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Exotic interactions between solitons of the (2+1)-dimensional asymmetric Nizhnik–Novikov–Veselov system*

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Starting from the extended tanh-function method (ETM) based on the mapping method, the variable separation solutions of the (2+1)-dimensional asymmetric Nizhnik–Novikov–Veselov (ANNV) system are derived. By further study, we find that these variable separation solutions are seemingly independent of but actually dependent on each other. Based on the variable separation solution and by choosing appropriate functions, some novel and interesting interactions between special solitons, such as bell-like compacton, peakon-like compacton and compacton-like semi-foldon, are investigated.

Keywords: extended tanh-function method, (2+1)-dimensional ANNV system, exotic interactions between solitons

PACC: 0365, 0340, 0545

1. Introduction

During the past 35 years, our knowledge of solitons has developed into a mature theory, and the concept of solitons has entered into various fields of natural sciences such as chemistry, biology, mathematics, communication and particularly into almost all branches of physics like fluid dynamics, plasma physics, field theory, nonlinear optics and condensed matter physics, etc. Many soliton equations describing nonlinear systems are known and solitons themselves have been observed (directly or indirectly) in various media.

As one of the effective methods in linear physics, the variable separation approach (VSA) has been successfully extended to nonlinear domains. The so-called multilinear variable separation approach (MLVSA) has also been established for various (2+1)-dimensional models.[1] Recently, along the lines of linear variable separation idea and using the extended tanh-function method (ETM) based on the mapping method, Zheng *et al* have realized variable separation for some (2+1)-dimensional systems, such as Broer–Kaup–Kupershmidt (BKK) system,[2] modified dispersive water-wave (MDWW) equation,[3] Boiti–Leon–Pempinel (BLP) model[4] and breaking soliton system.[5] And then we extended this method in order to obtain variable separation solutions of the (2+1)-dimensional Korteweg–de Vries (KdV) and asymmetric Nizhnik–Novikov–Veselov (ANNV) systems.[6,7] Moreover, some authors[8,9] successfully generalized ETM to the (1+1)-dimensional and (3+1)-dimensional nonlinear physical models. However, in fact, various solutions including solitary wave solutions, periodic wave solutions and rational function solutions derived by using ETM in Refs.[2–9], are seemingly independent of but actually dependent on each other. This viewpoint has been proven in Ref.[10]. More recently, by using the non-standard truncated expansion, Fang and his groups have also generalized the ETM and obtained various so-called new symmetrical variable separation solutions for the (2+1)-dimensional Broer–Kaup–Kupershmidt (BKK),[11] generalized Broer–Kaup (GBK),[12] Boiti–Leon–Pempinelli (BLP),[13] dispersive long wave (DLW),[14] Nizhnik–Novikov–Veselov (NNV)[15] and Korteweg–de Vries (KdV)[16] systems. However, similar to our reports in Ref.[10], these so-called new solutions in Refs.[11–16], we think, also depend on each other and the effective solution is identical to the universal formula in Ref.[1], which has been given in Ref.[10]. To prove our viewpoint, we consider the following (2+1)-dimensional asymmetric Nizhnik–Novikov–Veselov (ANNV) system:

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$$u_t + u_{xxx} - 3(uv)_x = 0,$$
$$v_y = u_x, \tag{1}$$

which was first derived by Boiti *et al*[17] using the weak Lax pair in Ref.[18] and may be considered as a model for an incompressible fluid. Here u and v are the components of the dimensionless velocity.[19] The spectral transformation for this system has been investigated in Refs.[17,20]. This system has also been considered in Ref.[21] as a generalization to (2+1)-dimensions of the results from Hirota and Satsuma.[22] Moreover, Hu *et al* used the Darboux transformations (DT) to find the variable separation solutions of this system in Ref.[23].

(2+1)-dimensional localized coherent structures, such as dromions, peakons, compactons and foldons, etc, have been extensively discussed.[1−16] However, our nature is colorful and has some novel localized coherent structures that have not yet been revealed. For example, some combined structures of dromions, peakons, compactons and foldons have hardly reported on. In this paper, based on the variable separation solution of (2+1)-dimensional ANNV system and selected appropriate functions, we discuss some combined structures, including bell-like compacton, peakon-like compacton and compacton-like semi-foldon.

