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Explicit solutions of Boussinesq–Burgers equation*

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Darboux transformation with multi-parameters for the Boussinesq–Burgers (B–B) equation is derived. For an application, some important explicit solutions of the B–B equation are obtained, including $2N$-soliton solution and periodic solution. Finally, some elegant and interesting figures are plotted.

Keywords: Darboux transformation, B–B equation, explicit solution
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1. Introduction

The Bossinesq–Burgers (B–B) soliton equation is given as[1−3]

$$u_t = -2uu_x + \frac{1}{2}v_x,$$
$$v_t = \frac{1}{2}u_{xxx} - 2(uv)_x, \tag{1}$$

which is a nonlinear long-wave equation. A good understanding of its solutions is very helpful for coastal and civil engineers to apply the nonlinear water wave model to harbour and coastal designs. Therefore, finding more new solutions of the B–B equation is of great value in fluid dynamics. This motives us to seek for various types of solutions. Many systematic methods have been developed to obtain exact solutions of soliton equations, such as the inverse scattering transformation, the bilinear transformation of Hirota,[4,5] the Darboux transformation (DT),[6−8] the homogeneous balance method,[9] the Lie symmetry approach,[10−13] the algebraic curve method and the nonlinearization approach of eigenvalue problem.[14−18]

The aim of this paper is to obtain more kinds of solutions of the B–B equation, apart from its algebraic-geometry solutions and $2N-1$ soliton solutions obtained in Refs.[1,2]. To this end, we start from the B–B spectral problem and achieve an N-order DT in Section 2, which is different from the DT in Ref.[2]. Resorting to DT in Section 2, a $2N$-soliton solution and periodic solution of the B–B equation (1) are obtained. In Section 3, the DT is used to generate new solutions of the B–B equation. We can directly obtain the $2N$-soliton solution and its periodic solution from a trivial solution. By appropriately choosing the eigenvalue λ, we obtain a multi-soliton solutions with two peaks, which is not of bell type but appears to be a new phenomenon for soliton equations associated with 2×2 matrix spectral problems. Finally, we plot some beautiful figures.

2. Darboux transformation

In this section, we construct another DT of the B–B equation

$$u_t = -2uu_x + \frac{1}{2}v_x,$$
$$v_t = \frac{1}{2}u_{xxx} - 2(uv)_x, \tag{2}$$

which has the Lax representation

$$\phi_x = U\phi, \quad U = \begin{pmatrix} \lambda + u & u_x + v \\ 1 & -\lambda - u \end{pmatrix}, \tag{3}$$
$$\phi_t = V\phi, \tag{4}$$

$$V = \begin{pmatrix} \lambda^2 - \frac{1}{2}u_x - u^2 & \lambda(u_x + v) + \frac{1}{2} - uu_x - uv + \frac{1}{2}v_x \\ \lambda - u & -\lambda^2 + \frac{1}{2}u_x + u^2 \end{pmatrix},$$

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where u and v are potentials, and λ is a constant spectral parameter.

In order to construct a DT of Eq.(2), we first derive a gauge transformation for the spectral problem (3). Here we use the method in Refs.[6,7] to construct the gauge transformation. To this end, we choose $\phi = (\phi_1, \phi_2)^{\mathrm{T}}$ and $\psi = (\psi_1, \psi_2)^{\mathrm{T}}$ to be two basic solutions of Eq.(3) and use (ϕ, ψ) to determine a 2×2 matrix T as

$$T = \begin{pmatrix} A_N\Big(\lambda^N + \sum\limits_{j=0}^{N-1} A_j\lambda^j\Big) & A_N \sum\limits_{j=0}^{N-1} B_j\lambda^j \\ \dfrac{1}{A_N}\sum\limits_{j=0}^{N-1} C_j\lambda^j C & \dfrac{1}{A_N}\Big(\lambda^N + \sum\limits_{j=0}^{N-1} D_j\lambda^j\Big) \end{pmatrix}, \tag{5}$$

where A_k, B_k, C_k and $D_k (0 \le k \le N-1)$ are determined by the following linear system:

$$\begin{aligned} &\sum_{k=0}^{N-1}(A_k + B_k\sigma_j)\lambda_j^k = -\lambda_j^N, \\ &\sum_{k=0}^{N-1}(C_k + D_k\sigma_j)\lambda_j^k = -\sigma_j\lambda_j^N, \end{aligned} \tag{6}$$

where

$$\sigma_j = \frac{\phi_2(\lambda_j) - \gamma_j\psi_2(\lambda_j)}{\phi_1(\lambda_j) - \gamma_j\psi_1(\lambda_j)}, \quad 1 \le j \le 2N, \tag{7}$$

and A_N will be determined later.

