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Mathematical concepts and their physical foundation in the nonstandard analysis theory of turbulence*

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Main mathematical concepts and their physical foundation in the nonstandard analysis theory of turbulence are presented and discussed. The underlying fact is that there does not exist the absolute zero fluid-volume. Therefore, the physical object corresponding to the absolute point is just the uniform fluid-particle. The fluid-particle, in general, corresponds to the monad. The uniform fluid-particle corresponds to the uniform monad, while the nonuniform fluid-particle to the nonuniform monad. There are two kinds of the differentiations, one is based on the absolute point, and the other based on the monad. The former is adopted in the Navier–Stokes equations, and the latter in the fundamental equations presented in this paper for the nonstandard analysis theory of turbulence. The continuity of fluid is elucidated by virtue of the concepts of the fluid-particle and fluid-particle at a lower level. Furthermore, the characters of the continuity in two cases, i.e. in the standard and nonstandard analyses, are presented in this paper. And the difference in discretization between the Navier–Stokes equations and the fundamental equations given herein is also pointed out.

Keywords: turbulence, monad, fluid-particle at a lower level, nonstandard analysis theory of turbulence

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1. Introduction

In Refs.[1–3], a new description of turbulence is presented and the preliminary discussion on some important and key concepts about this description is given. The new description of turbulence is named the nonstandard analysis theory of turbulence (NATT). In this article, we have a discussion on the reasonableness of the basic mathematical concepts in the NATT. Some of these concepts are available in the traditional standard analysis, but they have acquired new meanings in the NATT. And the others do not appear in the standard analysis at all.

Obviously, the reasonableness of these mathematical concepts should be based on their physical foundation. The mathematical logic is important, but should not be given the first place. In the mathematical logic, the introduction of these concepts should only meet the requirements for compatibility with existing concepts.

The mathematical concepts involved in this paper are the point, the structure of point, the point-average, the infinitesimal, the monad, the new derivative, the continuity of fluid, etc. We discuss their physical meanings in order to show the reasonableness of these mathematical concepts in the remainder of this paper.

2. Point and continuity of fluid

Fluid is composed of numerous moleculae, but could be assumed to be continuum provided that the spatial scales of any physical process in the fluid are much greater than the free path of molecule.[4] Then the fluid flow is usually called a flow field, i.e. the fluid in motion is modelled as a flow field. The flow field is a mathematical concept, which is continuously composed of the real number points and is also referred to as real number space mathematically. Yet after contemplation it is obvious that there does not exist the 'point' of real number space in a real fluid. Therefore, the physical meaning of the point in the real number space (i.e. flow field), in other words, the physical reality from which the concept of point is drawn, should be clarified. In fact, the logic of the process, with which the flowing fluid composed of moleculae is ab-

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stracted into the real number space (i.e. flow field), should be as follows.

It is well known that the motion of the moleculae contained in a very small volume will reach, through their collisions with one another, an equilibrium state in a very short period of time. Based on this fact, the concept of the fluid-particle has been presented. The fluid is, in fact, physically composed of the fluid-particles. The fluid-particle is a very small volume of fluid and is real physical object. Hence, the concept of the mathematical point is drawn from the fluid-particle, i.e. an abstracted fluid-particle is a point. After the abstraction of the fluid-particles, the flow field composed of the points is obtained. So the point forming a flow field is an abstracted concept and is drawn from the fluid-particle, that is to say, the concept of point is based on the fluid-particle.

Obviously, it is necessary to make this abstraction in order to know the properties of flow. On the basis of real number space composed of the real number points, the mathematical manipulation, by which the law and properties of fluid flow can be obtained, is made reasonably. Otherwise, there could hardly be a mathematical deduction based on the fluid-particle.

Actually, the fluid-particle is a very small volume of fluid. However small as the scale of the fluid-particle is, it is still finite from the perspective of physical practice. We point out some features about the scale of the fluid-particle below.

The physical world possesses a hierarchical structure. The scales of the fluid-particles at different levels are different accordingly. Surely, the scale of the fluid-particle at a higher level should be much greater than that at a lower level. The scale of the fluid-particle at the lowest level in a physical problem should not be smaller than a certain scale ξ. ξ is the lowest scale in all the scales, in which any volume formed can contain an enough number of moleculae for a stationary statistical average of physical quantity. The fluid volume, of which the linear scale is less than the scale ξ, does not have certain average values of physical quantities. Therefore, that fluid volume cannot be called a fluid-particle.

But the scale of a fluid-particle at any level cannot be too large to show the macroscopic variation of physical quantities with time and space.

It is important to note that the scales of the fluid-particles at any level are all objective. The scale of the fluid-particle is not determined by people at will, but by the nature of the specific physical problem under study. In other words, the scale of the fluid-particle should be on the basis of physics and correspond to the nature of the specific physical problem under discussion. For example, the scale of the fluid-particle in atmospheric flow is much greater than that in the flow around the wings of an aircraft.