2. Variable separation solutions for the (2+1)-dimensional ANNV system

By using the EMT and balancing procedure similar to that in Refs.[11–16], we take an ansatz for Eq.(1) as follows:

$$\begin{aligned} u = \;& a_0 + a_1\phi(R) + a_{-1}\phi^{-1}(R) + a_2\phi^2(R) \\ & + a_{-2}\phi^{-2}(R), \\ v = \;& b_0 + b_1\phi(R) + b_{-1}\phi^{-1}(R) + b_2\phi^2(R) \\ & + b_{-2}\phi^{-2}(R), \end{aligned} \tag{2}$$

where $R \equiv R(x,\ y,\ t)$, $a_i \equiv a_i(x,\ y,\ t)$, $b_i \equiv b_i(x,\ y,\ t)$, $(i = -2,\ -1,\ 0,\ 1,\ 2)$, and $\phi(R) \equiv \phi$ satisfying the Riccati equation $\frac{\mathrm{d}\phi}{\mathrm{d}R} = l_0 + \phi^2$, with l_0 being an arbitrary constant. Inserting Eq.(2) into Eq.(1), selecting the variable separation ansatz

$$R = p(x,t) + q(y,t), \tag{3}$$

and eliminating all the coefficients of polynomials of ϕ, one obtains a set of partial differential equations of a_i, b_i and p, q. Through careful calculations with the aid of Maple, we acquire an exact solution as follows:

$$\begin{aligned} & a_0 = 4l_0p_xq_y, \quad a_1 = 0, \quad a_{-1} = 0, \quad a_2 = 2p_xq_y, \\ & a_{-2} = 2l_0^2p_xq_y, \quad b_0 = \frac{p_t + p_{xxx} + c_t - 4l_0p_x^3}{3p_x}, \\ & b_1 = 2p_{xx}, \quad b_{-1} = -2l_0p_{xx}, \quad b_2 = 2p_x^2, \\ & b_{-2} = 2l_0^2p_x^2, \end{aligned} \tag{4}$$

where $q(y,t) = F(y) + c(t)$, $c \equiv c(t)$ and $F \equiv F(y)$. Substituting Eq.(4) together with the solutions of Riccati equation[11−16] into Eq.(2), the variable separation solutions of the (2+1)-dimensional ANNV system (1) possess the following forms:

Case 1 For $l_0 < 0$,

$$u_1 = 4l_0p_xq_y - 2l_0p_xq_y\left\{\tanh^2[\sqrt{-l_0}(p+q)] + \tanh^{-2}[\sqrt{-l_0}(p+q)]\right\}, \tag{5}$$

$$\begin{aligned} v_1 = \;& \frac{p_t + p_{xxx} + c_t - 4l_0p_x^3}{3p_x} - 2\sqrt{-l_0}p_{xx}\left\{\tanh[\sqrt{-l_0}(p+q)] + \tanh^{-1}[\sqrt{-l_0}(p+q)]\right\} \\ & -2l_0p_x^2\left\{\tanh^2[\sqrt{-l_0}(p+q)] + \tanh^{-2}[\sqrt{-l_0}(p+q)]\right\}, \end{aligned} \tag{6}$$

$$u_2 = 4l_0p_xq_y - 2l_0p_xq_y\left\{\coth^2[\sqrt{-l_0}(p+q)] + \coth^{-2}[\sqrt{-l_0}(p+q)]\right\}, \tag{7}$$

$$\begin{aligned} v_2 = \;& \frac{p_t + p_{xxx} + c_t - 4l_0p_x^3}{3p_x} - 2\sqrt{-l_0}p_{xx}\left\{\coth[\sqrt{-l_0}(p+q)] + \coth^{-1}[\sqrt{-l_0}(p+q)]\right\} \\ & -2l_0p_x^2\left\{\coth^2[\sqrt{-l_0}(p+q)] + \coth^{-2}[\sqrt{-l_0}(p+q)]\right\}; \end{aligned} \tag{8}$$