Now, we introduce a gauge transformation

$$\bar{\phi} = T\phi, \tag{8}$$

which transforms the spectral problem (3) and the auxiliary problem (4) into a spectral problem and its auxiliary one of $\bar{\phi}$

$$\bar{\phi}_x = \bar{U}\bar{\phi}, \quad \bar{U} = (T_x + TU)T^{-1}, \tag{9}$$

$$\bar{\phi}_t = \bar{V}\bar{\phi}, \quad \bar{V} = (T_t + TV)T^{-1}. \tag{10}$$

We will verify the following facts:

Proposition 1 Let A_N satisfy the first-order ordinary differential equation

$$\partial_x \ln A_N = \frac{C_{N-1,x}}{1 + 2C_{N-1}}, \tag{11}$$

then the matrix $\bar{U}$ determined by Eq.(9) will have the same form as U, that is $\bar{U} = U(\bar{u}, \bar{v}, \lambda)$, where the transformation formulae from the old potentials u and v into new ones are given by

$$\begin{cases} \bar{u} = u + \partial_x \ln A_N = u + \dfrac{C_{N-1,x}}{1 + 2C_{N-1}}, \\ \bar{v} = v - 2B_{N-1}(1 + 2C_{N-1}) + 2(u_x + v)C_{N-1} \\ \qquad -\Big(\dfrac{C_{N-1,x}}{1 + 2C_{N-1}}\Big)_x. \end{cases} \tag{12}$$

Proof

Let $T^{-1} = T^*/\det T$ and

$$(T_x + TU)T^* = \begin{pmatrix} f_{11}(\lambda) & f_{12}(\lambda) \\ f_{21}(\lambda) & f_{22}(\lambda) \end{pmatrix}. \tag{13}$$

It is easy to see that $f_{11}(\lambda)$, $f_{22}(\lambda)$ or $f_{12}(\lambda)$, $f_{21}(\lambda)$ are $(2N+1)$-order polynomials in λ or $2N$-order polynomials in λ. By using Eqs.(3) and (7), we obtain a Riccati equation

$$\sigma_{jx} = 1 - 2(\lambda_j + u)\sigma_j - (u_x + v)\sigma_j^2, \ 1 \le j \le 2N. \tag{14}$$

From Eqs.(5) and (6), we know that

$$A(\lambda_j) = -\sigma_j B(\lambda_j), \ C(\lambda_j) = -\sigma_j D(\lambda_j). \tag{15}$$

Resorting to Eqs.(14) and (15), it can be verified that all λ_j $(1 \le j \le 2N)$ are roots of f_{sl} $(s, l = 1, 2)$. Then we conclude

$$\det T | f_{sl}, \quad (s, l = 1, 2),$$

which together with Eq.(13) gives

$$(T_x + TU)T^* = (\det T)P(\lambda) \tag{16}$$

with

$$P(\lambda) = \begin{pmatrix} p_{11}^{(1)}\lambda + p_{11}^{(0)} & p_{12}^{(0)} \\ p_{21}^{(0)} & p_{22}^{(1)}\lambda + p_{22}^{(0)} \end{pmatrix},$$

and $p_{kj}^{(l)}$, $k, j = 1, 2$, $l = 0, 1$ being independent of λ. Now Eq.(16) can be written in the form

$$T_x + TU = P(\lambda)T. \tag{17}$$

By comparing the coefficients of λ^{N+1} and λ^N in Eq.(17) and using Eq.(12), we find that

$$p_{11}^{(1)} = -p_{22}^{(1)} = 1, \tag{18}$$

$$p_{11}^{(0)} = u + \partial_x \ln A_N = \bar{u}, \tag{19}$$

$$p_{22}^{(0)} = -u - \partial_x \ln A_N = -\bar{u}, \tag{20}$$

$$p_{12}^{(0)} = A_N^2(u_x + v - 2B_{N-1}) = \bar{u}_x + \bar{v}, \tag{21}$$

$$p_{21}^{(0)} = \frac{1}{A_N^2}(1+2C_{N-1}) = 1. \tag{22}$$

So, we have proved that $\bar{U} = U(\bar{u}, \bar{v}, \lambda)$.