In practice, one is generally unable to give how large exactly the fluid-particle is, yet we believe that the scale of the fluid-particle is not set by people's will, but it is objective.

In the light of the concept of the fluid-particle, the clearer physical meaning of the continuity of a fluid can be given, that is, if there exists a space scale of ξ, an enough number of moleculae are contained in the fluid volume with the linear scale ξ, so that the statistical averaging over the motion of these moleculae can give a stationary value, i.e. the fluid-volume with linear scale ξ is just a fluid-particle at a certain level. And, the spatial scales of any physical process in the fluid are all greater than the scale ξ. Furthermore, the motion of the moleculae contained in the volume with the linear scale ξ, through molecule collision, has already reached a thermodynamic equilibrium in a time interval less than or equal to the time scale of any physical process in the fluid. Then the fluid can be regarded as continuum. In this case, the effect of fluid composed of moleculae can be ignored, and the real flow of fluid could be modelled as a flow field.

3. Uniform and non-uniform points

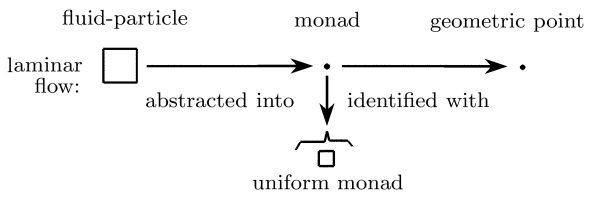
The fundamental physical idea in the NATT is that the fluid-particle, corresponding to the physical nature of a practical problem, is wholly uniform in a laminar flow, but not uniform in the case of turbulence, i.e. there exists an internal structure in the fluid-particle. By what has been mentioned above, though the fluid-particle in the turbulence can be divided logically into smaller and uniform fluid-particles, we cannot ignore the fact that the fluid-particle cannot be divided at will in physical practice. The scale of the fluid-particle has the objective physical basis in practice.

What is the mathematical meaning of this fundamental physical consideration in the NATT?

As is stated above, a point (the absolute geometric point) should be mathematically drawn from

only the uniform fluid-particle. If the non-uniform fluid-particle was, in mathematical abstraction, also thought of as an absolute geometric point, the internal structure of the non-uniform fluid-particle would be unable to be described. So this will be against the physical reality. Thus, a non-uniform fluid-particle should be, in mathematical abstraction, taken as a monad. The linear scale of the monad is ε, an infinitesimal one: $\varepsilon \neq 0$, but $\varepsilon > 0$. The internal structure can be permitted in the monad. From the nonstandard analysis, the infinitesimal ε is also a number, i.e. a nonstandard number. The real number is called the standard number.

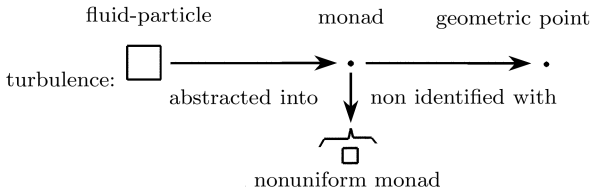
Even in the case of a laminar flow, in fact, the monad is also the mathematically abstracted uniform fluid-particle, and it is uniform and has no internal structure inside, which is essentially identical with the absolute point. This concept can be shown in Figs.1 and 2. Therefore, the physical meaning of mathematical monad is that the monad is the mathematically abstracted fluid-particle. In the case of a laminar flow, the fluid-particle is wholly uniform, so the monad is also uniform. However, in turbulence, the fluid-particle is not wholly uniform, so the abstract monad is non-uniform. In Ref.[1], the monad is also called the standard point, different from the absolute point. Thus, correspondingly there are also uniform and non-uniform standard points (usually called uniform and non-uniform points for short). In a laminar flow, the uniform standard point drawn from the uniform fluid-particle is essentially identical with the absolute point. So in physical meaning, the absolute point is also the uniform fluid-particle.



The whole flow field is just mathematically real number space composed of the absolute real number points.

Fig.1. Mathematically abstracted fluid-particle in a laminar flow.

The structure of the point in the NATT represents just the internal structure of the standard point, rather than the absolute point, which does not have the internal structure. Obviously, the internal structure of the standard point is the mathematically abstracted internal structure of the fluid-particle. So in physical meaning, the structure in a point is just the internal structure of the fluid-particle.



The whole flow field is just mathematically hyper real number space composed of monads (i.e. standard points).

Fig.2. Mathematically abstracted fluid-particle in turbulence.

Finally, it should be pointed out that the fluid-particle is the physical reality and has a finite scale. Because the scale of the fluid-particle is very small, the fluid-particle is considered, in the mathematical abstraction, as a monad, whose scale is infinitesimal. A uniform fluid-particle could be even considered as an absolute point with a zero scale.

In Ref.[1], it is shown that the fluid-particle, whether uniform or not, is composed of numerous fluid-particles at a lower level (see Fig.3), and a fluid-particle at a lower level is uniform and contains still an enough number of fluid moleculae for the stationary statistical average.

