Vol 16 No 5, May 2007
1009-1963/2007/16(05)/1182-04

Chinese Physics

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The construction of conserved quantities for linearly coupled oscillators and study of symmetries about the conserved quantities*

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(Received 23 September 2006; revised manuscript received 23 October 2006)

In this paper, the conserved quantities are constructed using two methods. The first method is by making an ansatz of the conserved quantity and then using the definition of Poisson bracket to obtain the coefficients in the ansatz. The main procedure for the second method is given as follows. Firstly, the coupled terms in Lagrangian are eliminated by changing the coordinate scales and rotating the coordinate axes, secondly, the conserved quantities are obtain in new coordinate directly, and at last, the conserved quantities are expressed in the original coordinates by using the inverse transform of the coordinates. The Noether symmetry and Lie symmetry of the infinitesimal transformations about the conserved quantities are also studied in this paper.

Keywords: linear coupled oscillator, conserved quantity, Noether symmetry, Lie symmetry
PACC: 0320

1. Introduction

Researches on the conserved quantity (or the first integral) of dynamical systems have already become an important and active field of modern mathematics, mechanics and physics, in which are employed traditional methods such as the Lie symmetry, the Noether symmetry and the form invariance.[1−14] It should be pointed out that these traditional methods to construct the conserved quantity are realized by introducing infinitesimal transformations and then solving the differential equations to obtain the generators of infinitesimal transformations. However, the Lagrangian is usually coupled and the corresponding differential equations are too difficult to solve so that more conserved quantities cannot be found.

The linearly coupled oscillators is one of the fundamental systems in mechanics and molecular physics. In this paper, we construct the conserved quantities by using two methods. The first method is making an ansatz of the conserved quantity and then using the definition of Poisson bracket to obtain the coefficients in the ansatz. The main procedure for the second method is as follows. Firstly, the coupled terms in Lagrangian are eliminated by changing the coordinate scales and rotating the coordinate axes, secondly, the conserved quantities are obtain in new coordinates directly, and at last, the conserved quantities are expressed in the original coordinates by using the inverse transform of the coordinates. Furthermore, according to inverse Noether theorem for Lagrange system, we can also obtain the generators of infinitesimal transformations and discuss the Noether symmetry and Lie symmetry of the infinitesimal transformations about conserved quantities.

2. Two methods of construction of conserved quantity

The main idea of the first method we use to construct the conserved quantity is making an ansatz of the conserved quantity and then using the definition of Poisson bracket to obtain the coefficients in the ansatz.[15]

The Hamiltonian of linearly coupled oscillators is

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + kx_1^2 + kx_2^2 - kx_1x_2, \tag{1}$$

whose Hamilton's equations of motion can be written as

$$\dot{x}_1 = \frac{p_1}{m},\ \dot{p}_1 = -2kx_1 + kx_2, \tag{2a}$$

$$\dot{x}_2 = \frac{p_2}{m},\ \dot{p}_2 = -2kx_2 + kx_1. \tag{2b}$$

*Project supported by the Science Research Program from Education Bureau of Zhejiang Province, China (Grant No 20061155).
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We can make an ansatz of the conserved quantity in the form[15]

$$I_1 = a_0 + \sum_{i=1}^{2} a_i \dot{x}_i + \sum_{i=1,j=1}^{2} a_{ij} \dot{x}_i \dot{x}_j, \tag{3}$$

where the coefficients a_0, a_i and a_{ij} are functions of the variables x_1 and x_2, and satisfy the condition given by $a_{ij} = a_{ji}$. According to the definition of Poisson bracket, the conserved quantity I_1 can be determined by