Let $\phi = (\phi_1, \phi_2)^{\mathrm{T}}$ and $\psi = (\psi_1, \psi_2)^{\mathrm{T}}$ be two basic solutions of Eqs.(3) and (4), then we can prove the following assertion:

Proposition 2 Assume A_N to satisfy the first-order ordinary differential equation with respect to the variable t

$$\begin{aligned}\partial_t \ln A_N = {} & (u_x+v)C_{N-1} - B_{N-1}(1+2C_{N-1}) - 2u \\ & \times \frac{C_{N-1,x}}{1+2C_{N-1}} - \Big(\frac{C_{N-1,x}}{1+2C_{N-1}}\Big)^2 \\ & - \frac{1}{2}\Big(\frac{C_{N-1,x}}{1+2C_{N-1}}\Big)_x,\end{aligned} \tag{23}$$

then the matrix $\bar{V}$ determined by Eq.(10) will have the same form as V under the same transformation of (12).

Proof

Let $T^{-1} = T^*/\det T$ and

$$(T_t + TV)T^* = \begin{pmatrix} g_{11}(\lambda) & g_{12}(\lambda) \\ g_{21}(\lambda) & g_{22}(\lambda) \end{pmatrix}. \tag{24}$$

A direct calculation shows that $g_{11}(\lambda)$, $g_{22}(\lambda)$ or $g_{12}(\lambda), g_{21}(\lambda)$ are $(2N+2)$-order polynomials in λ or $(2N+1)$-order polynomials in λ. By virtue of Eqs.(7) and (4), we obtain

$$\begin{aligned}\sigma_{jt} = {} & \lambda_j - u - 2\Big(\lambda_j^2 - \frac{1}{2}u_x - u^2\Big)\sigma_j - \Big[\lambda_j(u_x+v) \\ & + \frac{1}{2}u_{xx} - uu_x - uv + \frac{1}{2}v_x\Big]\sigma_j^2.\end{aligned} \tag{25}$$

Through direct calculation, all $\lambda_j (1 \le j \le 2N)$ are roots of $g_{sl}(\lambda), s, l = 1, 2$; therefore, it together with Eq.(24) gives

$$(T_t + TV)T^* = (\det T)Q(\lambda) \tag{26}$$

with

$$Q(\lambda) = \begin{pmatrix} q_{11}^{(2)}\lambda^2 + q_{11}^{(1)}\lambda + q_{11}^{(0)} & q_{12}^{(1)}\lambda + q_{12}^{(0)} \\ q_{21}^{(1)}\lambda + q_{21}^{(0)} & q_{11}^{(2)}\lambda^2 + q_{11}^{(1)}\lambda + q_{11}^{(0)} \end{pmatrix}$$

and $q_{k,j}^{(l)}, k,\ j = 1, 2,\ l = 0, 1, 2$ being independent of λ. Now Eq.(26) can be written as

$$T_t + TV = Q(\lambda)T. \tag{27}$$

By comparing the coefficients of λ^{N+2}, λ^{N+1} and λ^N in Eq.(27), we have

$$q_{11}^{(2)} = -q_{22}^{(2)} = 1, \quad q_{11}^{(1)} = q_{22}^{(1)} = 0, \tag{28}$$

$$q_{12}^{(1)} = A_N^2(u_x + v - 2B_{N-1}) = \bar{u}_x + \bar{v}, \tag{29}$$

$$q_{21}^{(1)} = \frac{1}{A_N^2}(1+2C_{N-1}), \tag{30}$$

$$\begin{aligned}q_{11}^{(0)} = {} & \partial_t \ln A_N - \frac{1}{2}u_x - u^2 - (u_x+v)C_{N-1} \\ & + B_{N-1}(1+C_{N-1}),\end{aligned} \tag{31}$$

$$\begin{aligned}q_{12}^{(0)} = {} & A_N^2\Big[\frac{1}{2}u_{xx} - uu_x - uv + \frac{1}{2}v_x + (u_x+v) \\ & \times (A_{N-1} - D_{N-1}) + 2B_{N-1}D_{N-1} - 2B_{N-2}\Big],\end{aligned} \tag{32}$$

$$q_{21}^{(0)} = \frac{1}{A_N^2}(2C_{N-2} + D_{N-1} - u) - A_{N-1}, \tag{33}$$

$$\begin{aligned}q_{22}^{(0)} = {} & -\partial_t \ln A_N + \frac{1}{2}u_x + u^2 + (u_x+v)C_{N-1} \\ & - B_{N-1}(1+C_{N-1}).\end{aligned} \tag{34}$$

