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Generalized projective synchronization of a class of chaotic (hyperchaotic) systems with uncertain parameters*

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In this paper is investigated the generalized projective synchronization of a class of chaotic (or hyperchaotic) systems, in which certain parameters can be separated from uncertain parameters. Based on the adaptive technique, the globally generalized projective synchronization of two identical chaotic (hyperchaotic) systems is achieved by designing a novel nonlinear controller. Furthermore, the parameter identification is realized simultaneously. A sufficient condition for the globally projective synchronization is obtained. Finally, by taking the hyperchaotic Lü system as example, some numerical simulations are provided to demonstrate the effectiveness and feasibility of the proposed technique.

Keywords: Chaotic system, hyperchaotic system, generalized projective synchronization, parameter identification

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1. Introduction

Since Lorenz proposed a classical Lorenz chaotic system in 1963,[1] the chaotic phenomena in nonlinear science have been widely studied. Many new chaotic systems have been gradually revealed in the process of research on the classical Lorenz system, such as the Chen system,[2] which is dual but not topologically equivalent to the classic Lorenz system; the Lü system,[3] which is critical between the Lorenz and Chen systems; the unified system,[4] which evolves continuously from the Lorenz to Chen systems, and so on. Recently, the hyperchaotic Chen system[5] and hyperchaotic Lü system[6] have been successively investigated.

Chaos synchronization was introduced for the first time by Pecora and Carroll in 1990,[7] and an effective control method was proposed. In the past years, along with the further research, many schemes for chaos synchronization have been developed, for example, linear coupling method,[8] feedback approach,[9,10] adaptive technique,[11−13] backstepping method,[14] etc.[15−18] Owing to the potential applications in security communication and oscillator design,[19,20] the chaos control and synchronization have become an important area of nonlinear dynamics. At present, the researches are concentrating on the following aspects: the complete (precise) synchronization, generalized synchronization, phase synchronization, lag synchronization, mutual synchronization and so on.[20−24] The generalized projection synchronization (GPS) means that the state vectors of two generalized synchronous chaotic systems evolve proportionally—the vectors become proportional.[20,23,25]

In this paper, we investigate the GPS and parameter identification of a class of chaotic (hyperchaotic) systems. Based on the adaptive technique, a novel nonlinear controller is designed such that the parameter identification is realized, and two identical chaotic (hyperchaotic) systems with uncertain parameters achieve the globally GPS simultaneously. The conclusion is proved theoretically via the Lyapunov method.[26] Finally, some numerical simulations with the hyperchaotic Lü system are given to demonstrate the effectiveness and feasibility of the proposed technique.

2. Description of system model

Consider a class of chaotic (or hyperchaotic) systems described by

$$\dot{x} = Ax + Bf(x), \tag{1}$$

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where $x \in R^n (n \geq 3)$ is the state vector, $A \in R^{n\times n}$ and $B \in R^{n\times s}$ are the constant matrices, $f : R^n \to R^s$ is a continuous nonlinear function. A contains the system parameters and all parameters are linear. $Bf(x)$ does not contain any system parameters. The linear part Ax of Eq.(1) can be rewritten as $Ax = \tilde{A}x + g(x)\alpha$, that is to say, those certain parameters and uncertain parameters can be separated from each other. here, $\alpha \in R^r$ is an uncertain parameter vector, $\tilde{A} \in R^{n\times n}$ is a constant matrix which does not contain any uncertain parameters. $g : R^n \to R^{n\times r}$ is a linear function matrix. Thus Eq.(1) can be rewritten as

$$\dot{x} = \tilde{A}x + g(x)\alpha + Bf(x). \tag{2}$$

The system (1) has one positive Lyapunov exponent at least. It is chaotic when there exists only one positive Lyapunov exponent ($n \geq 3$) and the sum of all Lyapunov exponents is negative; it is hyperchaotic when there exists more than one positive Lyapunov exponent ($n \geq 4$) and the sum of all Lyapunov exponents is negative. The main characteristics of the model (1) are as follows. (i) All of the system parameters are contained in the linear part, and the nonlinear part of the system does not contain any system parameters. (ii) The system is linearly dependent on the system parameters. (iii) The certain parameters are separate from the uncertain parameters. In fact, many chaotic and hyperchaotic systems can be written in the form of system (1). For instance, the Lorenz system, the Chen system, the Lü system, the unified chaotic system, hyperchaotic Chen system, hyperchaotic Lü system, etc. Therefore, the further research on the synchronization of such a class of chaotic (hyperchaotic) systems is very significant.

