in the sense of Li and Yorke. Finally, the simulation examples are used to illustrate the design and the success.

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## 2. Problem description

Without loss of generality, we consider an $n$-dimensional linear time-invariant controllable discrete dynamic system:

$$\begin{cases} x_{k+1} = Ax_k + Bu_k, \\ x_0 \in R^n \quad \text{(given)}, \end{cases} \tag{1}$$

where $A$ is an $n \times n$ nonsingular constant matrix and $B$ is an $n \times 1$ constant vector. They are in Brunowsky controllable canonical form, namely,

$$A = \begin{bmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ \vdots & & & 1 \\ -a_1 & \cdots & \cdots & -a_n \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 1 \end{bmatrix}. \tag{2}$$

The characteristic equation of system (1) has the form

$$f(s) = s^n + a_n s^{n-1} \cdots + a_1. \tag{3}$$

Our purpose is to design a control input sequence $\{u_k\}_{k=0}^{\infty}$, which are functions of the system states

$$u_k = u_k(x_k) \tag{4}$$

to make the following closed-loop system

$$x_{k+1} = g(x_k, u_k(x_k)) = Ax_k + Bu_k(x_k) \tag{5}$$

chaotic in the sense of Li–Yorke.

**Remark 1** If the dynamic system is controllable and not in Brunowsky controllable canonical form, there always exists a nonsingular coordinate transform which can make its coefficient matrices to be in canonical form,[21] like the form (2).

## 3. Preliminary knowledge

It is well known that the first mathematical definition of chaos was given by Li and Yorke, where a very simple condition for chaos in a one-dimensional discrete time system was obtained ("period three implies chaos').[20] Marotto extended that work to multidimensional systems by introducing the concept of snap-back repellers, and deduced that 'snap-back repellers imply chaos in $R^n$'.[22] Then the generalized theorem of Marotto was developed to include saddle points by Shiraiwa and Kurata.[23] Afterwards, a lot of scholars study chaos by applying the Marotto theorem. For example, Ushio and Hirai obtained sufficient conditions for chaos in nonlinear sampled-data control systems.[24,25] Chen and Aihara showed that both transiently chaotic neural networks and discrete-time recurrent neural networks are chaotic,[26] and Chen *et al* verified spatiotemporal chaos in wave propagation.[27] More recently, Lu also used this theorem to verify the method of anticontrol chaos in Takagi–Sugeno system.[28,29]

For the completeness of our presentation, we briefly introduce the Marotto theorem here. Consider an $n$-dimensional discrete-time autonomous system of the form

$$x_{k+1} = g(x_k), \tag{6}$$

where $g$ is a continuous function. Let $g^t$ denote the $t$ times compositions of $g$ with itself. A point, $x^*$, is said to be a $p$-periodic point of $g$, if $g^p(x^*) = x^*$ but $g^t(x^*) \neq x^*$ for $1 \leq t < p$. If $p = 1$, that is, $g(x^*) = x^*$, $x^*$ is called a fixed point. Let $Dg(x)$ and $\det(Dg(x))$ be the Jacobian of $g$ at the point $x$ and its determinant respectively, and let $B(x;r) = \{y \in R^n | \, \|y - x\| \leq r\}$ be a closed ball in $R^n$ of radius $r$ centred at the point $x$. In this paper, $\|\bullet\|$ denotes the Euclidean norm of a finite-dimensional vector or a matrix.

**Definition**[22] A fixed point $x^*$ of system (6) is said to be a snap-back repeller if

(i) there exists a real number, $r > 0$, such that $g$ is differentiable with all eigenvalues of $Dg(x)$ exceeding the unity in absolute value for all $x \in B(x^*;r)$;

(ii) there exists a point $x_0 \in B(x^*;r)$ with $x_0 \neq x^*$, such that for some positive integer $m$, $g^m(x_0) = x^*$ and $g^m(x_0)$ is differentiable at $x_0$ with $\det(Dg^m(x_0)) \neq 0$.