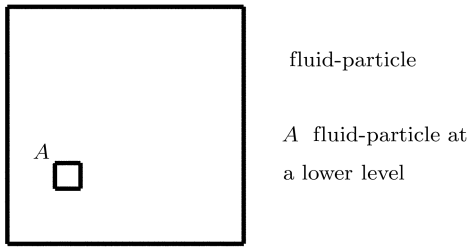


Fig.3. Fluid-particle composed of fluid-particles at a lower level.

A mathematically abstracted fluid-particle at a lower level is a nonstandard point (i.e. the internal point of a monad). Therefore, the nonstandard points (i.e. the internal points of a monad) are uniform points.

Thus the hierarchical structure of a turbulent field could be elucidated as follows. The global turbulent field, composed of the monads, i.e. the abstracted

fluid-particles, is at a level. And the monad field (i.e. the abstraction from the flow in the fluid-particle), composed of the nonstandard points (the mathematical abstraction from the fluid-particles at a lower level), is at another lower level.

4. Two kinds of differentiations

Given a function $f(x)$, whose derivative is

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}. \tag{1}$$

Here mathematically $\Delta x \to 0$ means that Δx approaches an absolute zero. But, in many cases, one cannot clearly give the physical meaning of the absolute zero. For example, it makes no sense to say: 'A fluid volume approaches an absolute zero'. In many cases, in fact, there exists no 'absolute zero' in physics. The physical meaning of the statement of 'a fluid volume approaches an absolute zero' is ambiguous. Therefore, the statement that Δx approaches an absolute zero (i.e. $\Delta x \to 0$) in mathematics does not mean that any physical quantity approaches an absolute zero, but corresponds to, for example, in the case of fluid flow, the fact that physical quantity Δx approaches a linear scale of the uniform fluid-particle.

In the NATT, given a function $f(x, x')$, there exists another derivative as follows:

$$\frac{\partial f}{\partial x} = \frac{f(x + \varepsilon, x') - f(x, x')}{\varepsilon}. \tag{2}$$

Here ε is nonstandard number infinitesimal. ε is not arbitrarily infinitesimal, but a linear scale of a monad. The monad is an abstracted fluid-particle. So the infinitesimal ε, the linear scale of the monad, is logically the linear scale of the abstracted fluid-particle, namely, in physical meaning, the infinitesimal ε is the linear scale of the fluid-particle.

The definition (2) has already been presented in the nonstandard analysis.[5] But there the infinitesimal is arbitrary, while in the NATT ε is a certain infinitesimal, i.e. a linear scale of a monad.

The definition (1), in general, implies mathematically that the $\dfrac{\partial f}{\partial x}$ is the rate of increment of function f at some point (absolute geometric point), i.e. the slope of tangent to the curve of function f. But, in many cases, the absolute point does not exist in physics, let alone the rate of increment of function at this point. In the case of fluid, the physical meaning corresponding to the definition (1) can be understood to be the increment of the physical quantity f between two neighbouring uniform fluid-particles at some level, divided by the linear scale of the fluid-particle at the same level. Let A and B be two neighbouring uniform fluid-particles, and the linear scale of the fluid-particle be L, then the physical meaning of $\dfrac{\partial f}{\partial x}$ in definition (1) will be expressed as

$$\frac{f(B) - f(A)}{L}.$$

Therefore, from the perspective of physics, the physical quantity $\dfrac{f(B) - f(A)}{L}$ is defined on only the uniform fluid-particle at a certain level. Correspondingly, in mathematics, the definition (1) is applicable only to the uniform point at the same level.

The right side of the definition (2), from the perspective of mathematics, represents the increment of function f between two corresponding internal points of two infinitely close monads, divided by the linear scale of the monad. And in physical meaning, the right side of definition (2) means the increment of physical quantity function f between two corresponding fluid-particles at a lower level, which are uniform in NATT, of two neighbouring fluid-particles, divided by the linear scale of the fluid-particle. Let L be the linear scale of the fluid-particle, b and a be the two corresponding fluid-particles at a lower level of two neighbouring fluid-particles B and A, i.e. b and a are located at the same place in respective fluid-particles (see Fig.4).

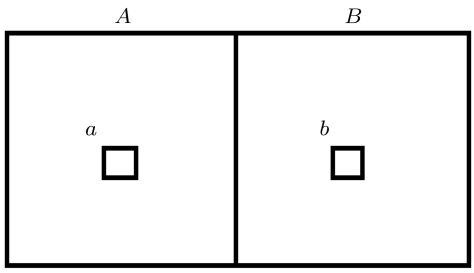


Fig.4. Positions of a and b in fluid-particles A and B.

So the $\dfrac{\partial f}{\partial x}$ in the definition (2) is the mathematically abstracted physical quantity $\dfrac{f(b) - f(a)}{L}$. The reasonableness of the mathematical concept $\dfrac{\partial f}{\partial x}$ in the expression (2) is based on the real physical quantity $\dfrac{f(b) - f(a)}{L}$.