Take the hyperchaotic Lü system as example, which is described below:

$$\begin{cases} \dot{x}_1 = a(x_2 - x_1) + x_4, \\ \dot{x}_2 = -x_1x_3 + cx_2, \\ \dot{x}_3 = x_1x_2 - bx_3, \\ \dot{x}_4 = x_1x_3 + dx_4. \end{cases} \tag{3}$$

It is hyperchaotic when $a = 36$, $b = 3$, $c = 20$ and $-0.35 < d \leq 1.3$.

Assuming that a and b are certain parameters and c and d are uncertain parameters, then system (3) can be rewritten in the form of systems (1) or (2) as indicated below:

$$\dot{x} = Ax + Bf(x) \quad \text{or} \quad \dot{x} = \tilde{A}x + g(x)\alpha + Bf(x), \tag{4}$$

where $x = [x_1, x_2, x_3, x_4]^{\mathrm{T}}$, $A = \begin{bmatrix} -a & a & 0 & 1 \\ 0 & c & 0 & 0 \\ 0 & 0 & -b & 0 \\ 0 & 0 & 0 & d \end{bmatrix}$,

$B = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $f(x) = \begin{bmatrix} x_1x_3 \\ x_1x_2 \\ x_1x_3 \end{bmatrix}$,

$\tilde{A} = \begin{bmatrix} -a & a & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -b & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $g(x) = \begin{bmatrix} 0 & 0 \\ x_2 & 0 \\ 0 & 0 \\ 0 & x_4 \end{bmatrix}$,

and $\alpha = [c, d]^{\mathrm{T}}$, where $[\cdot]^{\mathrm{T}}$ denotes the transposition of $[\cdot]$.

3. GPS and parameter identification of two identical chaotic (hyperchaotic) systems with uncertain parameters

Now we consider the problems of the GPS and parameter identification of two identical chaotic (hyperchaotic) systems with model (1). The drive (master) system is

$$\begin{aligned} &\dot{x}_{\mathrm{m}} = Ax_{\mathrm{m}} + Bf(x_{\mathrm{m}}) \\ \text{or} \quad &\dot{x}_{\mathrm{m}} = \tilde{A}x_{\mathrm{m}} + g(x_{\mathrm{m}})\alpha + Bf(x_{\mathrm{m}}). \end{aligned} \tag{5}$$

The controlled response (slave) system is

$$\dot{x}_{\mathrm{s}} = \tilde{A}x_{\mathrm{s}} + g(x_{\mathrm{s}})\hat{\alpha} + Bf(x_{\mathrm{s}}) + U, \tag{6}$$

where $Ax_{\mathrm{m}} = \tilde{A}x_{\mathrm{m}} + g(x_{\mathrm{m}})\alpha$, $x_{\mathrm{m}} = [x_{m1}, x_{\mathrm{m}2}, \cdots, x_{\mathrm{m}n}]^{\mathrm{T}}$, and $x_{\mathrm{s}} = [x_{\mathrm{s}1}, x_{\mathrm{s}2}, \cdots, x_{sn}]^{\mathrm{T}}$. $\hat{\alpha} \in R^r$ is the uncertain parameter vector of the response system (6), α is the corresponding parameter vector of the drive system (5) with $\hat{\alpha}$. All other hypotheses are consistent with those in model (1). U is a nonlinear controller to be designed.

Our task is to design a suitable controller U and an adaptive control law about the uncertain parameter vector $\hat{\alpha}$ such that systems (5) and (6) achieve the GPS, and the parameter identification is realized simultaneously. That is, $\lim\limits_{t\to+\infty} e = 0$, and $\lim\limits_{t\to+\infty} (\hat{\alpha} - \alpha) = 0$ where the error vector is defined as $e = x_{\mathrm{s}} - \lambda x_{\mathrm{m}}$ and λ is a scaling factor.