By using expressions (23) and (18), we can prove

$$q_{11}^{(0)} = -q_{22}^{(0)} = -\bar{u}_x - \bar{u}^2.$$

At the same time, comparing the coefficients of λ^{N-1} in Eq.(17), we obtain

$$A_{N-1,x} = -B_{N-1}, \tag{35}$$

$$\begin{aligned}B_{N-1,x} = {} & -(u_x+v)A_{N-1} + 2uB_{N-1} + 2B_{N-2} \\ & + \frac{1}{A_N^2}D_{N-1}(\bar{u}_x + \bar{v}),\end{aligned} \tag{36}$$

$$D_{N-1,x} = -(u_x+v)C_{N-1} + A_N^2 B_{N-1}, \tag{37}$$

$$\begin{aligned}2C_{N-2} = {} & -2uC_{N-1} - C_{N-1,x} - D_{N-1} \\ & + A_N^2 A_{N-1}.\end{aligned} \tag{38}$$

Substituting expression (38) into Eq.(33) and noting expression (12), we can verify

$$q_{21}^{(0)} = -\bar{u}.$$

Through tedious calculations, we obtain from expressions (32), (36), (12) and (38) that

$$q_{12}^{(0)} = \frac{1}{2}\bar{u}_{xx} - \bar{u}\bar{u}_{xx} - \bar{u}\bar{v} + \frac{1}{2}\bar{v}_x.$$

So, we have proved that $\bar{V} = V(\bar{u}, \bar{v}, \lambda)$.

According to Propositions 1 and 2, the transformation (12) transforms the Lax pairs (3) and (4) into those of the same type as Eqs.(9) and (10). Therefore, both of the Lax pairs lead to the same equation (2). We call the transformation $(\phi, u, v) \to (\bar{\phi}, \bar{u}, \bar{v})$ a DT of Eq.(2). We obtain the following assertion:

Theorem 3 The solution (u, v) of the B–B equation is mapped into new one $(\bar{u}, \bar{v})$ under the DT (12), where B_{N-1} and C_{N-1} are given by the linear algebraic system (6).

3. Explicit solutions

In this section, we obtain some explicit solutions of the B–B equation by applying DT (12).

Substituting $(u = 0, v = -1)$ into the Lax pairs (3) and (4), we have two basic solutions:

$$\phi(\lambda_i) = (\mathrm{e}^{\xi_i}, (\lambda_i - c_i)\mathrm{e}^{\xi_i})^{\mathrm{T}},$$
$$\psi(\lambda_i) = (\mathrm{e}^{-\xi_i}, (\lambda_i + c_i)\mathrm{e}^{-\xi_i})^{\mathrm{T}}$$

with $\xi_i = c_i(x + \lambda_i t)$ and $c_i = (\lambda^2 - 1)^{1/2}, 1 \le i \le 2N$.

According to expression (7), we have

$$\sigma_i = \lambda_i - c_i \frac{\mathrm{e}^{\xi_i} + \gamma_i \mathrm{e}^{-\xi_i}}{\mathrm{e}^{\xi_i} - \gamma_i \mathrm{e}^{-\xi_i}}, \quad 1 \le i \le 2N.$$

Let λ_j, $1 \le j \le 2N$ be constants. Solving the linear algebraic system (6), we have

$$B_{N-1} = \frac{\Delta_{B_{N-1}}}{\Delta}, \quad C_{N-1} = \frac{\Delta_{C_{N-1}}}{\Delta} \tag{39}$$