The physical quantity $\dfrac{f(b)-f(a)}{L}$ can be defined on both the uniform and non-uniform fluid-particles in physics, therefore, the definition (2) can be applied to both the uniform and non-uniform points in mathematics.

If, in fluid mechanics, the conservation laws of physics (the conservation of mass, momentum and energy) are applied to uniform and nonuniform fluid-particles at some levels respectively, different results will be obtained. In the case of uniform fluid-particles, the mathematical abstraction of the results obtained yields the Navier–Stokes equations (for the incompressible fluid)

$$\frac{\partial U_i}{\partial x_i}=0, \tag{3}$$

$$\frac{\partial U_i}{\partial t}+\frac{\partial (U_iU_j)}{\partial x_j}=-\frac{1}{\rho}\frac{\partial P}{\partial x_i}+\nu\nabla^2U_i. \tag{4}$$

Equations (3) and (4) hold for only the uniform points.

In the case of the non-uniform fluid-particles, the mathematical abstraction of the results obtained leads to the equations (for the incompressible fluid)

$$\frac{\partial \tilde{U}_i}{\partial x_i}=0, \tag{5}$$

$$\frac{\partial \tilde{U}_i}{\partial t}+\frac{\partial (\widetilde{U_iU_j})}{\partial x_j}=-\frac{1}{\rho}\frac{\partial \tilde{P}}{\partial x_i}+\nu\nabla^2\tilde{U}_i, \tag{6}$$

where the sign '$\sim$' denotes the average over all the nonstandard points contained in a monad. Equations (5) and (6) hold for both the uniform and non-uniform points and will degenerate into the Navier–Stokes equations (3) and (4) in the case of the uniform points.

The uniform monad (point) is drawn from the uniform fluid-particle. And the fluid-particle is uniform in a laminar flow. Therefore, Eqs.(3) and (4) can be applied to a laminar flow. But the fluid particle is not uniform in turbulence. Then only Eqs.(5) and (6) can be applied to turbulence. So we call Eqs.(5) and (6) the fundamental equations in turbulence theory.

The derivative (1) is based on the limit, expressed as the statement of $\delta-\varepsilon$, and in the framework of $\delta-\varepsilon$, the derivative (2) is not based on the limit and jumps out of the framework of $\delta-\varepsilon$. The statement of $\delta-\varepsilon$ shows mathematically the endless process of infinite approach, and makes the expression of the process stricter mathematically. The statement of the process of infinite approach, for example infinite approach to zero, does not give any room to physics, i.e. it shows the process of approach to the absolute zero or the absolute nil. However, in the real physics, the process of infinite approach to zero is just a mathematically abstracted real process of gradually approaching the scale of the uniform fluid-particle at some level.

The statement of $\delta-\varepsilon$ obliterates this real uniform fluid-particle. In many cases, this obliteration does not affect the results, but sometimes it does cause some problems. However, if the fluid-particle is not uniform, in which the structure in the fluid-particle cannot be described by the absolute point as done in the case of turbulence, the statement of $\delta-\varepsilon$ should be abandoned and the derivative (2) is adopted. The derivative (2) is mathematically based on the infinitesimal. In the 1960s, the nonstandard analysis was proposed. The existence of an infinitesimal has been proved strictly in the nonstandard analysis, which gives the mathematical foundation to the new description of turbulence.

5. Continuity

In Section 1, we have mentioned the physical meaning of the fluid continuity. From the physical reality the mathematical model of fluid, i.e. continuous flow field, is obtained. In the continuous flow field, the physical properties (i.e. the various physical quantities at all points of the field) vary continuously with time and space. Yet the mathematical description of the continuous variation of the physical quantities in the nonstandard analysis is different from that in the standard analysis.

In the case of standard analysis, when a function representing a physical quantity is mathematically continuous, the physical quantity is considered to be continuous. In other words, if a physical quantity varies continuously with time and space, the function f representing the physical quantity is, in mathematics, a continuous function. The definition of continuous function is given as (see Fig.5)

$$\lim_{\Delta x\to 0}[f(x+\Delta x)-f(x)]=0.$$

Note that the function $f(x)$ is characterized only by the value of function.

In nonstandard analysis, however, it is another story. Here the function representing the physical quantity should be written as $f(x,x')$, where the letter x denotes the monad x in some hyperreal number

space, x' is the internal coordinates in the monad x, i.e. x' is the coordinates of the internal point of the monad x (see Fig.6). $f(x, x')$ denotes the value of the physical quantity at an internal point x' of the monad x. Then the continuous variation of the physical quantity expressed by the function $f(x, x')$ with time and space implies that the function $f(x, x')$ varies continuously not only in value but also in structure. In other words, the continuity of function $f(x, x')$ means that not only the increment of $f(x + \Delta x, x') - f(x, x')$ approaches an infinitesimal, but also the shape of the function $f(x + \Delta x, x')$ approaches infinitely the shape of the function $f(x, x')$, when Δx approaches an infinitesimal ε (the linear scale of a monad), i.e. the two monads (the monad $x + \Delta x$ and the monad x) are infinitely close to each other (see Fig.7).