From systems (5) and (6), one has

$$\begin{aligned}\dot{e} &= \tilde{A}(x_{\rm s}-\lambda x_{\rm m})+g(x_{\rm s})\hat{\alpha}-\lambda g(x_{\rm m})\alpha+B(f(x_{\rm s})-\lambda f(x_{\rm m}))+U\\ &= \tilde{A}e+[g(x_{\rm s})-\lambda g(x_{\rm m})]\alpha+g(x_{\rm s})(\hat{\alpha}-\alpha)+B(f(x_{\rm s})-\lambda f(x_{\rm m}))+U\\ &= \tilde{A}e+g(x_{\rm s}-\lambda x_{\rm m})\alpha+g(x_{\rm s})(\hat{\alpha}-\alpha)+B(f(x_{\rm s})-\lambda f(x_{\rm m}))+U\\ &= \tilde{A}e+g(e)\alpha+g(x_{\rm s})(\hat{\alpha}-\alpha)+B(f(x_{\rm s})-\lambda f(x_{\rm m}))+U\\ &= Ae+g(x_{\rm s})(\hat{\alpha}-\alpha)+B(f(x_{\rm s})-\lambda f(x_{\rm m}))+U.\end{aligned}$$

Then one can obtain the error dynamical system

$$\begin{aligned}\dot{e} = {} & Ae+g(x_{\rm s})(\hat{\alpha}-\alpha)\\ & +B(f(x_{\rm s})-\lambda f(x_{\rm m}))+U. \end{aligned} \tag{7}$$

Theorem 1 If there exists the matrix $K\in R^{s\times n}$ in systems (5) and (6), such that $P=\frac{1}{2}[(BK-A)+(BK-A)^{\rm T}]$ is positive definite, we may take the controller

$$U=-B(h(x_{\rm s})-\lambda h(x_{\rm m})), \tag{8}$$

where $h(x)=f(x)+Kx$, and the control law

$$\dot{\hat{\alpha}}=-(g(x_{\rm s}))^{\rm T}e-l(\hat{\alpha}-\alpha). \tag{9}$$

Then system (6) is globally GPS with system (5) for an arbitrary scaling factor $\lambda(\lambda\neq 0)$ and constant $l>0$, also $\hat{\alpha}$ tracks α. That is, there exist $\lim\limits_{t\to+\infty}e=0$, and $\lim\limits_{t\to+\infty}(\hat{\alpha}-\alpha)=0$ for any initial values $x_{\rm s}(0)$, $x_{\rm m}(0)$ and $\hat{\alpha}(0)$.

Proof Let $V(e)=\frac{1}{2}e^{\rm T}e$, then grad $V(e)=e$.

Construct the following Lyapunov function:

$V_1(e,\hat{\alpha})=V(e)+\frac{1}{2}(\hat{\alpha}-\alpha)^{\rm T}(\hat{\alpha}-\alpha)$, obviously, $V_1(e,\hat{\alpha})$ is positive and definite. Its time derivative along with systems (7) and (9) is

$$\begin{aligned}\dot{V}_1(e,\hat{\alpha}) &= ({\rm grad}V(e),\dot{e})+\dot{\hat{\alpha}}^{\rm T}(\hat{\alpha}-\alpha)\\ &= (e,Ae+g(x_{\rm s})(\hat{\alpha}-\alpha)+B(f(x_{\rm s})-\lambda f(x_{\rm m}))+U)+\dot{\hat{\alpha}}^{\rm T}(\hat{\alpha}-\alpha)\\ &= e^{\rm T}(A-BK)e-l(\hat{\alpha}-\alpha)^{\rm T}(\hat{\alpha}-\alpha)\\ &= -e^{\rm T}\frac{1}{2}[(BK-A)+(BK-A)^{\rm T}]e-l(\hat{\alpha}-\alpha)^{\rm T}(\hat{\alpha}-\alpha)\\ &= -e^{\rm T}Pe-l(\hat{\alpha}-\alpha)^{\rm T}(\hat{\alpha}-\alpha)\\ &\leq -\lambda_{\min}(P)||e||^2-l||\hat{\alpha}-\alpha||^2,\end{aligned}$$

where $||\cdot||$ is the Euclidean norm.