$$\frac{\mathrm{d}I_1}{\mathrm{d}t} = [I_1, H] = 0, \tag{4}$$

namely

$$\begin{aligned}&\Big(\frac{\partial a_0}{\partial x_1} + \frac{\partial a_1}{\partial x_1}\dot{x}_1 + \frac{\partial a_2}{\partial x_1}\dot{x}_2 + \frac{\partial a_{11}}{\partial x_1}\dot{x}_1^2 + 2\frac{\partial a_{12}}{\partial x_1}\dot{x}_1\dot{x}_2 + \frac{\partial a_{22}}{\partial x_1}\dot{x}_2^2\Big)\dot{x}_1 - \Big(\frac{a_1}{m} + 2\frac{a_{11}}{m}\dot{x}_1 + 2\frac{a_{12}}{m}\dot{x}_2\Big)(2kx_1 - kx_2)\\ &+\Big(\frac{\partial a_0}{\partial x_2} + \frac{\partial a_1}{\partial x_2}\dot{x}_1 + \frac{\partial a_2}{\partial x_2}\dot{x}_2 + \frac{\partial a_{11}}{\partial x_2}\dot{x}_1^2 + 2\frac{\partial a_{12}}{\partial x_2}\dot{x}_1\dot{x}_2 + \frac{\partial a_{22}}{\partial x_2}\dot{x}_2^2\Big)\dot{x}_2 - \Big(\frac{a_2}{m} + 2\frac{a_{12}}{m}\dot{x}_1 + 2\frac{a_{22}}{m}\dot{x}_2\Big)(2kx_2 - kx_1)\\ &= 0.\end{aligned} \tag{5}$$

Equation (5), being in a polynomial form, implies that the coefficients of all terms containing the form $\dot{x}_1^m \dot{x}_2^n (m \neq n)$ must be identically zero. Therefore, a set of partial differential equations about a_0, a_i and a_{ij} can be obtained by

$$-\frac{k}{m}a_1(2x_1 - x_2) - \frac{k}{m}a_2(2x_2 - x_1) = 0, \tag{6a}$$

$$\frac{\partial a_0}{\partial x_1} - 2\frac{k}{m}a_{11}(2x_1 - x_2) - 2\frac{k}{m}a_{12}(2x_2 - x_1) = 0, \tag{6b}$$

$$\frac{\partial a_0}{\partial x_2} - 2\frac{k}{m}a_{12}(2x_1 - x_2) - 2\frac{k}{m}a_{22}(2x_2 - x_1) = 0, \tag{6c}$$

$$\frac{\partial a_1}{\partial x_1} = 0, \tag{6d}$$

$$\frac{\partial a_2}{\partial x_2} = 0, \tag{6e}$$

$$\frac{\partial a_{11}}{\partial x_1} = 0, \tag{6f}$$

$$\frac{\partial a_{22}}{\partial x_2} = 0, \tag{6g}$$

$$2\frac{\partial a_{12}}{\partial x_1} + \frac{\partial a_{11}}{\partial x_2} = 0, \tag{6h}$$

$$2\frac{\partial a_{12}}{\partial x_2} + \frac{\partial a_{22}}{\partial x_1} = 0. \tag{6i}$$

From Eqs.(6a), (6d) and (6e), we can obtain

$$a_1 = a_2 = 0. \tag{7}$$

In terms of Eqs.(6f)–(6i), the solution of a_{ij} can be obtained immediately by

$$\begin{aligned}&a_{11} = c_1 x_2^2 + c_2 x_2 + c_3,\ a_{22} = c_1 x_1^2 + c_4 x_1 + c_5,\\ &a_{12} = -\left(c_1 x_1 x_2 + \frac{1}{2}c_2 x_1 + \frac{1}{2}c_4 x_2 - c_6\right).\end{aligned} \tag{8}$$

Substituting Eq.(8) into Eqs.(6b) and (6c) and comparing the coefficients of all terms containing the form $\dot{x}_1^m \dot{x}_2^n$, we can obtain

$$c_1 = c_2 = c_4 = 0, \quad c_5 = c_3, \quad c_6 = 2c_3 \tag{9}$$

and

$$a_0 = 6c_3 \frac{k}{m} x_1 x_2. \tag{10}$$

Inserting Eq.(9) into Eq.(8), we get

$$a_{11} = c_3,\ a_{22} = c_3,\ a_{12} = 2c_3.$$

Finally, the conserved quantity of the system is

$$I_1 = c_3\left(\dot{x}_1^2 + \dot{x}_2^2 + 4\dot{x}_1\dot{x}_2 + 6\frac{k}{m}x_1 x_2\right), \tag{11}$$

where c_3 is an arbitrary constant. The conserved quantity I_1 is the coupling energy of system.

