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Discrete integrable couplings associated with modified Korteweg–de Vries lattice and two hierarchies of discrete soliton equations

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A direct way to construct integrable couplings for discrete systems is presented by use of two semi-direct sum Lie algebras. As their applications, the discrete integrable couplings associated with modified Korteweg–de Vries (m-KdV) lattice and two hierarchies of discrete soliton equations are developed. It is also indicated that the study of integrable couplings using semi-direct sums of Lie algebras is an important step towards the complete classification of integrable couplings.

Keywords: discrete integrable system, m-KdV lattice equation, semi-direct sums of Lie algebras
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1. Introduction

In the past two decades, the continuous type of integrable system of the soliton theory has been extensively developed,[1−3] and has become a most important topic because of its applications in physics, chemistry and biology, etc. So it is necessary to pay more attention to the generation of the discrete integrable system and its relevant properties.

The integrable coupling is quite a new and significant subject in the theory of soliton, which originates from the investigation of centreless Virasoro symmetry algebras of soliton equations. For a given integrable lattice hierarchy $u_t = K(u)$, we can construct a new bigger triangular integrable system as follows:

$$\begin{cases} u_t = K(u), \\ v_t = S(u, v). \end{cases} \tag{1}$$

This is usually called the integrable couplings of $u_t = K(u)$ if the system (1) is still an integrable system in the Lax sense. Recently, a systematic approach which uses the trace identity as a powerful tool has been introduced for generating continuous integrable systems.[4−14] The result, however, is not so good for integrable lattice models. There are few references devoted to the integrable coupling of lattice equations. This is mainly due to the difficulty in doing it. In this paper, we provide two semi-direct sum Lie algebras to generate discrete integrable couplings. As their applications, the discrete integrable couplings associated with modified Korteweg–de Vries (m-KdV) lattice and two hierarchies of discrete soliton equations are developed.

Usually a discrete isospectral problem is given by

$$E\varphi = U(u, \lambda)\varphi, \tag{2}$$

and the auxiliary problem

$$\varphi_t = V\varphi = V(u, \lambda)\varphi, \tag{3}$$

where shift operator E is determined by

$$\begin{aligned} &Ef(n) = f(n+1), \quad (E-1)f(n) = f_{n+1} - f_n, \\ &E^{-1}f(n) = f(n-1), \quad E^k f = f^{(k)}, \\ &(E-1)^{-1}f(n) = -\sum_{k=0}^{\infty} f_{n+k}, \\ &Df(n) = f(n+1) - f(n), \end{aligned}$$

and $U(u, \lambda)$ is a matrix with $u = (u_1, u_2, \cdots, u_p)^{\mathrm{T}}$ as the potential, λ is the spectral parameter with $\lambda_t = 0$.

After introducing the matrix Γ, we have a hierarchy nonlinear differential–difference equation

$$U_{tm} = (E\Gamma)U - U\Gamma, \tag{4}$$

which can be written as

$$u_{tm} = K(u, Eu, E^{-1}u, \cdots). \tag{5}$$

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The Gateaux derivative and the variational derivative are given by

$$J'(u)[f] = \frac{\partial}{\partial \varepsilon} J(u + \varepsilon f\ )|_{\varepsilon=0}\,,$$

$$\frac{\delta \tilde{H}}{\delta u} = \sum_{m\in Z} E^{-m}\Big(\frac{\partial H}{\partial u^{(m)}}\Big)$$

with $\tilde{H} = \sum_{n\in Z} H(u(n))$.

To construct an integrable coupling of the lattices Eq.(5), we use semi-direct sums of Lie algebras to expand the original Lie algebra G_1. Again, we take another matrix Lie algebra G_2 closed under matrix multiplication and then perform a semi-direct sum G of G_1 and G_2

$$G = G_1 \oplus G_2. \tag{6}$$

We also require the closure between G_1 and G_2 under matrix multiplication

$$G_1G_2, G_2G_1 \subseteq G_2, \tag{7}$$

where $G_1G_2 = \{U_1U_2 | U_1 \in G_1, U_2 \in G_2\}$, to ensure a Lax pair for forming the semi-direct sum G_1 generating a coupling system.