Case 2 For $l_0 > 0$,

$$u_3 = 4l_0p_xq_y + 2l_0p_xq_y\left\{\tan^2[\sqrt{l_0}(p+q)] + \tan^{-2}[\sqrt{l_0}(p+q)]\right\}, \tag{9}$$

$$v_3 = \frac{p_t + bp_{xxx} + c_t - 4l_0p_x^3}{3p_x} + 2\sqrt{l_0}p_{xx}\left\{\tan[\sqrt{l_0}(p+q)] - \tan^{-1}[\sqrt{l_0}(p+q)]\right\} + 2l_0p_x^2\left\{\tan^2[\sqrt{l_0}(p+q)] + \tan^{-2}[\sqrt{l_0}(p+q)]\right\}, \tag{10}$$

$$u_4 = 4l_0p_xq_y + 2l_0p_xq_y\left\{\cot^2[\sqrt{l_0}(p+q)] + \cot^{-2}[\sqrt{l_0}(p+q)]\right\}, \tag{11}$$

$$v_4 = \frac{p_t + bp_{xxx} + c_t - 4l_0p_x^3}{3p_x} - 2\sqrt{l_0}p_{xx}\left\{\cot[\sqrt{l_0}(p+q)] - \cot^{-1}[\sqrt{l_0}(p+q)]\right\} + 2l_0p_x^2\left\{\cot^2[\sqrt{l_0}(p+q)] + \cot^{-2}[\sqrt{l_0}(p+q)]\right\}; \tag{12}$$

Case 3 For $l_0 = 0$,

$$u_5 = \frac{2p_xq_y}{(p+q)^2}, \tag{13}$$

$$v_5 = \frac{p_t + p_{xxx} + c_t}{3p_x} - \frac{2p_{xx}}{p+q} + \frac{2p_x^2}{(p+q)^2}, \tag{14}$$

where p is an arbitrary functions of $\{x,t\}$, and $q(y,t) = F(y) + c(t)$.

Through careful analysis, we find that by redefining $p = \exp[-(4\sqrt{-l_0}p + \mathrm{i}\pi)]$ and $q = \exp[4\sqrt{-l_0}q]$ in solution (13), solutions (5) and (7) can be obtained. Similarly, by letting $p = \exp[-\mathrm{i}(4\sqrt{l_0}p + \pi)]$ and $q = \exp[4\mathrm{i}\sqrt{l_0}q]$ in solution (13), solutions (9) and (11) can also be recovered. Therefore, only solution (13) is essentially effective. Other results for the (2+1)-dimensional BKK,[11] GBK,[12] BLP,[13] DLW,[14] NNV[15] and KdV[16] systems are similar.

3. Novel localized structures in (2+1)-dimensional ANNV system

All rich localized coherent structures, such as non-propagating solitons, dromions, peakons, compactons, foldons, instantons, ghostons, ring solitons, and the interactions between these solitons, can be derived by using the quantity $U \equiv u_5$ expressed by expression Eq.(13). These abundant localized coherent structures are neglected in the present study since some similar situations have been reported in Refs.[1–16]. Now we pay our attention to the combined structures, including bell-like compactons, peakon-like compactons and compacton-like semi-foldons, for the field u_5 expressed by expression (13).

3.1. Special localized structures

Based on the physical quantity (13), multi-valued localized structures can be discussed, when the function p is a single-valued function and q is selected via the relations

$$q_y = \sum_{i=1}^{N}\kappa_i(\zeta - d_it),$$
$$y = \zeta + \sum_{i=1}^{N}\eta_i(\zeta - d_it),$$
$$q = \int^{\zeta} q_yy_\zeta \mathrm{d}\zeta, \tag{15}$$

where $d_i (i = 1, 2, \cdots, N)$ are arbitrary constants, κ_i and η_i are localized excitations with the properties $\kappa_i(\pm\infty) = 0$, $\eta_i(\pm\infty) = \text{consts}$. From Eq.(15), one can know that ζ may be a multi-valued function in some suitable regions of y by choosing the functions η_i appropriately. Therefore, the function q_y, which is obviously an interaction solution of N localized excitations due to the property $\zeta|_{y\to\infty} \to \infty$, may be a multi-valued function of y in these areas, though it is a single-valued function of ζ. Actually, most of the known multi-loop solutions are special cases of Eq.(15).