with

$$\Delta = \begin{vmatrix} 1 & \sigma_1 & \lambda_1 & \sigma_1\lambda_1 & \dots & \lambda_1^k & \sigma_1\lambda_1^k & \dots & \lambda_1^{N-1} & \sigma_1\lambda_1^{N-1} \\ 1 & \sigma_2 & \lambda_2 & \sigma_2\lambda_2 & \dots & \lambda_2^k & \sigma_2\lambda_2^k & \dots & \lambda_2^{N-1} & \sigma_2\lambda_2^{N-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \sigma_{2N} & \lambda_{2N} & \sigma_{2N}\lambda_{2N} & \dots & \lambda_{2N}^k & \sigma_{2N}\lambda_{2N}^k & \dots & \lambda_{2N}^{N-1} & \sigma_{2N}\lambda_{2N}^{N-1} \end{vmatrix},$$

$$\Delta_{B_{N-1}} = \begin{vmatrix} 1 & \sigma_1 & \lambda_1 & \sigma_1\lambda_1 & \dots & \lambda_1^k & \sigma_1\lambda_1^k & \dots & \lambda_1^{N-1} & -\lambda_1^N \\ 1 & \sigma_2 & \lambda_2 & \sigma_2\lambda_2 & \dots & \lambda_2^k & \sigma_2\lambda_2^k & \dots & \lambda_2^{N-1} & -\lambda_2^N \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \sigma_{2N} & \lambda_{2N} & \sigma_{2N}\lambda_{2N} & \dots & \lambda_{2N}^k & \sigma_{2N}\lambda_{2N}^k & \dots & \lambda_{2N}^{N-1} & -\lambda_{2N}^N \end{vmatrix},$$

$$\Delta_{C_{N-1}} = \begin{vmatrix} 1 & \sigma_1 & \lambda_1 & \sigma_1\lambda_1 & \dots & \lambda_1^k & \sigma_1\lambda_1^k & \dots & -\sigma_1\lambda_1^N & \sigma_1\lambda_1^{N-1} \\ 1 & \sigma_2 & \lambda_2 & \sigma_2\lambda_2 & \dots & \lambda_2^k & \sigma_2\lambda_2^k & \dots & -\sigma_2\lambda_2^N & \sigma_2\lambda_2^{N-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \sigma_{2N} & \lambda_{2N} & \sigma_{2N}\lambda_{2N} & \dots & \lambda_{2N}^k & \sigma_{2N}\lambda_{2N}^k & \dots & -\sigma_{2N}\lambda_{2N}^N & \sigma_{2N}\lambda_{2N}^{N-1} \end{vmatrix}.$$

Resorting to expression (12), we obtain a solution of the B–B equation

$$\begin{cases} \bar{u} = u + \dfrac{C_{N-1,x}}{1 + 2C_{N-1}}, \\ \bar{v} = v - 2B_{N-1}(1 + 2C_{N-1}) + 2(u_x + v)C_{N-1} - \Big(\dfrac{C_{N-1,x}}{1 + 2C_{N-1}}\Big)_x. \end{cases} \tag{40}$$

For simplicity, we shall discuss the two special cases, i.e. $N = 1$ and $N = 2$.

(I) For $N = 1$, we have a new solution of the B–B equation from solution (40)

$$\begin{cases} \bar{u}[1] = \dfrac{C_{0,x}}{1 + 2C_0}, \\ \bar{v}[1] = -1 - 2B_0(1 + 2C_0) - 2C_0 - \Big(\dfrac{C_{0,x}}{1 + 2C_0}\Big)_x, \end{cases} \tag{41}$$

with

$$B_0 = \frac{\lambda_1 - \lambda_2}{\sigma_2 - \sigma_1}, \quad C_0 = \frac{\lambda_2 - \lambda_1}{\sigma_2 - \sigma_1}\sigma_1\sigma_2.$$

If $\lambda_2 > \lambda_1 > 1$, the solution determined by expression (41) is two-overtaking collision soliton solution.

If $\lambda_1 > 1$, $\lambda_2 < -1$, the solution determined by expression (41) is two-head-on collision soliton solution (see

Fig.1.).