In this paper,we take the special Lie algebra $\tilde{G}_1 = \text{span}\{\tilde{h}_1, \tilde{h}_2, \tilde{e}, \tilde{f}\}$, where

$$\tilde{h}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \tilde{h}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tilde{e} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \tilde{f} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

along with the commutative operations

$$[\tilde{h}_1, \tilde{h}_2] = 0, \quad [\tilde{h}_1, \tilde{e}] = \tilde{e}, \quad [\tilde{h}_1, \tilde{f}] = -\tilde{f}, \quad [\tilde{h}_2, \tilde{e}] = -\tilde{e}, \quad [\tilde{h}_2, \tilde{f}] = \tilde{h}_1, \quad [\tilde{e}, \tilde{f}] = \tilde{h}_1 - \tilde{h}_2,$$

and expand G_1 as another Lie algebra

$$\tilde{G} = \text{span}\{h_1, h_2, e, f, e_+, e_-, f_+\},$$

where

$$h_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad h_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$e_+ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad f_+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

then we find that $\tilde{G}_1$ is isomorphic to $G_1 = \text{span}\{h_1, h_2, e, f\}$. Let $G_2 = \text{span}\{e_+, e_-, f_+\}$, then we will have

$$G = G_1 \oplus G_2, [G_1, G_2] \subseteq G_2,$$

where G_2 is an Abelian ideal of the Lie algebra G. Furthermore, we also see that G is closed under the following matrix multiplication: $h_1h_1 = h_1$, $h_1h_2 = h_2h_1 = 0$, $h_2h_2 = h_2$, $ee = ff = 0$, $f_+f_+ = f_+$, $h_1e = e$, $eh_1 = 0$, $h_2e = 0$, $eh_2 = e$, $h_1f = 0$, $fh_1 = f$, $h_2f = f$, $fh_2 = 0$, $ef = h_1$, $fe = h_2$, $h_1e_+ = e_+$, $e_+h_1 = 0$, $h_2e_- = e_-$, $e_-h_2 = 0$, $ee_- = e_+$, $e_-e = 0$, $fe_+ = e_-$, $e_+f = 0$, $e_+f_+ = e_+$, $f_+e_+ = 0$, $e_-f_+ = e_-$, $f_+e_- = 0$, $h_1e_- = e_-h_1 = h_1f_+ = f_+h_1 = h_2e_+ = e_+h_2 = 0$, $h_2f_+ = f_+h_2 = ee_+ = e_+e = ef_+ = f_+e = fe_- = e_-f = f_+f = ff_+ = e_+e_- = e_-e_+ = 0$.

It is easy to find that the spectral matrices U and V given in Eqs.(2) and (3) can be presented in terms of h_1, h_2, e, f, e_+, e_- except f_+ as follows:

$$M = \begin{pmatrix} U & \tilde{U}_1 \\ 0 & 0 \end{pmatrix},$$

where U is a 2×2 matrix, and $\tilde{U} = (e, f)^{\mathrm{T}}$.

2. Discrete integrable couplings

The (1+1)-dimensional and (2+1)-dimensional m-KdV lattice equations refer to respectively

$$r_{n_t} = r_n^2(r_{n+1} - r_{n-1}), \tag{8}$$

$$r_{n_t} = [r_n^2(r_{n+1} + r_{n-1})]_y. \tag{9}$$

We now expand Eqs.(8) and (9) into a large discrete integrable system which is the integrable coupling. Consider the following isospectral problem by resorting to the Lie algebra $\tilde{G}$:

$$\begin{aligned} U(u,\lambda) &= \begin{pmatrix} 0 & r & p \\ \lambda s & \lambda & q \\ 0 & 0 & 0 \end{pmatrix} \\ &= \lambda h_2 + re + \lambda f + pe_+ + qe_-, \end{aligned} \tag{10}$$

where $s = s(n,t)$, $q = q(n,t)$, $r = r(n,t)$, and $p = p(n,t)$ is potential depending on $n \in Z$ and $t \in R$.

The isospectral problem is

$$E\varphi = U(u,\lambda)\varphi, \tag{11}$$

in which $\varphi = (\varphi^1(n,t),\ \varphi^2(n,t),\ \varphi^3(n,t))^{\mathrm{T}} \equiv (\varphi_n^1, \varphi_n^2, \varphi_n^3)^{\mathrm{T}}$ is an eigenfunction.