Since P is positive definite, its minimum eigenvalue $\lambda_{\min}(P)>0$, also $l>0$, then $\dot{V}_1(e,\hat{\alpha})$ is negative definite. Based on the Lyapunov stability theory, the error system (7) with (9) is globally asymptotically stable in the equilibrium points $e=x_{\rm s}-\lambda x_{\rm m}=0$, and $\hat{\alpha}-\alpha=0$. Therefore, we have $\lim\limits_{t\to+\infty}e=0$, and $\lim\limits_{t\to+\infty}(\hat{\alpha}-\alpha)=0$ for any initial values $x_{\rm s}(0)$, $x_{\rm m}(0)$ and $\hat{\alpha}(0)$. Now the proof is completed.

4. Example and simulations

4.1. GPS and parameter identification of two identical hyperchaotic Lü systems with uncertain parameters

In this section, the proposed technique for the GPS and parameter identification is illustrated by two identical hyperchaotic Lü systems with uncertain parameters.

Consider the hyperchaotic Lü system described as systems (3) and (4), the drive and controlled response systems are respectively

$$\dot{x}_{\rm m}=\tilde{A}x_{\rm m}+g(x_{\rm m})\alpha+Bf(x_{\rm m}) \tag{10}$$

and

$$\dot{x}_{\rm s} = \tilde{A}x_{\rm s} + g(x_{\rm s})\hat{\alpha} + Bf(x_{\rm s}) + U, \tag{11}$$

where $\tilde{A}x + g(x)\alpha = Ax$, and $\alpha = [c, d]^{\rm T}$. $\hat{\alpha} = [\hat{c}, \hat{d}]^{\rm T}$ is the uncertain parameter vector of the response system (11).

If we take the parameters $a = 36$, $b = 3$, $c = 20$, $d = 1$ then

$$A = \begin{bmatrix} -36 & 36 & 0 & 1 \\ 0 & 20 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$g(x) = \begin{bmatrix} 0 & 0 \\ x_2 & 0 \\ 0 & 0 \\ 0 & x_4 \end{bmatrix}, \quad \text{and} \quad f(x) = \begin{bmatrix} x_1x_3 \\ x_1x_2 \\ x_1x_3 \end{bmatrix}.$$

Taking $K = \begin{bmatrix} 0 & -30 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$, the matrix $P = \frac{1}{2}[(BK - A) + (BK - A)^{\rm T}]$ is positive definite because its eigenvalues are 45.2081, 3.0000, 1.1444, and 0.6475.

Thus $h(x) = f(x) + Kx = [x_1x_3 - 30x_2, x_1x_2, x_1x_3 + 2x_4]^{\rm T}$, one may take the controller

$$\begin{aligned} U &= -B(h(x_{\rm s}) - \lambda h(x_{\rm m})) \\ &= \begin{bmatrix} 0 \\ x_{\rm s1}x_{\rm s3} - \lambda x_{\rm m1}x_{\rm m3} - 30e_2 \\ -x_{\rm s1}x_{\rm s2} + \lambda x_{\rm m1}x_{\rm m2} \\ -x_{\rm s1}x_{\rm s3} + \lambda x_{\rm m1}x_{\rm m3} - 2e_4 \end{bmatrix} \end{aligned} \tag{12}$$

and the control law

$$\begin{aligned} \dot{\hat{\alpha}} &= -(g(x_{\rm s}))^{\rm T}e - l(\hat{\alpha} - \alpha) \\ &= \begin{bmatrix} -x_{\rm s2}e_2 - l(\hat{c} - 20) \\ -x_{\rm s4}e_4 - l(\hat{d} - 1) \end{bmatrix}. \end{aligned} \tag{13}$$

By using theorem 1 and conditions (12) and (13), systems (5) and (6) can achieve a globally GPS, and the parameter $\hat{\alpha}$ tracking α.

4.2. Numerical simulations

In this section, some numerical simulations about the GPS between systems (10) and (11) are given to demonstrate the effectiveness and feasibility of the proposed technique. In all simulations, we take the parameter $l = 1$ and all the differential equations are solved by using the fourth-order Runge–Kutta method.

