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Analysis of a kind of Duffing oscillator system used to detect weak signals*

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The stability of the periodic solution of the Duffing oscillator system in the periodic phase state is proved by using the Yoshizaw theorem, which establishes a theoretical basis for using this kind of chaotic oscillator system to detect weak signals. The restoring force term of the system affects the weak-signal detection ability of the system directly, the quantitative relationship between the coefficients of the linear and nonlinear items of the restoring force of the Duffing oscillator system and the SNR in the detection of weak signals is obtained through a large number of simulation experiments, then a new restoring force function with better detection results is established.

Keywords: Duffing oscillator system, periodic phase state, restoring force function, SNR, weak signal detection

PACC: 4350, 4610, 0545

1. Introduction

The phase states of the Duffing oscillator system include chaos, bifurcation, period doubling and so on, which have been studied using Poincaré section, the method of Melnikov, and Lyapunov exponent.[1,2] There is a close relationship between the stable periodic phase state and some application fields such as chaos control[3], signal detection[4,5] and cryptographic communications. When the Duffing oscillator in the large-scale periodic phase state is disturbed by noise, the orbit of the periodic phase state is rough,[4,5] the group of the orbits of the periodic phase state has some width,[4,5] which behaves as an approximate period, namely the pseudo-period.[5] The detection of the amplitude,[5] frequency[4] and phase[6] of a weak signal by Duffing oscillator is based on such dynamic state of the system. The existence[7−11] of the periodic solution of the system has been proved when the dynamic system used to detect periodic signals is in the periodic phase state. The periodic phase state of the Duffing oscillator with x^3 is analysed theoretically by the stability of the conceptual solutions of the system in this paper, which establishes a theoretical basis for using the periodic phase state of this kind of chaos oscillator system to detect weak signals. For the Duffing equation with x^3 as the nonlinear term of the restoring force, its detection performance is affected directly by the restoring force.[12] In this paper, the influence of the restoring force of this kind of Duffing oscillator on the detection signal-to-noise ratio (SNR) of the system is discussed, then a new restoring force with better detection performance is established in order to improve the weak-signal detection ability of the system in the background of strong noise.

2. Analysis of the stability of the phase-state solutions of the Duffing equation with x^3

The basis of using the Duffing oscillator with x^3 to detect weak signals is to confirm that the system has a periodic solution.[7] Assume that the Duffing equation with x^3 is

$$\ddot{x} + 0.5\dot{x} - x + x^3 = \gamma\cos(\omega t), \qquad ()$$

where $(-x + x^3)$ is the restoring force.

Equation (1) is equivalent to the set of equations:

$$\dot{x} = y - 0.5x, \quad \dot{y} = x - x^3 + \gamma\cos(\omega t), \qquad (2)$$

$$f(t, x, y) = y - 0.5x,$$

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$$g(t,x,y)=x-x^3+\gamma\cos(\omega t). \tag{3}$$

The stability of the phase-state solution of the system is analysed based on the necessary and sufficient conditions[13] of the Yoshizaw theorem.

The conditions of the theorem about the stability of the solutions consists of five main points, which are proved one by one below.

For the given area Δ_2: $t_0\le t\le+\infty$, $|x|\le A$, $|u|\le A$, $|y|\le B$, $|v|\le B$, where $t_0>0$,

If there exists a function defined at $\bar{\Delta}$: $|x|\le A$, $|u|\le A$, $|y|\le B$, $|v|\le B$:

$$\Phi(x,u,y,v)=(-x+x^3+u-u^3)(x-u)+(y-v)^2-2C(x-u)(y-v),\quad (C\text{ is a constant}). \tag{4}$$

The first point, when $|x-u|+|y-v|>0$, $\Phi(x,u,y,v)>0$.

Proof $\Phi(x,u,y,v)=(x-u)^2(x^2+u^2+xu-1)+(y-v)^2-2C(x-u)(y-v)$.