To derive the discrete integrable system, we first solve the equation

$$(E\Gamma)U(u,\lambda) - U(u,\lambda)\Gamma = 0, \tag{12}$$

where

$$\Gamma = \Sigma_{n\geq 0} \begin{pmatrix} a_m & b_m & d_m \\ \lambda c_m & -a_m & g_m \\ 0 & 0 & 0 \end{pmatrix} \lambda^{-m}$$

$$= a(h_1 - h_2) + be + \lambda cf + de_+ + ge_-.$$

Equation(12) can be written as

$$\begin{aligned} &sb^{(1)} = rc, \quad a^{(1)} - a = rc^{(1)} - sb, \\ &r(a^{(1)} + a) + \lambda b^{(1)} = 0, \quad s(a^{(1)} + a) + c\lambda = 0, \\ &pa^{(1)} + qb^{(1)} = rg, \quad \lambda pc^{(1)} - qa^{(1)} = \lambda sd + \lambda g, \end{aligned} \tag{13}$$

which lead to $sb^{(1)} = rc$. Take $a_0 = -1/2$, $b_0 = c_0 = d_0 = g_0 = 0$, and let $a_j|_{[u]=0} = 0$, $b_j|_{[u]=0} = 0$, $c_j|_{[u]=0} = 0$, $d_j|_{[u]=0} = 0$, and $g_j|_{[u]=0} = 0 \quad (j \geq 1)$, where $[u] = (u, u^{(1)}, u^{(2)}, \cdots)$, the first coefficients will be given as follows: $a_1 = r^{(-1)}s$, $b_1 = r^{(-1)}$, $c_1 = s$, $b_2 = -r^{(-1)}(r^{(-2)}s^{(-1)} + r^{(-1)}s)$, $c_2 = -s(r^{(-1)}s + r^{(1)}s)$, and $a_2 = -r^{(-2)}r^{(-1)}s^{(-1)} - r^{(-1)}rss^{(1)} - (r^{(-1)})^2s^2, \cdots$. In this way, the recursion relation (13) determines uniquely a_m, b_m, c_m, d_m, g_m, $m \geq 0$. We also obtain that a_m $(m \geq 1)$ are all local, i.e. $a_m (m \geq 1)$ are all polynomials of $q, s, p, r, q^{(1)}$, $s^{(1)}, \cdots$.

Let $\Gamma_+^{(n)} = \Sigma_{i=0}^n (a_i(h_1 - h_2) + b_e + \lambda c_i f + d_i e_+ + g_i e_-)\lambda^{n-i} = \lambda^n \Gamma - \Gamma_-^{(n)}$, then Eq.(12) will be written as

$$(E\Gamma_+^{(n)})U - U\Gamma_+^{(n)} = -(E\Gamma_-^{(n)})U - U_n\Gamma_-^{(n)}.$$

A direct calculation leads to

$$\begin{aligned} (E\Gamma_+^{(n)})U - U\Gamma_+^{(n)} &= b_{n+1}^{(1)}e - c_{n+1}f + (pc_{n+1}^{(1)} - sd_{n+1} - g_{n+1})e_- \\ &= \begin{pmatrix} 0 & b_{n+1}^{(1)} & 0 \\ -c_{n+1} & 0 & pc_{n+1}^{(1)} - sd_{n+1} - g_{n+1} \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Take

$$\Delta_1 = \begin{pmatrix} v_1 & v_2 & v_3 \\ v_4 & v_5 & v_6 \\ 0 & 0 & 0 \end{pmatrix}, \quad (E\Delta_1)U_n - U_n\Delta_1 = \begin{pmatrix} \Delta_{11} & \Delta_{12} & \Delta_{13} \\ \Delta_{21} & \Delta_{22} & \Delta_{23} \\ 0 & 0 & 0 \end{pmatrix}.$$

then we will have

$$\begin{aligned} &\Delta_{11} = \lambda sv_2^{(1)} - rv_4, \quad \Delta_{12} = rv_1^{(1)} + \lambda v_2^{(1)} - rv_5, \quad \Delta_{13} = pv_1^{(1)} + qv_2^{(1)} - rv_6, \\ &\Delta_{21} = \lambda sv_5^{(1)} - \lambda v_4 - \lambda sv_1, \quad \Delta_{22} = rv_4^{(1)} + \lambda v_5^{(1)} - \lambda v_5 - \lambda sv_2, \quad \text{and} \quad \Delta_{23} = pv_4^{(1)} + qv_5^{(1)} - \lambda v_6 - \lambda sv_3. \end{aligned}$$