**Theorem of Marotto**[22] If system (6) has a snap-back repeller then the system is chaotic in the sense of Li–Yorke:

(i) there exists a positive integer $n$ such that for every integer $p \geq n$, system (6) has a periodic orbit with period $p$;

(ii) there exists a scrambled set (an uncountable invariant set $S$ containing no periodic points) such that

(a) $g(S) \subset S$,

(b) for every $X, Y \in S$ with $X \neq Y$,

$$\lim_{k\to\infty} \sup \left\|g^k(X) - g^k(Y)\right\| > 0, \tag{7}$$

(c) for every $X \in S$ and any periodic point Y of system (6),

$$\lim_{k\to\infty} \sup \left\|g^k(X) - g^k(Y)\right\| > 0; \tag{8}$$

(iii) there exists an uncountable subset $S_0$ of $S$ such that for any $X, Y \in S_0$,

$$\lim_{k\to\infty} \inf \left\|g^k(X) - g^k(Y)\right\| = 0. \tag{9}$$

## 4. Generate chaos via state feedback in linear controllable system

To make the closed-loop system (5) chaotic, we design a single-input state feedback controller

$$u_k(x_k) = K\Phi(x_k), \tag{10}$$

where $K$ is a $1 \times n$ constant vector $K = (k_1, \ldots, k_n)$ and $\Phi(x_k)$ has the form

$$\Phi(x) = (\varphi(x_1), \varphi(x_2), \ldots, \varphi(x_n))^{\mathrm{T}}. \tag{11}$$

The $\varphi(y)$ is overflow function defined by (see Fig.1)

$$\varphi(y) = y - 2m\varepsilon \quad \text{for} \quad (2m-1)\varepsilon \le y < (2m+1)\varepsilon,$$
$$m = 0, \pm 1, \pm 2, \ldots. \tag{12}$$

Then we can rewrite the closed-loop system as

$$g(x) = Ax + BK\,\Phi(x). \tag{13}$$

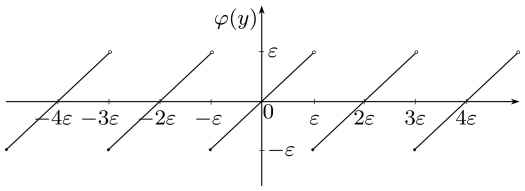



**Fig.1.** The overflow function.

**Theorem 1** If the constant vector $K$ satisfies the two following conditions:

(i) all eigenvalues of $A + BK = \bar{A}$ exceed the unity in absolute value,

(ii) $k_i = 0$, if $a_i = 0 (1 < i \le n)$, otherwise

$$-1 < \frac{a_1}{k_1} = \frac{a_2}{k_2} \cdots = \frac{a_n}{k_n} < \frac{1}{3}, \tag{14}$$

then the controllable system (1) with controller (10) is chaotic in the sense of Li–Yorke.

**Proof** Obviously, $x^* = 0$ is the fixed point of the system (13), and now we will prove that the $x^*$ is a snapback repeller point.

First, we show that there exists a neighbourhood $B(x^*; r)$ of $x^* = 0$ such that $g$ is expanding in it. Indeed, since $x = (x^1, x^2, \ldots, x^n) \in B(x^*; r)$, $\|x\| < r < \varepsilon$ and $|x^i| < r < \varepsilon (i = 1, 2, \ldots, n)$, we have

$$g(x) = Ax + BK\,\Phi(x) = Ax + BKx, \tag{15}$$

and

$$\mathrm{D}g(x) = A + BK = \bar{A}. \tag{16}$$

According to the linear control systems theory, we can configure the eigenvalues of the matrix $\bar{A}$ arbitrarily. It means that the condition of Theorem 1 about snap-repeller (i) is easy to satisfy by choosing suitable vector $K$ to let all eigenvalues of $\bar{A}$ exceed the unity in absolute value.