Figures 1 and 2 display a comparison between the attractors in R^3 and the evolution process of the scaling factor when the GPS is realized between systems (10) and (11), respectively, with $\lambda = -1$ and 3, the initial values $x_{\rm m}(0) = (1, 5, 5, 10)^{\rm T}$, $x_{\rm s}(0) = (2, 20, -5, 5)^{\rm T}$, $\hat{\alpha}(0) = (1, 10)^{\rm T}$ and $x_{\rm m}(0) = (5, 10, 15, 2)^{\rm T}$, $x_{\rm s}(0) = (10, 2, 0, 5)^{\rm T}$, $\hat{\alpha}(0) = (15, 10)^{\rm T}$. From those figures, one can see that the ratios of $x_{\rm s}(i)/x_{\rm m}(i)(i = 1, 2, 3, 4)$ converge to the scaling values given in advance. That is, the two systems quickly achieve the GPS. At the same time, Figs.3 and 4 show that the parameters $\hat{c}$ and $\hat{d}$, respectively, track c and d when $\lambda = -1$ and 3. That is to say, the parameter identification is realized simultaneously.

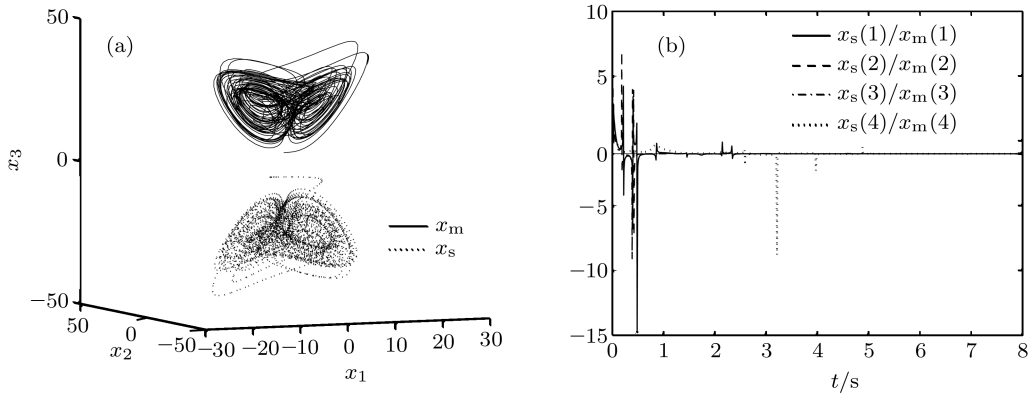


Fig.1. The GPS of the systems (10) and (11) with $\lambda = -1$. (a) Comparison between the attractors in R^3, (b) Time evolution of scaling factors.

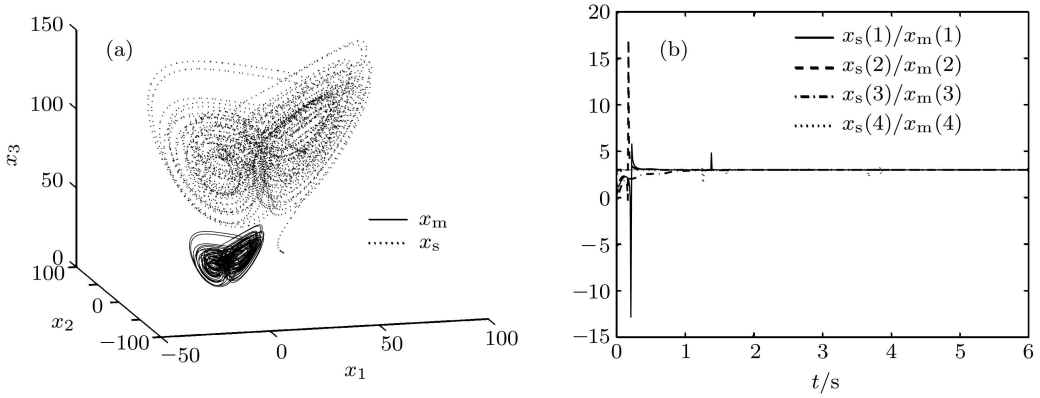


Fig.2. The GPS of the system (10) and (11) with $\lambda = 3$, (a) Comparison between the attractors in R^3, (b) Time evolution of scaling factors.