Specifically, p and q are chosen as

$$p = 8 + \begin{cases} 0, & x \le -\dfrac{\pi}{2}, \\ \sin(x) + 1, & -\dfrac{\pi}{2} < x \le \dfrac{\pi}{2}, \\ 2, & x > \dfrac{\pi}{2}, \end{cases} \tag{16}$$

$$q_y = 0.5\mathrm{sech}^2(\zeta), \quad y = \zeta - A\tanh(\zeta), \tag{17}$$

where A is a characteristic parameter, determining the localized structure. Figures 1–3 describe three new localized structures, i.e. bell-like compacton, peakon-like compacton and compacton-like semi-foldon with $A = 0.05, 0.95, 1.5$, respectively. They localize as a compacton in the x-direction and a bell-like soliton, peakon and loop soliton in the y-direction, respectively.

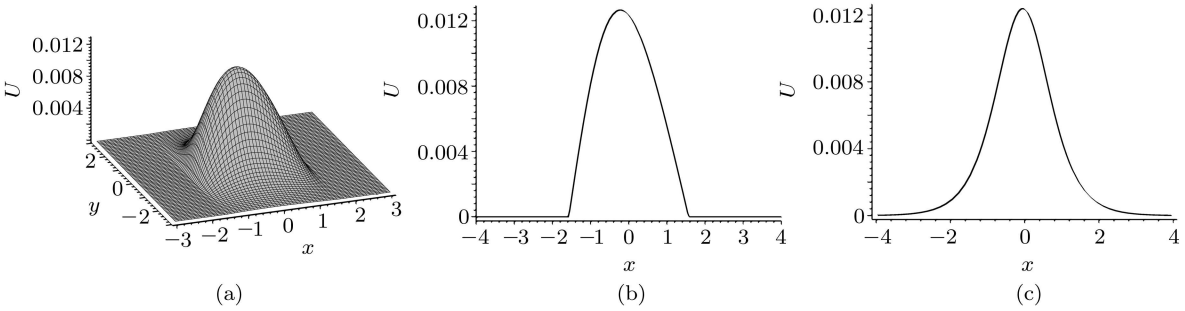


Fig.1. (a) Bell-like compacton structure for U under conditions (16), (17) and $A = 0.05$, (b) a sectional view related to (a) at $y = 0$, (c) a sectional view related to (a) at $x = 0$.

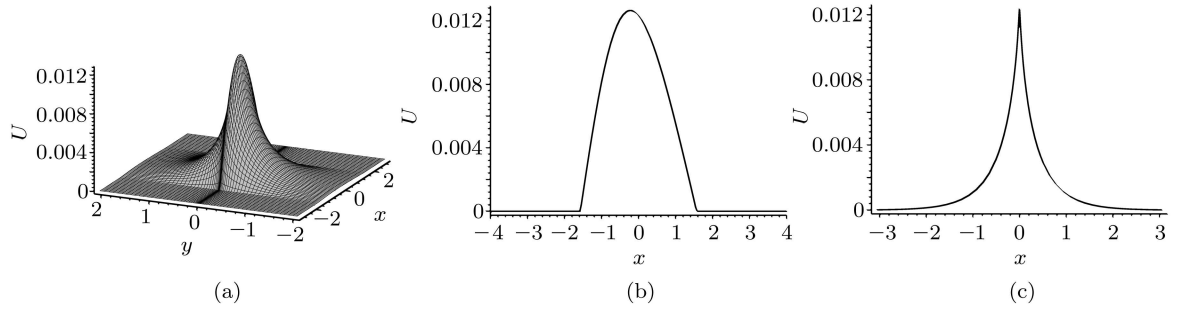


Fig.2. (a) Peakon-like compacton structure for U under conditions (16), (17) and $A = 0.95$, (b) a sectional view related to (a) at $y = 0$, (c) a sectional view related to (a) at $x = 0$.