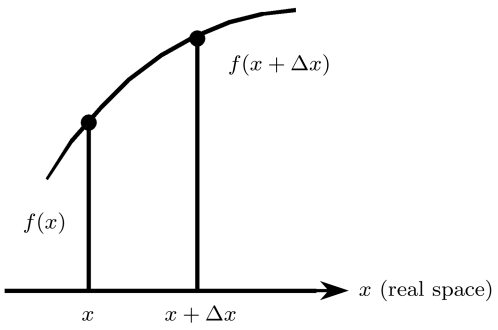


Fig.5. Continuous function $f(x)$.

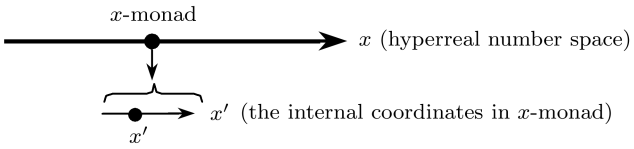


Fig.6. The internal point x' of monad x.

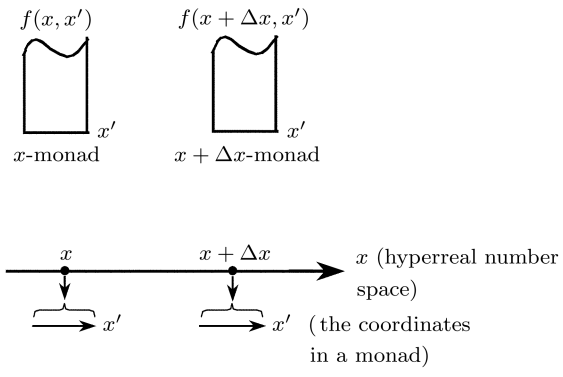


Fig.7. Continuous function $f(x, x')$.

Here the function $f(x, x')$ possesses both the value and the shape (i.e. the variation of the function $f(x, x')$ with x' in the monad x). So the variation in not only function value but also function shape should be continuous, if the function of physical quantity is continuous. Therefore, the continuity of the fluid should be the continuity for both the function shape and the function value. These are the two aspects in the continuity of fluid in the case of the nonstandard analysis.

In the standard analysis, the mathematical continuity of a function shows, in physics, a certain relationship between physical quantities of only two neighbouring uniform fluid-particles. When one says that the function representing the physical quantity is continuous, the continuity means in physics that the difference in the value of the physical quantity between two neighbouring uniform fluid-particles is very small.

Similarly, in the nonstandard analysis, the mathematical concept of continuity depicts, in physics, just a certain relationship between physical quantities of two neighbouring fluid-particles (both uniform and non-uniform). The difference in physical quantity, i.e. in value and structure of a function representing the physical quantity between the two neighbouring fluid-particles is very small. In the NATT, this is called the assumption of close property between two neighbouring fluid-particles, or in abstraction, between two infinitely close monads. The assumption is just the continuity (in physics, corresponding to the case of both uniform and non-uniform fluid-particles) in the nonstandard analysis, which is the extension of the general continuity (in physics, corresponding only to the case of uniform fluid-particle) in the standard analysis. The legitimacy of this assumption should be verified by the rationality of the results from the assumption.

Discontinuity is the opposite of continuity. In order to show these concepts clearly, the concept of the fluctuation in a fluid will be discussed further as follows.

Will the data obtained from the measurement be continuous (with no fluctuation) if the accuracy of the measurement in turbulence is very high?

As stated above, the concept of the monad is drawn from the fluid-particle. Here, the monad and fluid-particle are in the space of four dimensions: time and space. It is clear what is the fluid-particle in space (three dimensions). Yet the meaning of the fluid-particle in the time-direction needs to be elucidated. Let the time scale of the fluid-particle be δ_{t},

and the time scale of the fluid-particle at a lower level be τ. Surely, the scales of δ_t and τ are all objective, and they are determined by the physical nature of the problem under discussion. The scale τ needs to meet the requirements for the following conditions: The motion of the fluid moleculae contained in a spatial fluid-particle at a lower level through molecule collision has already reached a thermodynamic equilibrium in the time period of τ. The time-direction scale which is infinitesimal, of the monad in four dimensions, and the time-direction scale which is a higher order infinitesimal, of the monad-interior point (nonstandard point) in four dimensions are the mathematically abstracted δ_t and τ respectively.

It is known that if a measurement in a fluid is carried out at the levels of atom and molecule, i.e. the measurement is performed on the motions of atom and molecule, the measurements will show that they are in random motion, governed by their corresponding quantum laws. The discussion in this respect is out of the scope of this paper. Now the objective of the measurement is restricted to the mean value of the collective motion of numerous moleculae.