View the equation above as a unitary quadratic polynomial about the variable $(y-v)$, then

$$\Delta=(x-u)^2[4C^2-(x^2+u^2-xu+1)]<0,$$

namely $4C^2<(x^2+u^2-xu+1)$, whereas $(x^2+u^2-xu+1)<3A^2-1$, so if only $C^2<(3A^2-1)/4$, the first point is satisfied.

The second point, when $|x-u|+|y-v|=0$, $\Phi(x,u,y,v)=0$.

Proof Substitute $|x-u|+|y-v|=0$ into $\Phi(x,u,y,v)$, then $\Phi(x,u,y,v)=0$.

The third point, when the Lipschitz condition is satisfied, namely there exists $L>0$, which makes $|\Phi(x,u,y,v)-\Phi(x_1,u_1,y_1,v_1)|<L(|x-x_1|+|u-u_1|+|y-y_1|+|v-v_1|)$ true.

Proof

$$\begin{aligned}
&|\Phi(x,u,y,v)-\Phi(x_1,u_1,y_1,v_1)|=|(x^2+u^2+xu-1)(x-u)^2+(y-v)^2-2C(x-u)(y-v)\\
&\quad-(x_1^2+u_1^2+x_1u_1-1)(x_1-u_1)^2+(y_1-v_1)^2-2C(x_1-u_1)(y_1-v_1)|\\
&\quad\le|3A^2-1||x+x_1-u-u_1|(|x-x_1|+|u-u_1|)+|y+y_1-v-v_1|(|y-y_1|+|v-v_1|)\\
&\quad+2C(B|x-x_1|+B|u-u_1|+A|y-y_1|+A|v-v_1|).
\end{aligned}$$

Since $|x+x_1-u-u_1|<4A$, $|y+y_1-v-v_1|<4B$, thus set $L_1=\max(4A|3A^2-1|,4B)$, $L\ge L_1+2C\max(A,B)$.

Hence $|\Phi(x,u,y,v)-\Phi(x_1,u_1,y_1,v_1)|<L(|x-x_1|+|u-u_1|+|y-y_1|+|v-v_1|)$, the condition is satisfied.

The fourth point, for each interior point in Δ_2, there exists

$$\overline{\lim_{h\to0}}\frac{1}{h}\{\Phi[x+hf(t,x,y),u+hf(t,u,v),y+hg(t,x,y),v+hg(t,u,v)]-\Phi(x,u,y,v)\}\le0.$$

The fifth point, for any $\lambda>0$, there exists a number $\chi(\lambda)$ correspondingly which makes each point that satisfies $|x-u|+|y-v|\ge\lambda$ have

$$\overline{\lim_{h\to0}}\frac{1}{h}\{\Phi[x+hf(t,x,y),u+hf(t,u,v),y+hg(t,x,y),v+hg(t,u,v)]-\Phi(x,u,y,v)\}\le\chi(\lambda)<0.$$

Proof If the formula below is satisfied, then both the fourth and fifth point of the conditions will be satisfied,

$$\frac{\partial\Phi}{\partial x}f(t,x,y)+\frac{\partial\Phi}{\partial u}f(t,u,v)+\frac{\partial\Phi}{\partial y}g(t,x,y)+\frac{\partial\Phi}{\partial v}g(t,u,v)<0. \tag{5}$$

Prove equation (5) is true below.

Substitute equation (3) and (4) into the left side of equation (4) and coordinate it, then one obtains

$$\begin{aligned}
&(x-u)^2[(2x+u)(y-0.5x)+(2u+x)(v-0.5u)]-2[(y-v)-C(x-u)][(x-x^3)-(u-u^3)]\\
&\quad+[2(x^2+u^2+xu-1)(x-u)-2C(y-v)][(y-0.5x)-(v-0.5u)]\\
&\le(x-u)^2[6AB-(x^2+u^2+xu)]+[2(x-u)(x^2+u^2+xu-1)-2C(y-v)][y-v-0.5(x-u)]\\
&\quad-[2(y-v)-2C(x-u)](x-u)(x^2+u^2+xu-1).
\end{aligned}$$

Let $x^2 + u^2 + xu - 1 = \theta$, then $-1 \leq \theta \leq 3A^2 - 1$, thus the last formula$=-2C(y-v)^2 + C(x-u)(y-v) + (6AB - 2\theta + 2C\theta - 1)(x-u)^2$.