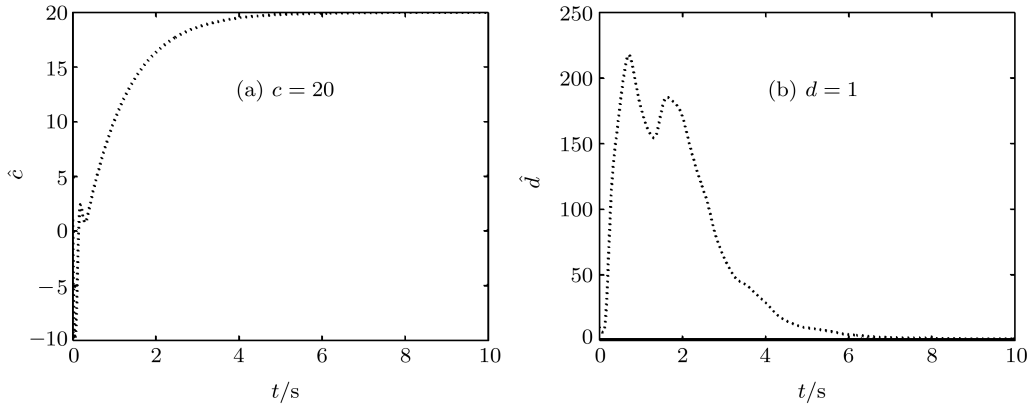


Fig.3. The parameter identification when $\lambda = -1$, (a) $c = 20$, (b) $d = 1$.

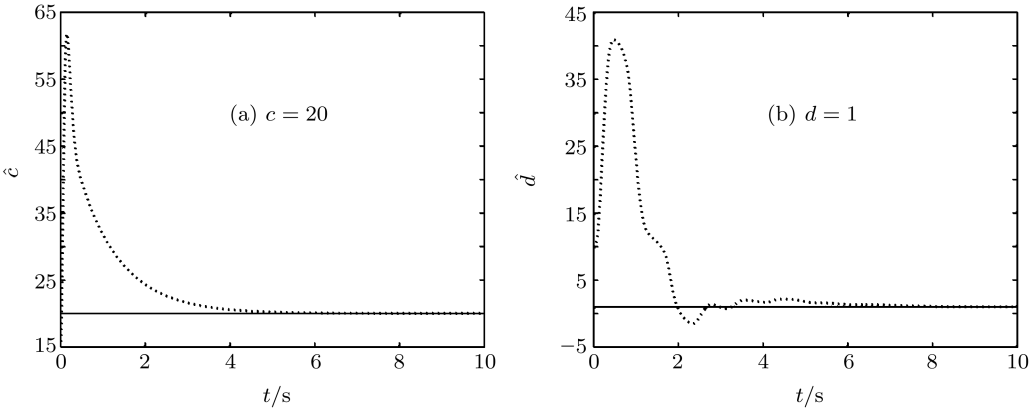


Fig.4. The parameter identification when $\lambda = 3$, (a) $c = 20$, (b) $d = 1$.

Remark 1 The gain matrix K should be appropriately chosen to make the matrix P positive definite. It is independent of the scaling factor λ, therefore without any change of K, one can adjust the scaling factor to any desired scale to realize the GPS. Specially, we can obtain the precise synchronization when $\lambda = 1$.

Remark 2 Matrix K is generally not unique, it should be chosen to make the controller as simple as possible.

5. Conclusions

In this paper, we have discussed the problems about the GPS and the parameter identification of a class of chaotic (hyperchaotic) systems. A novel nonlinear controller is designed based on the adaptive technique to realize the parameter identification, and two identical chaotic (hyperchaotic) systems have achieved the globally GPS simultaneously. A sufficient condition for the globally GPS is given and proved theoretically via the Lyapunov method. Finally, the numerical simulations with the hyperchaotic Lü system have demonstrated the effectiveness and feasibility of the proposed technique.

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