Assume that the motion of numerous moleculae contained in a spatial fluid-particle at a lower level, through molecule collision, has already reached a thermodynamic equilibrium in the time interval greater than or equal to τ. The time interval between two consecutive measurements is t_0. If $t_0 < \tau$, the data of the measurement are not able to show the physical properties of this fluid-particle at a lower level, but the average values, which are nonstationary, of the motion properties of the moleculae contained in the fluid volume smaller than the volume of the fluid-particle at a lower level. So the data are fluctuant. If $t_0 > \tau$, but $t_0 \ll \delta_t$, the data of measurement should be continuous (with no fluctuation) and obey the Navier–Stokes equations. The measurement, in fact, is carried out in the range of time scale of a fluid-particle.

But usually the measurement in turbulence is not performed in such a way as described above. In the turbulence measurement at present, the different measurements are conducted at different fluid-particles with four dimensions. The results of one measurement are just the statistical average values of the motion properties, which reach an equilibrium through intermolecular collisions in the interval τ, of the numerous moleculae contained in the spatial fluid-particle at a lower level. In other words, in the mathematical abstract, the different measuring operations are performed on different monads. The results of one measurement are just the statistical average values of the motion properties of the numerous moleculae contained at the spatial internal point in one monad (see Fig.8). The data obtained by the measurements represent the statistical average values of physical quantities at different internal points, for example, the internal points: 1, 2, 3, $\cdots, n$ (see Fig.8). The essential character of this measurement is that the measurer cannot know which internal point of a monad has been measured, but can know which monad contains the measured internal point only. So the measurer knows only the range over the monad, but not the exact intra-monad position of the measured objective. This is an uncertainty in measurement on turbulence. Therefore, the obtained data will show the fluctuation provided that the motion properties of different internal points of a monad are obviously different.

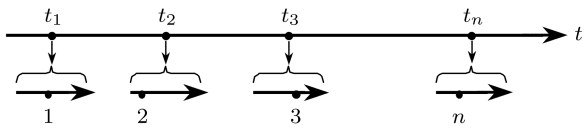


Fig.8. The schematic of the measurement in turbulence.

It is possible in principle that the measurements can be carried out continuously at the different internal points of one monad, and continuous data can be obtained in the monad. The required accuracy in the measurements can be reached by modern technologies. And the consecutive measurement can be performed on the next monad. In the same way the continuous measurements can proceed to be made on various monads one by one. However, the practical manipulation in measurement can hardly be carried out, because it is nearly a process in which infinite measurements are performed. This process of measurement, just like a process of solving an infinite number of equations, is impracticable. Therefore, this kind of measurement exists only in theory, but not in reality.

Hence the practical measurement of turbulence will give, as mentioned above, a fluctuant result. This fluctuation results from the measurement of a regular flow (the turbulence is also regular). Note that the fluid flow itself is one thing, the result from the measurement of the fluid flow is another story. They are closely related to each other, but are not the same. The irregular results of measurement do not represent

the irregularity and discontinuity of the fluid flow itself, and conversely, the continuity and regularity of the fluid flow do not exclude the possibility of the irregular results (or called the fluctuation) from the measurement on the fluid flow. The measurement carried out on turbulence implies that the irregular and fluctuant data are obtained from the measurement on a regular and continuous fluid flow. It may be said that the turbulence-fluctuation is, in fact, just the fluctuation of the data from the measurement on turbulence.

As to the fluctuation, besides the random results of measurements at a molecule level, there are the fluctuation results from the statistical average of a state that has not yet become stationary, i.e. from the measurement in the range of the scale less than that of the fluid-particle at the lowest level, and the fluctuation arises from the aforementioned uncertainty in the measurement on turbulence (at the level of global flow). Only the turbulence-measurement in the fluid particle, from which the concept of monad is drawn, can possibly give the continuous results.

In the NATT, there is a mathematical concept of the monad-average (point-average). From its physical meaning, the point-average means the average over the motion of numerous lower-level fluid-particles in a fluid-particle. The point-average is drawn from the average over the fluid-particle. Here the point-average is different from the conventional average in turbulence research, and it is obtained without averaging over measurement data. Surely, we can easily relate the point-average in the NATT with the conventional average in the common sense. The value of the point-average can be obtained by the numeric computation, and then the conventional average value comparable with the average value from the measurement is easily obtained from the values of the point-average.