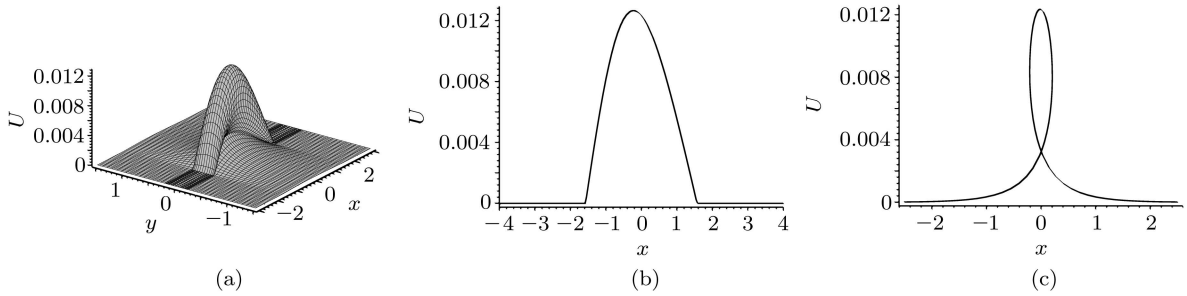


Fig.3. (a) Compacton-like semi-foldon structure for U under conditions (16), (17) and $A = 1.5$, (b) a sectional view related to (a) at $y = 0$, (c) a sectional view related to (a) at $x = 0$.

3.2. Interaction between localized coherent excitations produced by multi-valued functions

Now we discuss some novel localized coherent structures for the physical quantity U, and focus our attention on the interaction between these localized coherent excitations produced by multi-valued functions in Section 3.1. In order to discuss them conveniently and reveal the phase shift more clearly and visually, it is expedient to fix one of them to have a zero velocity. That is to say, if we take the specific choice $N = 2, d_1 = 0.5$ and $d_2 = 0$ in Eq.(15), one has

$$\begin{aligned} q_y &= \mathrm{sech}^2(\zeta - 0.5t) + 0.5\mathrm{sech}^2(\zeta), \\ y &= \zeta - A\tanh(\zeta - 0.5t) - B\tanh(\zeta), \end{aligned} \tag{18}$$

where A and B are characteristic parameters, and they determine the types of interaction and the phase shift of solitons. Moreover, p is given by Eq.(16). From the expression U with Eqs.(18) and (16), one can obtain two solitons, one is static and the other is moving. Generally, the phase shift of the static soliton is $\eta_1(-\infty) - \eta_1(+\infty) = 2A$ in the y-direction, and the phase shift of the moving soliton is $\eta_2(-\infty) - \eta_2(+\infty) = -2B$ in the y-direction.

3.2.1. Non-completely elastic interaction between bell-like compacton and peakon-like compacton

If we take the specific values $A = 0.95, B = 0.05$ in Eq.(18) and p satisfying Eq.(16), then we can successfully construct interaction between bell-like compacton and peakon-like compacton, which both possess phase shifts for the physical quantity U depicted in Fig.4. From Fig.4, one can find that the interaction between these two types of compactons may exhibit a novel property, which is non-completely elastic since their shapes and amplitudes are not completely maintained any more after the interaction. The phase shift can also be observed. Prior to the interaction, the small soliton has been set to be $\{v_{0x} = 0, v_{0y} = d_2 = 0\}$, however, the small soliton site changes from $y = -0.95$ to $y = 0.95$, then resides at $y = 0.95$ and maintains its initial velocities $\{v_x = v_{0x}, v_y = v_{0y}\}$ after the interaction. Therefore the phase shift of the static small soliton is 1.9. The final velocities V_x and V_y of the moving large soliton also completely maintain its initial velocities $\{V_x = V_{0x} = 0, V_y = V_{0y} = d_1 = 0.5\}$. The phase shift of the moving large soliton is -0.1.