Second, we show that there exists a point $x_0 \in B(x^*; r)$ and $x_0 \ne x^*$ such that $g^{n+1}(x_0) = x^* = 0$. Let $x_0 = (x_0^1, 0, \ldots, 0)^{\mathrm{T}} \in B(x^*; r)$, where $x_0^1 = (2k_1\varepsilon)/(k_1 - a_1)^2$ and choose the vector $K$ to let $|x_0^1| < r < \varepsilon$ and satisfy the condition (14). Thence we can get

$$g(x_0) = x_1 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ x_1^n = -a_1 x_0^1 + k_1 x_0^1 = \dfrac{2k_1\varepsilon}{k_1 - a_1} \end{bmatrix}, \tag{17}$$

and $\varepsilon < x_1^n < 3\varepsilon$ by using Eq.(14). Afterwards, we obtain

$$g^2(x_0) = x_2 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ x_1^n \\ -a_n x_1^n + k_n x_1^n - 2k_n\varepsilon = 0 \end{bmatrix}, \tag{18}$$

$$g^n(x_0) = x_n = \begin{bmatrix} x_1^n \\ 0 \\ \vdots \\ 0 \\ -a_2 x_1^n + k_2 x_1^n - 2k_2\varepsilon = 0 \end{bmatrix} \tag{19}$$

and

$$g^{n+1}(x_0) = x_{n+1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -a_1 x_1^n + k_1 x_1^n - 2k_1\varepsilon = 0 \end{bmatrix} = x^*. \tag{20}$$

Finally, we prove that the $\det(\mathrm{D}(g^{n+1}(x_0))) \ne 0$. Clearly, according to the differential chain rule, $|x_0^1| < r < \varepsilon$ and $\varepsilon < x_1^n < 3\varepsilon$, we have

$$\begin{aligned} \det(\mathrm{D}(g^{n+1}(x_0))) = {} & \det(\mathrm{D}g(x_0)) \det(\mathrm{D}g(x_1)) \\ & \cdots \det(\mathrm{D}g(x_n)) \\ = {} & (\det(A + BK))^{(n+1)} \ne 0. \end{aligned} \tag{21}$$

In the end, we conclude that $x^* = 0$ is a snapback repeller of closed-loop system. So the states of system (13) will be chaotic in sense of Li–Yorke.

**Remark 2** It is a very important problem how to choose the feedback gain $K$ of Theorem 1, but there is no general way to do it. Therefore, we have to calculate $K$ for each system heuristically.

## 5. Simulation examples

Here we use two simulation examples to illustrate the effectiveness of the proposed method. First, consider a two-dimensional (2D) controllable system with the form:

$$\begin{bmatrix} x_{k+1}^1 \\ x_{k+1}^2 \end{bmatrix} = Ax_k + Bu_k = \begin{bmatrix} 0 & 1 \\ 0.72 & 0.1 \end{bmatrix} \begin{bmatrix} x_k^1 \\ x_k^2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k. \quad (22)$$

The eigenvalues of the matrix $A$ are $\lambda_1 = -0.8, \lambda_2 = 0.9$. Therefore, the uncontrolled system (with $u_k = 0$) is asymptotically stable. To make the system (22) chaotic, the controller is taken as

$$u_k = k_1\varphi(x_k^1) + k_2\varphi(x_k^2). \quad (23)$$

In the simulation, we choose $-1 < a_1/k_1 = a_2/k_2 = -0.24 < 1/3$ and $\varepsilon = 0.75$ such that $k_1 = 3, k_2 = 0.41667$. Then they will satisfy the condition that all the eigenvalues of the matrix $\bar{A}$ exceed 1 in norm. Figure 2 shows the simulation result which represents the evolution of the states $x_k^1$ and $x_k^2$, because they are only one step in difference. Figure 3 shows the strange attractor of the controlled system.