The main procedure for the second method is as follows. Firstly, the coupled terms in Lagrangian are eliminated by changing the coordinate scales and rotating the coordinate axes, secondly, the conserved quantities are obtain in the new coordinates directly, and at last, the conserved quantities are expressed in the original coordinates by using the inverse transform of the coordinates.

The Lagrangian of this system is

$$L = \frac{m}{2}(\dot{x}_1^2 + \dot{x}_2^2) - kx_1^2 - kx_2^2 + kx_1 x_2. \tag{12}$$

By rescaling the coordinates such that

$$y_i = \sqrt{m}x_i \quad (i = 1, 2), \tag{13}$$

the Lagrangian (12) turns into

$$L_1 = \frac{1}{2}(\dot{y}_1^2 + \dot{y}_2^2) - \frac{k}{m}(y_1^2 + y_2^2 - y_1 y_2). \tag{14}$$

By rotating the coordinates defined as

$$q_1 = \frac{\sqrt{2}}{2}y_1 + \frac{\sqrt{2}}{2}y_3,\ q_2 = \frac{\sqrt{2}}{2}y_1 - \frac{\sqrt{2}}{2}y_2, \quad (15)$$

the uncoupled Lagrangian in new coordinates is

$$L_2 = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) - \frac{k}{2m}(q_1^2 + 3q_2^2). \quad (16)$$

According to Eq.(16), two conserved quantities of the system in new coordinates can be obtained immediately as

$$I_2 = \frac{1}{2}\dot{q}_1^2 + \frac{k}{2m}q_1^2,\ I_3 = \frac{1}{2}\dot{q}_2^2 + \frac{3k}{2m}q_2^2. \quad (17)$$

By using inverse transform of coordinates, these two conserved quantities are expressed in the original coordinates as

$$\begin{aligned} I_2 &= \frac{m}{4}(\dot{x}_1 + \dot{x}_2)^2 + \frac{k}{4}(x_1 + x_2)^2, \\ I_3 &= \frac{m}{4}(\dot{x}_1 - \dot{x}_2)^2 + \frac{3k}{4}(x_1 - x_2)^2. \end{aligned} \quad (18)$$

We find that these two conserved quantities are not independent; the relationship between I_2 and I_3 is

$$I_2 + I_3 = I = H, \quad (19)$$

where H is the Hamiltonian of the system.

3. Noether symmetry and Lie symmetry of the three conserved quantities

Let us consider infinitesimal transformations,

$$\begin{aligned} t^* &= t + \varepsilon\tau_\alpha(t, x_s, \dot{x}_s), \\ x_s^* &= x_s + \varepsilon\xi_s^\alpha(t, x_s, \dot{x}_s) \\ &(s = 1, 2,\ \alpha = 1, 2, 3), \end{aligned} \quad (20)$$

where ε is the infinitesimal parameter, and τ_α and ξ_s^α are the generators of infinitesimal transformations. Under infinitesimal transformations (20), we introduce the generator vector for infinitesimal transformations

$$X^{(0)} = \tau_\alpha\frac{\partial}{\partial t} + \sum_{s=1}^{2}\xi_s^\alpha\frac{\partial}{\partial x_s}, \quad (21)$$

whose first and second extended vector are respectively given by

$$X^{(0)} = X^{(0)} + \sum_{x=1}^{2}(\dot{\xi}_s^\alpha - \dot{x}_s\dot{\tau}_\alpha)\frac{\partial}{\partial \dot{x}_s}, \quad (22)$$

$$X^{(2)} = X^{(1)} + \sum_{s=1}^{n}[(\ddot{\xi}_s^\alpha - \dot{x}_s\ddot{\tau}_\alpha - 2\ddot{x}_s\dot{\tau}_\alpha)\frac{\partial}{\partial \ddot{x}_s}. \quad (23)$$