The standard and nonstandard analyses have been mentioned in the NATT. In a laminar flow, the standard analysis is employed; while in turbulence the nonstandard analysis is adopted. In both laminar and turbulent flows, in fact, either the standard or nonstandard analysis can be used. The standard analysis means that there is a description at one level, while the nonstandard analysis means that there is a description at multi-levels. In the standard analysis, the description of a fluid flow is based on a uniform fluid-particle (e.g. a fluid-particle in a laminar flow, and a fluid-particle at a lower level in a turbulent flow). The fluid-particle as well as the fluid-particle at a lower level is wholly uniform in the case of a laminar flow, so the fluid-particle can be thought of as a basic element in a laminar flow. And just a fluid-particle corresponds to a physical objective. Therefore, the standard analysis, the description of one level, is employed in a laminar flow. However, in turbulence, the fluid-particle is not wholly uniform and the fluid-particles at a lower level, contained in the fluid-particle, are possessed of corresponding different moving properties. Obviously, there exists a multi-level structure in turbulence. This is why the nonstandard analysis is employed in turbulence by the author. Theoretically, the standard analysis, in which the single-level description of flow and the Navier–Stokes equations are adopted, can be also employed in turbulence, but the fluid-particle at a lower level must be the basic element. The fluid-particle at a lower level does not correspond to a physical objective, so there appears the unsuitability in practice. The evident example of this unsuitability is that the difficulty of an enormous amount of computation work appears currently in front of us, if the Navier-Stokes equations are solved numerically in the case of turbulence.

6. Discretization of the equations

In an incompressible fluid, there are three sets of closed equations that may be chosen to describe a turbulent flow in the NATT.[1]

Choice one

$$\frac{\partial \tilde{U}_i}{\partial x_i} = 0, \quad \frac{\partial \tilde{U}_i}{\partial t} + \frac{\partial (\tilde{U}_i \tilde{U}_j)}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \tilde{P}}{\partial x_i} + \nu \nabla^2 \tilde{U}_i + o(\varepsilon^2); \tag{7}$$

Choice two

$$\frac{\partial U_i}{\partial x_i} = 0, \quad \frac{\partial U_i}{\partial t} + \frac{\partial (U_i U_j)}{\partial x_j} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu \nabla^2 U_i, \tag{8}$$

$$\frac{\partial u_i}{\partial x_i} = 0, \quad \frac{\partial u_i}{\partial t} + U_j \frac{\partial u_i}{\partial x_j} + u_j \frac{\partial U_i}{\partial x_j} - 2\frac{\partial (u_i u_j)}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \nabla^2 u_i + o(\varepsilon^3); \tag{9}$$

Choice three

$$\frac{\partial \tilde{U}_i}{\partial x_i} = 0, \quad \frac{\partial \tilde{U}_i}{\partial t} + \frac{\partial (\tilde{U}_i \tilde{U}_j)}{\partial x_j} + \frac{\partial (u_i u_j)}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \tilde{P}}{\partial x_i} + \nu \nabla^2 \tilde{U}_i + o(\varepsilon^3), \tag{10}$$

$$\frac{\partial u_i}{\partial x_i} = 0, \quad \frac{\partial u_i}{\partial t} + \tilde{U}_j \frac{\partial u_i}{\partial x_j} + u_j \frac{\partial \tilde{U}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \nabla^2 u_i + o(\varepsilon^3), \tag{11}$$

where ε is the linear scale of a monad and is infinitesimal (nonstandard number); U_i and P are the instantaneous velocity and pressure; u_i and p are the fluctuant velocity and pressure, which are the functions of$(x_1, x_2, x_3, t, x_1', x_2', x_3', t')$; $\tilde{U}_i$ and $\tilde{P}$ are the point-averages (monad-averages) of the instantaneous velocity and pressure; the independent variables of the $\tilde{U}_i$ and $\tilde{P}$ are (x_1, x_2, x_3, t). Note that here the fluctuant quantities are different from the common fluctuations in a measurement on turbulence. The definitions of the fluctuations u_i and p are $u_i = U_i - \tilde{U}_i$ and $p = P - \tilde{P}$. Obviously, u_i and p are defined in a monad. Therefore, u_i and p would be called the fluctuations in a monad, or the internal fluctuations.

The derivative (2) has been applied to $\dfrac{\partial}{\partial t}$ and $\dfrac{\partial}{\partial x}$ in these equations. So the equations can be applied to both the uniform and non-uniform monads. What is the physical meaning of the discretization of Eqs.(7)–(11)? Based on the fact that the infinitesimal ε in the definition (2) is the linear scale of a monad, the discretization of Eqs.(7)–(11) means the coarsening of the monad. But the discretization of the Navier–Stokes equations means, in the case of turbulence, the coarsening of the uniform internal point, which is essentially identified with the absolute point, of a monad. They have different meanings in the two cases. The monad is the abstracted fluid-particle. Normally, a monad can be coarsened as a fluid-particle. But the scale of grid of discretization of Eqs.(7)–(11) is, in general, not equal to that of fluid-particle in reality. If the physical relation between neighbouring grids is very close to the real physical relation between the neighbouring fluid-particles, then the results of computation will be approximately correct, and the discretization is reasonable. Conversely, if the physical relation between the neighbouring grids is not close to that between neighbouring fluid-particles, the correct results cannot be obtained and the discretization is unreasonable.