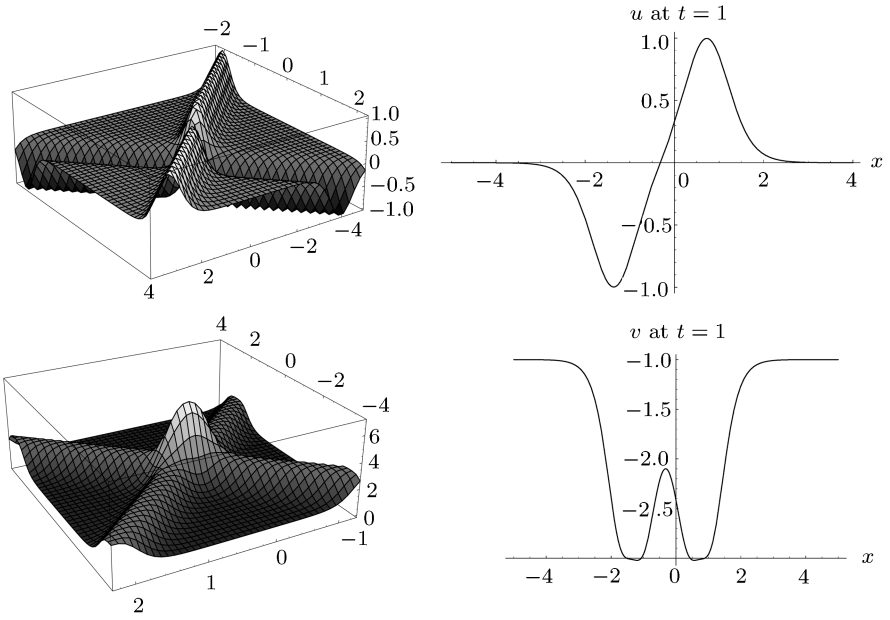


Fig.1. Two-soliton solution (41) with $\lambda_1 = -\lambda_2 = 2$, $\gamma_1 = \gamma_2 = -0.5$.

If $|\lambda_1| < 1$, the solution determined by expression (41) is periodic solution (see Fig.2.).

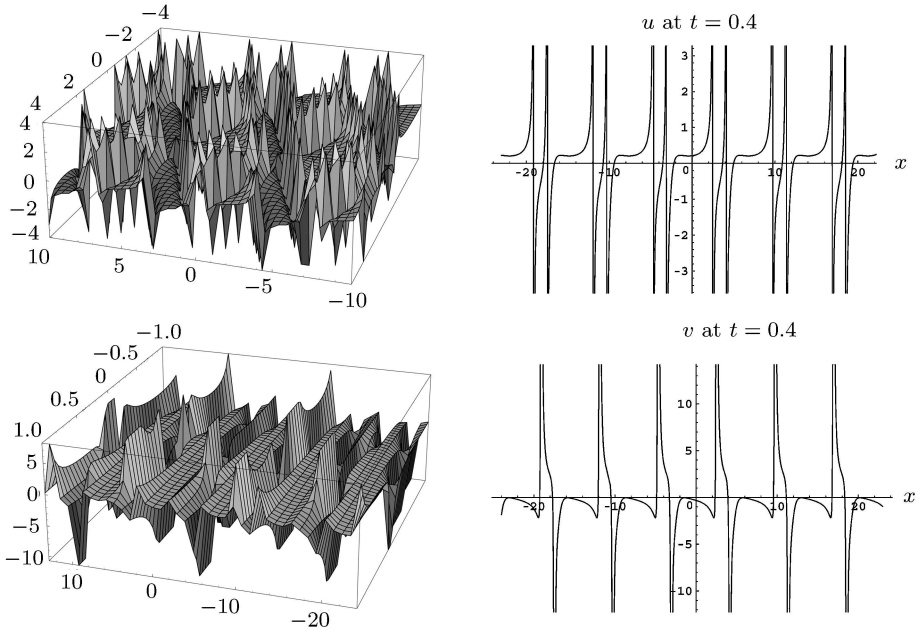


Fig.2. Periodic-solution (41) with $\lambda_1 = -\lambda_2 = -0.9$, $\gamma_1 = -\gamma_2 = -0.5$.