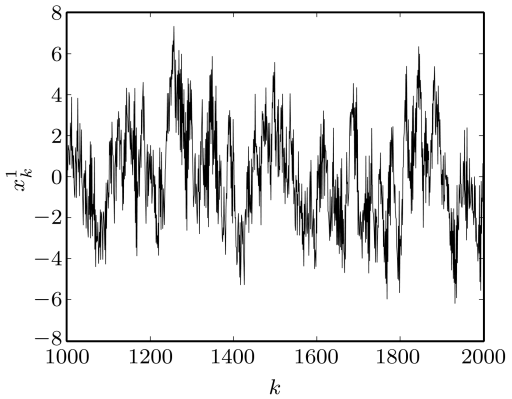



**Fig.2.** Simulation result of the system state $x_k^1(x_k^2)$ $(1000 \le k \le 2000)$.

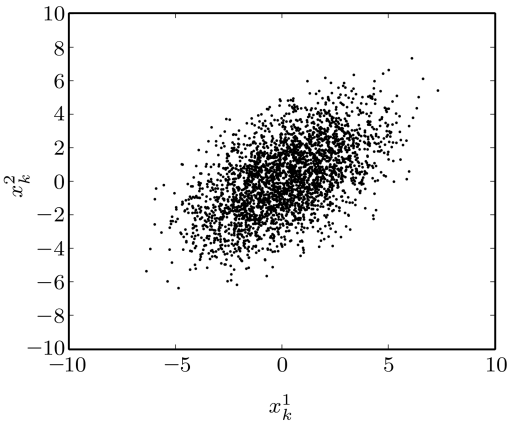



**Fig.3.** The strange attractor of the controlled system $(1000 \le k \le 2000)$.

Second, we consider a three-dimensional (3D) controllable system as:

$$\begin{bmatrix} x_{k+1}^1 \\ x_{k+1}^2 \\ x_{k+1}^3 \end{bmatrix} = Ax_k + Bu_k = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.6 & 0.3 & 0 \end{bmatrix} \begin{bmatrix} x_k^1 \\ x_k^2 \\ x_k^3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_k. \quad (24)$$

Since the eigenvalues of the matrix $A$ are $\lambda_1 = 0.961, \lambda_2 = -0.481 + 0.627j, \lambda_3 = -0.481 - 0.627j$, the uncontrolled system (with $u_k = 0$) is asymptotically stable. To chaotify the system (24), the controller is taken as

$$u_k = k_1\varphi(x_k^1) + k_2\varphi(x_k^2) + k_3\varphi(x_k^3). \quad (25)$$

According to theorem 1. (ii), we choose the $-1 < a_1/k_1 = a_2/k_2 = -0.1 < 1/3$ and $\varepsilon = 0.75$ such that $k_1 = 6, k_2 = 3, k_3 = 0$. Then the theorem 1. (i) is satisfied well, i.e., all the eigenvalues of the matrix $\bar{A}$ exceed 1 in norm. Figure 4 shows the simulation result which represents the evolution of the states $x_k^1$ and $x_k^2$. And Fig.5 shows the strange attractor of the controlled system. We can see that the controlled systems displayed complex chaotic behaviours.

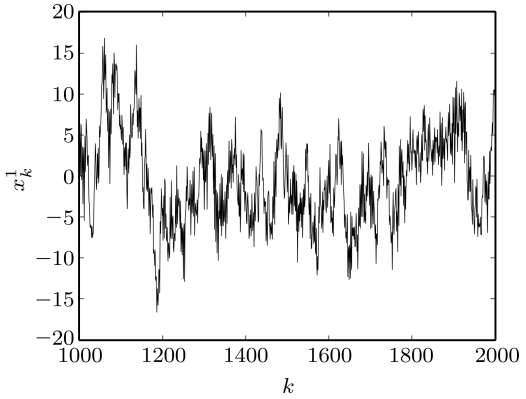



**Fig.4.** Simulation result of the system state $x_k^1(x_k^2)$ $(1000 \leq k \leq 2000)$.

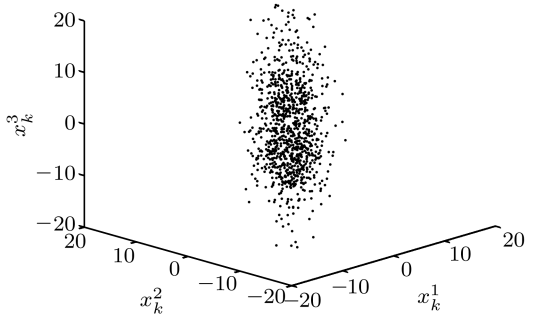



**Fig.5.** The attractor of the controlled system $(1000 \leq k \leq 2000)$.

## 6. Conclusions

This paper has presented a chaotification algorithm for a stable controllable linear system only with single input. Thanks to the useful Marotto theorem, we have proved that the closed-loop system with designed controller is indeed chaotic in sense of Li and Yorke. Intuitively, a stable linear system tends to converge to the origin, while the designed controller introduces an expanding effect into the controlled system against this inherent asymptotic stability. The combination of these two effects yields complex chaotic dynamics within a bounded region of the controlled system orbits. The simulation results have illustrated the effectiveness of the proposed method.

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