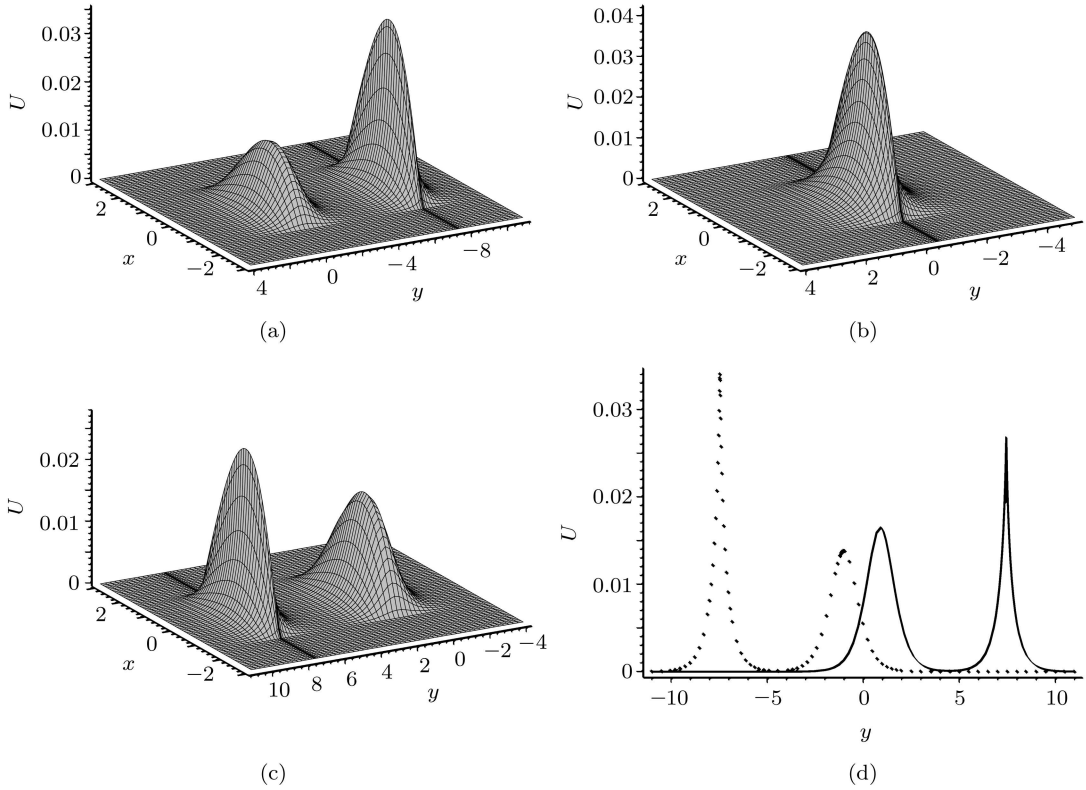


Fig.4. Interaction between bell-like compacton and peakon-like compacton for U with conditions (18) under $A = 0.95, B = 0.05$ and (16) (a) $t = -15$, (b) $t = -1$, (c) $t = 15$. (d) The corresponding sectional view at $x = 0$ (dotted and solid lines denote before and after collision, respectively.)

3.2.2. Non-completely elastic interaction between peakon-like compacton and compacton-like semi-foldon

If we choose the specific values $A = 0.95, B = 1.5$ in Eq.(18) and p satisfying Eq.(16), then we can successfully construct the interaction between peakon-like compacton and compacton-like semi-foldon, which both possess phase shifts for the physical quantity U depicted in Fig.5. From Fig.5, we can see that these two special localized excitations possess a novel property, that is, there exists a multi-valued foldon in the

process of their collision, which is different from the case in Section 3.2.1. One can find that the interaction between them may also exhibit another novel property, which is non-completely elastic since their shapes and amplitudes are not completely maintained any more after the interaction. Through careful analysis similar to that in Section 3.2.1, we know that the phase shift of the static small soliton is 1.9, and the moving big soliton has a phase shift of -3.