It is known from the above analysis that the Navier–Stokes equations are based on the limit $\Delta x \to 0$, and $\Delta t \to 0$, and can be applied only to the uniform point (i.e. the abstracted uniform fluid-particle). Yet Eqs.(7)–(11) can be applied to the non-uniform point (i.e. the abstracted non-uniform fluid-particle). In turbulence, the fluid-particle, which is abstracted into the monad, is not uniform, but the fluid-particle at a lower level, which is abstracted into the nonstandard point (i.e. the internal point of the monad), is uniform. Therefore, in turbulence, the Navier–Stokes equations can be applied only to the internal points of a monad rather than the monad itself. But Eqs.(7)–(11) can be used on the monad itself. When the computation of these equations is carried out, the discretization of the Navier–Stokes equations means the coarsening of the internal point of a monad physically corresponding to the fluid-particle at a lower level, and the discretization of Eqs.(7)–(11) means the coarsening of the monad itself physically corresponding to the fluid-particle. Therefore, in the turbulence computation, the discretized grid corresponds to the fluid-particle at a lower level in the case of the computation of the Navier–Stokes equations, but corresponds to the fluid-particle itself in the case of the computation of Eqs.(7)–(11). In other words, only the very fine grids, which lead to an enormous amount of computational work, can meet the needs for obtaining the reasonable solutions of the Navier–Stokes equations, but the coarsening grids can meet the needs for the reasonable solutions of Eqs.(7)–(11) in the case of the computation on turbulence. There is no need to have fine grids, so there is no difficulty in the computation on turbulence based on Eqs.(7)–(11) in the sense of the amount of computation work.

First, the mathematical concepts are drawn from the physical reality. And by virtue of the mathematical concepts a series of the mathematical deductions and calculations is performed in order to obtain the various results about the physical properties, for example, their governing equations. Then, after the discretization of the governing equations the computation of the discretized equations can be carried out and the various physical properties of the real problem are obtained. There are two procedures: from the

physical reality to the mathematical abstract and from the mathematical abstract to the physical reality, and the latter is an inverse procedure of the former.

7. Conclusions

In the above sections, the discussion on some important mathematical concepts and their corresponding physical meanings is carried out. The corresponding physical meanings herein are the physical objectives from which the mathematical concepts are drawn. The correspondences between them are summarized in Table 1.

Finally, what has been stated in this paper is that the legitimacy of the mathematical concepts, especially in physics, is based just on their corresponding physical objectives. Every reasonable mathematical concept, even mathematical deduction method, must rely on its physical foundation, i.e. it is abstracted physical objective (physical reality). In physics, there does not exist a mathematical concept separated from the physical objective. One sometimes ignores this fact, and is restricted to and only stays in the stage of the pure mathematics. As they are accustomed, for example, to the absolute point in mathematical abstraction, some researchers usually do not pay attention to the effect of no absolute point existing in physical reality on the description of physical problem such as the turbulence. Therefore some physical problems become very difficult to comprehend. An understanding of the physical meanings of these mathematical concepts, i.e. what are the real physical objectives from which these mathematical concepts are drawn, may easily enlighten us on them.

Table 1. The correspondences between the mathematical concepts and the corresponding physical objects.

MATHEMATICAL CONCEPTS	CORRESPONDING PHYSICAL OBJECTS
the absolute point	the uniform fluid-particle
the uniform point (uniform monad)	the uniform fluid-particle
the nonuniform point (nonuniform monad)	the nonuniform fluid-particle
the nonstandard point (internal point of a monad)	the fluid-particle at a lower level
$\Delta x \to 0$	Δx approaches a linear scale of uniform fluid-particle
$\dfrac{\partial f}{\partial x} = \lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$	the difference between values of f at two neighbouring uniform fluid-particles, divided by the linear scale of the fluid-particle
the infinitesimal ε	the linear scale of the fluid-particle
$\dfrac{\partial f}{\partial x} = \dfrac{f(x+\varepsilon, x') - f(x, x')}{\varepsilon}$	the difference between values of f at the corresponding two lower-level fluid-particles of the neighbouring fluid-particles, divided by the linear scale of the fluid-particle
the Navier–Stokes equations	the conservation equations (i.e. the mass, momentum and energy conservation equations) only for the uniform fluid-particle
the fundamental equations in the NATT (Eqs.(7)–(11) in the closed form)	the conservation equations for both the uniform and nonuniform fluid-particles
the standard analysis	the flow fluid composed of only uniform fluid-particles, only one level
the nonstandard analysis	the flow fluid composed of the fluid-particles, multi-level analysis
the discretization of the Navier–Stokes equations in the computation on turbulence	the coarsening of the nonstandard point into the grid corresponding to the fluid-particle at a lower level
the discretization of the equations (7)–(11) in the computation on turbulence	the coarsening of the monad into the grid corresponding to the fluid-particle itself
the continuity of fluid	the physical description of fluid-continuity (in the last paragraph of the section 1 in this article)

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