(II) For $N = 2$, we have a new solution of the B–B equation

$$\begin{cases} \bar{u}[2] = \dfrac{C_{1,x}}{1+2C_1}, \\ \bar{v}[2] = -1 - 2B_1(1+2C_1) - 2C_1 - \Big(\dfrac{C_{1,x}}{1+2C_1}\Big)_x, \end{cases} \tag{42}$$

with $B_1 = \Delta_{B_1}/\Delta$, $C_1 = \Delta_{C_1}/\Delta$,

$$\Delta = \begin{vmatrix} 1 & \sigma_1 & \lambda_1 & \sigma_1\lambda_1 \\ 1 & \sigma_2 & \lambda_2 & \sigma_2\lambda_2 \\ 1 & \sigma_3 & \lambda_3 & \sigma_3\lambda_3 \\ 1 & \sigma_4 & \lambda_4 & \sigma_4\lambda_4 \end{vmatrix}, \quad \Delta_{B_1} = \begin{vmatrix} 1 & \sigma_1 & \lambda_1 & -\lambda_1^2 \\ 1 & \sigma_2 & \lambda_2 & -\lambda_2^2 \\ 1 & \sigma_3 & \lambda_3 & -\lambda_3^2 \\ 1 & \sigma_4 & \lambda_4 & -\lambda_4^2 \end{vmatrix}, \quad \Delta_{C_1} = \begin{vmatrix} 1 & \sigma_1 & -\sigma_1\lambda_1^2 & \sigma_1\lambda_1 \\ 1 & \sigma_2 & -\sigma_2\lambda_2^2 & \sigma_2\lambda_2 \\ 1 & \sigma_3 & -\sigma_3\lambda_3^2 & \sigma_3\lambda_3 \\ 1 & \sigma_4 & -\sigma_4\lambda_4^2 & \sigma_4\lambda_4 \end{vmatrix}.$$

If $|\lambda_j| > 1$, $j = 1, 2, 3, 4$, the solution determined by expression (42) is a four-soliton solution (see Fig.3.).

If $|\lambda_j| < 1$, $j = 1, 2, 3, 4$, the solution determined by (42) is a periodic solution.

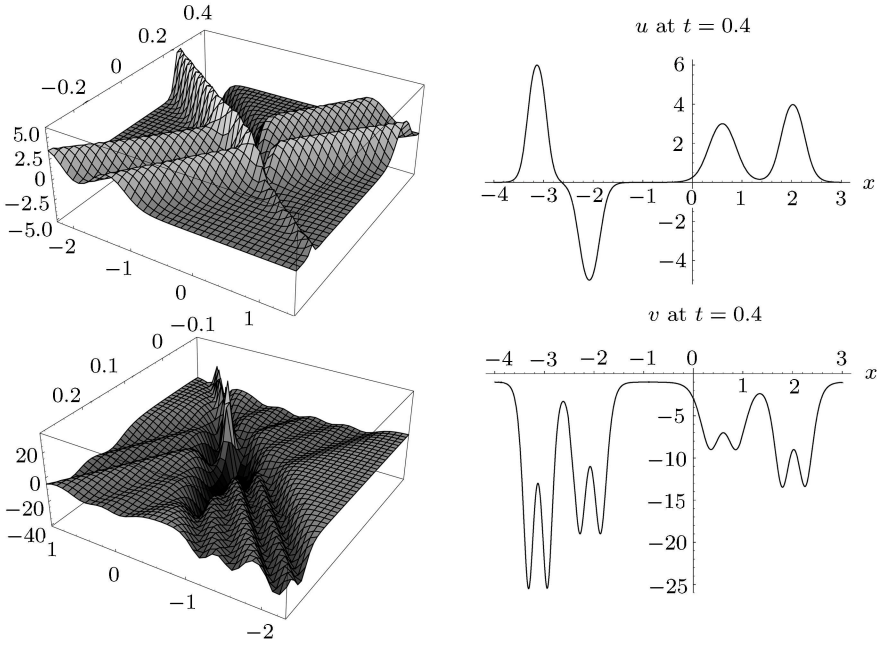


Fig.3. Four-soliton solution (42) with $\lambda_1 = 7$, $\lambda_2 = 6$, $\lambda_3 = -5$, $\lambda_4 = -4$, $\gamma_1 = -6$, $\gamma_2 = \gamma_4 = 0$, $\gamma_3 = 2$.

4. Conclusion

To conclude this paper we make the following remarks. Darboux transformation provides an effective way to obtain solutions of soliton equations. This transformation is usually used to yield two-soliton solutions or a series of multi-soliton solutions from a trivial 'seed'. But now, we use it to generate different types of solutions, and we have had a success. In addition to two-soliton and four-soliton solutions, periodic solutions of the B–B equation are obtained. And beautiful pictures of them are also presented here in this paper. It is interesting that every soliton of four-soliton solutions (in Fig.3.) has two peaks, which are rarely seen, however.

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