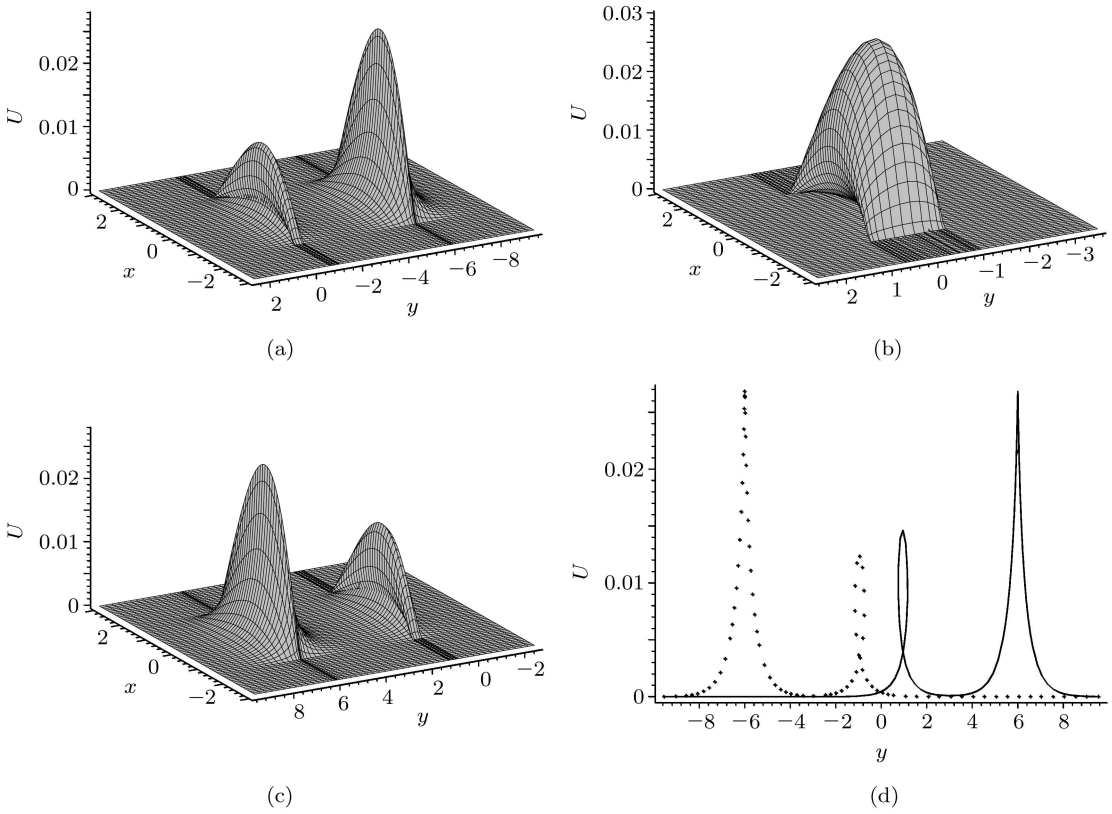


Fig.5. Non-completely elastic interaction between peakon-like compacton and compacton-like semi-foldon for U under conditions (18) and (16) with $A = 0.95, B = 1.5$ (a) $t = -15$, (b) $t = -1$, (c) $t = 15$. (d) The corresponding sectional view at $x = 0$ (dotted and solid lines denote before and after collision, respectively.)

Similarly, we can also discuss the interaction between bell-like compacton and compacton-like semifoldon by choosing $A = 0.05$ and $B = 1.5$ in Eq.(18). The properties of this interaction is similar to the case discussed above. Owing to the limited length of this paper, we neglect them here.

3.2.3. Completely elastic interaction between compacton-like semifoldons

It is interesting to note that although the above selections result in non-completely elastic interaction behaviours for the (2+1)-dimensional solutions, we can also construct localized coherent structures with completely elastic interaction behaviours by appropriately selecting the values of A and B in Eq.(18).

If we select the specific values $A = B = 1.5$ in Eq.(18) and p satisfying Eq.(16), then we can successfully construct the interaction between compacton-like semifoldons that possess phase shifts for the physical quantity U depicted in Fig.6. From Fig.6, one can find that the interaction between them may exhibit a novel property, which is completely elastic since their shapes, amplitudes and velocities are completely maintained after the interaction. The phase shift can also be observed. Before the interaction, the static small semifoldon is located at $y = -1.5$ and after the interaction, it is shifted to $y = 1.5$ and then resides at $y = 1.5$ and maintains its initial velocities $\{v_x = v_{0x} = 0, v_y = v_{0y} = 0\}$ after the interaction. While the moving big semifoldon also completely maintains its initial velocities $\{V_x = V_{0x} = 0, V_y = V_{0y} = d_1 = 0.5\}$. Through careful analysis similar to that in Section 3.2.1, we know that the phase shift of the static small semifoldon is 3, and the moving large semifoldon has a phase shift of -3.