Solving the following equations for v_i:

$$\lambda sv_2^{(1)} - rv_4 = 0, \ \ v_2^{(1)} = 0, \ \ rv_4^{(1)} = 0, \quad Dv_5 - sv_2 = 0, \ \ sv_3 + v_6 = 0, \tag{14}$$

gives $v_2 = v_5 = v_4 = 0$, $v_3 = qg_{n+1}$, and $v_6 = -sqg_{n+1}$. Choose $v_1 = -2a_n$ and let $\Gamma^{(n)} = \Gamma_+^{(n)} + \Delta_1$, then we will have

$$(E(\lambda^n\Gamma)_+)U - U(\lambda^n\Gamma)_+ = \begin{pmatrix} 0 & b_{n+1}^{(1)} + 2ra_n & 2pa_{n+1} + qsrg_{n+1} \\ -c_{n+1} - 2sa_n & 0 & pc_{n+1}^{(1)} - sd_{n+1} - g_{n+1} \end{pmatrix}.$$

Therefore, the discrete zero curvature equation

$$U_{nt} = (E\Gamma^{(n)})U - U\Gamma^{(n)} \tag{15}$$

determines the following Lax integrable discrete nonlinear systems:

$$\begin{cases} r_{tn} = -b_{n+1}^{(1)} - 2ra_n^{(1)} = r(a_n - a_n^{(1)}), \\ s_{tn} = c_{n+1} + 2sa_n = s(a_n - a_n^{(1)}), \\ p_{tn} = 2pa_{n+1} + srqg_{n+1}, \\ q_{tn} = pc_{n+1}^{(1)} - sd_{n+1} - g_{n+1} = qa_{n+1}^{(1)}. \end{cases} \tag{16}$$

Taking $p = q = 0$, Eq.(16) reduces to an integrable differential–difference equation

$$\begin{cases} r_{tn} = -b_{n+1}^{(1)} - 2ra_n^{(1)} = r(a_n - a_n^{(1)}), \\ s_{tn} = c_{n+1} + 2sa_n = s(a_n - a_n^{(1)}). \end{cases} \tag{17}$$

Equation (17) can be written as

$$\begin{pmatrix} r \\ s \end{pmatrix}_{nt} = J \begin{pmatrix} \dfrac{a_n^{(1)}}{r} \\ \dfrac{a_n}{s} \end{pmatrix} = JL \begin{pmatrix} \dfrac{a_{n-1}^{(1)}}{r} \\ \dfrac{a_{n-1}}{s} \end{pmatrix}, \tag{18}$$

where Hamiltonian operator

$$J = \begin{pmatrix} 0 & r(1-E)s \\ s(E^{-1}-1)r & 0 \end{pmatrix}$$

and recurrence operator

$$L = \begin{pmatrix} \dfrac{1}{r}(1-E^{-1})^{-1}(sE^{-1}r - rEs)r & \dfrac{1}{r}(1-E^{-1})^{-1}(sE^{-1}r - rEs)s \\ \dfrac{1}{s}(E-1)^{-1}(sE^{-1}r - rEs)r & \dfrac{1}{s}(E-1)^{-1}(sE^{-1}r - rEs)s \end{pmatrix}.$$

When $m = 1$, the first system is

$$\begin{cases} r_{t1} = r(r^{(-1)}s - rs^{(1)}), \\ s_{t1} = s(r^{(-1)}s - rs^{(1)}). \end{cases} \tag{19}$$

Taking $r = s$, $u \to u_n$, the system (19) reduces to m-KdV Lattice equation (8).
When $m = 2$, the second system is

$$\begin{cases} r_{t2} = r[rs^{(1)}(r^{(1)}s^{(2)} + rs^{(1)}) - r^{(-1)}s(r^{(-2)}s^{(-1)} + r^{(-1)}s)], \\ s_{t2} = s[rs^{(1)}(r^{(1)}s^{(2)} + rs^{(1)}) - r^{(-1)}s(r^{(-2)}s^{(-1)} + r^{(-1)}s)]. \end{cases} \tag{20}$$