By means of inverse Noether theorem for Lagrange system, we can obtain the generators of infinitesimal transformations:[1]

$$\xi_s^\alpha = \tau^\alpha\dot{x}_s + \frac{1}{m}\frac{\partial I_\alpha}{\partial \dot{x}_s}(s = 1, 2,\ \alpha = 1, 2, 3), \quad (24)$$

$$\tau_\alpha = \frac{1}{L}\left[I_\alpha - \frac{\partial L}{\partial \dot{x}_s}(\xi_s^\alpha - \dot{x}_s\tau_\alpha) - G_\alpha\right](\alpha = 1, 2, 3), \quad (25)$$

where G_α is a gauge function satisfying the following Noether equalities:

$$\begin{aligned} &\frac{\partial L}{\partial t}\tau_\alpha + \frac{\partial L}{\partial x_s}\xi_s^\alpha + \frac{\partial L}{\partial \dot{x}_s}\dot{\xi}_s^\alpha + \left(L - \frac{\partial L}{\partial \dot{x}_s}\dot{x}_s\right)\dot{\tau}_\alpha \\ = \ & -\dot{G}_\alpha, \\ & (s = 1, 2, \quad \alpha = 1, 2, 3). \end{aligned} \quad (26)$$

Substituting Eqs.(11) and (18) into Eqs.(24) and (25) respectively and noticing Eq.(26), we have the following solutions:

$$\begin{aligned} &\tau_1 = 1,\ \xi_1^1 = \dot{x}_1 + \frac{c_3}{m}(2\dot{x}_1 + 4\dot{x}_2), \\ &\xi_2^1 = \dot{x}_2 + \frac{c_3}{m}(2\dot{x}_2 + 4\dot{x}_1), \\ &G_1 = -I_1 - L + c_3\frac{12k}{m}x_1x_2; \end{aligned} \quad (27)$$

$$\begin{aligned} &\tau_2 = -\frac{1}{2},\ \ \xi_1^2 = \frac{\dot{x}_2}{2},\ \xi_2^2 = \frac{\dot{x}_1}{2}, \\ &G_2 = -\frac{1}{2}m\dot{x}_1\dot{x}_2 - \frac{k}{4}x_1^2 - \frac{k}{4}x_2^2 + kx_1x_2; \end{aligned} \quad (28)$$

$$\begin{aligned} &\tau_3 = -\frac{1}{2},\ \xi_1^3 = -\frac{\dot{x}_2}{2},\ \xi_2^3 = -\frac{\dot{x}_1}{2}, \\ &G_3 = \frac{1}{2}m\dot{x}_1\dot{x}_2 + \frac{k}{4}x_1^2 + \frac{k}{4}x_2^2 - kx_1x_2. \end{aligned} \quad (29)$$

So we can see that generators (27)–(29) of the three conserved quantities have Noether symmetry.

Furthermore, we find that Eqs.(27)–(29) also satisfy the following Lie symmetrical determining equations:

$$\begin{aligned} &\ddot{\xi}_s^\alpha - \dot{x}_s\ddot{\tau}_\alpha - 2\dot{\tau}_\alpha a_s = X^{(1)}(a_s) \\ &(s = 1, 2,\ \alpha = 1, 2, 3), \end{aligned} \quad (30)$$

where α_s are generalized accelerations and can be written as

$$\begin{aligned} a_1 &= \frac{1}{m}(-2kx_1 + kx_2), \\ a_2 &= \frac{1}{m}(-2kx_2 + kx_1). \end{aligned} \quad (31)$$

It is indicated that generators (27)–(29) of the three conserved quantities have also Lie symmetry.

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