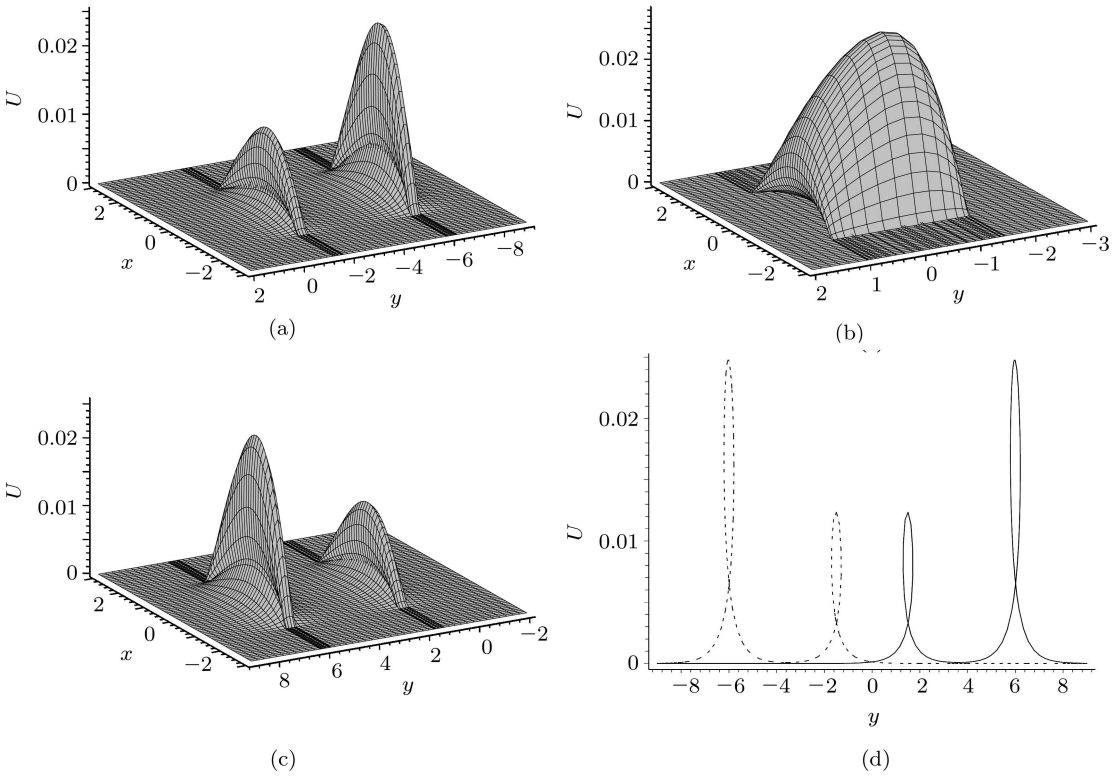


Fig.6. Non-completely elastic interaction between peakon-like compactons for U under conditions (18) and (16) with $A = B = 1.5$ (a) $t = -15$, (b) $t = -1$, (c) $t = 15$. (d) The corresponding sectional view at $x = 0$ (dotted and solid lines denote before and after collision, respectively.)

4. Summary and discussion

The extended tanh-function method is applied to variable separation solutions of (2+1)-dimensional ANNV system, and some lower-dimensional arbitrary functions can be included in their exact solutions. By further studying, we find that these variable separation solutions are seemingly independent of but actually dependent on each other. In fact, there exist similar cases in these variable separation solutions for the (2+1)-dimensional BKK,[11] GBK,[12] BLP,[13] DLW,[14] NNV[15] and KdV[16] systems. Based on the field (13), new types of interactions between special solitons, such as bell-like compactons, peakon-like compactons and compacton-like semi-foldons, are investigated. Some novel properties and interesting behaviours show that the interaction between compacton-like semi-foldons is completely elastic and possesses phase shifts, and the interactions between bell-like compacton and peakon-like compacton, between bell-like compacton and compacton-like semifoldon, and between peakon-like compacton and compacton-like semifoldon are incompletely elastic and dependent on the details of the solutions. Moreover, there exists a multi-valued foldon in the process of the interactions between bell-like compacton and compacton-like semifoldon, and between peakon-like compacton and compacton-like semifoldon, which is reported here for the first time, to the best of our knowledge. Although we have reported on some novel interactions in (2+1)-dimensions, it is obvious that there are still many significant and interesting problems left to be further discussed. For example, how is the notion of complete or incomplete elasticity quantified more appropriately? What is the measure of the deviation of a solution from elasticity? What is the general equation describing the distribution of the energy and momentum for these exotic interactions?

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