Taking $t_1 \to y$, $t_2 \to t$, and $u \to u_n$, we obtain the (2+1)-dimensional Lattice equation

$$\begin{cases} r_{nt} = -[r_n(r_{n-1}s_n + r_n s_{n+1})]_y, \\ s_{nt} = -[s_n(r_{n-1}s_n + r_n s_{n+1})]_y, \end{cases} \tag{21}$$

and taking $r_n = s_n$, the above equation reduces to Eq.(9). Therefore, we call Eq.(16) the discrete integrable couplings of the m-KdV Lattice equation according to the definition of integrable coupling.

The choice of the modified term Δ_1 is not unique. For example, we can take

$$\Delta_2 = \begin{pmatrix} v_1 & v_2 & v_6 \\ \lambda v_3 & \lambda + v_5 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad (E\Delta_1)U_n - U_n\Delta_1 = \begin{pmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ 0 & 0 & 0 \end{pmatrix},$$

then we have

$$\delta_{11} = \lambda(sv_2^{(1)} - rv_3), \quad \delta_{12} = \lambda(v_2^{(1)} - r) + rv_1^{(1)} - rv_5, \quad \delta_{13} = pv_1^{(1)} + qv_2^{(1)},$$
$$\delta_{21} = \lambda^2(s - v_3) + \lambda(sv_5^{(1)} - sv_1), \quad \delta_{22} = \lambda(rv_3^{(1)} - sv_2 + v_5^{(1)} - v_5), \quad \text{and} \quad \delta_{23} = pv_5^{(1)} + \lambda(q + pv_3^{(1)} - sv_6).$$

Solving the following equations for v_j:

$$sv_2^{(1)} - rv_3 = 0, \quad v_2^{(1)} - r = 0, \quad s - v_3 = 0, \quad rv_3^{(1)} - sv_2 + Dv_5 = 0, \quad q + pv_3^{(1)} - sv_6 = 0, \tag{22}$$

gives $v_2 = r^{(-1)}$, $v_3 = s$, $v_5 = -sr^{(-1)}$, and $v_6 = (q + pr^{(1)})/s$. Choose $v_1 = -2a_n$ and let $Gamma^{(n)} = Gamma_+^{(n)} + \Delta_1$, then we will have

$$\begin{aligned}&(E(\lambda^n Gamma)_+)U - U(\lambda^n Gamma)_+\\ &= \begin{pmatrix} 0 & b_{n+1}^{(1)} + 2ra_n + rr^{(-1)}s & -2pa_{n+1} + qr^{(-1)} \\ -c_{n+1} - 2sa_n + ss^{(1)}r & 0 & pc_{n+1}^{(1)} - sd_{n+1} - g_{n+1} - psr^{(-1)} \end{pmatrix}.\end{aligned}$$

Therefore, the discrete zero curvature equation

$$U_{nt} = (EGamma^{(n)})U - UGamma^{(n)} \tag{23}$$

determines the following Lax integrable discrete nonlinear system:

$$\begin{cases} r_{tn} = -b_{n+1}^{(1)} - 2ra_n^{(1)} - rr^{(-1)}s = -rDa_n - rr^{(-1)}s, \\ s_{tn} = c_{n+1} + 2sa_n = -sDa_n - ss^{(1)}r, \\ p_{tn} = 2pa_{n+1} - qr^{(-1)}, \\ q_{tn} = -pc_{n+1}^{(1)} + sd_{n+1} + g_{n+1} + psr^{(-1)} = -qa_{n+1}^{(1)} + psr^{(-1)}. \end{cases} \tag{24}$$

The first two equations in Eq.(24) are similar to those in Eq.(16), but a bit different, and also integrable couplings of m-KdV lattice equation.

In fact, in order to obtain the integrable couplings of discrete soliton, we also take the spectral matrices U and V given in Eqs.(2) and (3) as follows: $N = \left(\begin{smallmatrix} U & \tilde{U}_1 \\ 0 & \tilde{U}_2 \end{smallmatrix}\right)$, where $U, \tilde{U}_1$ and $\tilde{U}_2$ are all 2×2 matrices, 0 denotes a 2×2 zero matrix, which will be discussed in other papers.

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