To make the last formula < 0, only needs $\Delta = C^2 + 8C(6AB - 2\theta + 2C\theta - 1) < 0$, namely

$$16C(C+1)\theta + 8C(6AB-1) + C^2 < 0.$$

when

$$C > 1, \quad \theta < -\frac{8C(6AB-1)+C^2}{16C(C-1)},$$

only needs

$$-\frac{8C(6AB-1)+C^2}{16C(C-1)} \geq 3A^2 - 1,$$

namely

$$(48A^2 - 15)C + (18AB - 48A^2 + 8) \leq 0.$$

A, B and C that satisfy this condition must be able to be determined.

When $C = 1$, $48AB < 7$ is just needed to determine A and B that satisfy this condition.

When

$$C < 1, \quad \theta > -\frac{8C(6AB-1)+C^2}{16C(C-1)},$$

then only needs

$$-\frac{8C(6AB-1)+C^2}{16C(C-1)} \leq -1,$$

namely, $C > (48AB + 8)/15$, then A, B and C that satisfy the condition can also be determined.

Hence Eq.(4) is true, thus both the fourth and fifth points of the conditions are satisfied.

So the conditions of the stability of the solution are true based on the five points above.

3. Influence of the restoring force term of a kind of Duffing equation on the detection S/N of the system

To analyse the influence of the restoring force term on the detection S/N of the Duffing oscillator, the equation (1) is modified as

$$\ddot{x} + k\dot{x} - ax + bx^3 = \gamma\cos(\omega t) + \xi \cdot H(t), \quad (6)$$

where $a > 0$, $b > 0$, the changes of a, b can reflect the change of the restoring force. $H(t)$ contains coloured noise and the signal to be detected. Let $k = 0.5$, $\omega = 1$, time sampling interval is $\Delta t = 0.1$ ms, the simulation time is 500s. As the combination of a, b is unlimited, set $b = 1$ first and change a; then with the best a, change b to determine the optimal combination of a and b based on the S/N.

3.1. $b = 1$, the simulation experiment on the change of a

The change of a is divided into three parts, the first part is 0.5, 0.7, 0.8, 1, 1.2 and 1.5. One can obtain the detection S/N of the system correspondingly; the data (Table 1) show that near $a = 0.7$, the result of the detection is the best. Considering that the frequency-detection errors arising from the change of the damping ratio k have self-organization property,[14] it is not clear whether the change of restoring force has such property also. Take the second part of values at $1 < a < 1.2$, a =1.05, 1.1, 1.15, and the third part of values a =1.02, 1.03, 1.04, the results of the detection S/N of the system will be obtained correspondingly which are shown in Table 1.

Table 1. The simulation experiment results of the detection S/N of the system ($b = 1$).

a	0.5	0.7	0.8	1	1.02	1.03	1.04	1.05	1.1	1.15	1.2	1.5
serial number	1	1	1	1	3	3	3	2	2	2	1	1
(S/N)/dB	−23	−80	−9	−16	−16.99	−6.99	−6.99	−7.44	−13.4	−6.99	−14	−9

Figure 1 is plotted below to demonstrate the change of state expressed by these data more clearly. It can be seen from the figure that: a. a has been densified twice, the results show that for any pair of values of a such that $a_1 < a_2$, when $a_1 < a_x < a_2$, the S/N corresponding to a_1, a_x, a_2 can not be approximated by a linear relationship, namely that the S/N also has self-organization phenomenon which arises from the change of a. b. For the values of a obtained above, though the corresponding change of the S/N is hard to determine, $(S/N)_{\max}$ is about −80dB at $0.5 \leq a \leq 1.5$; this value appears near $a = 0.7$. Considering the complexity of the paired change of a and b, it is reasonable to set $a = 0.7$ in discussing the change of S/N induced by the change of b.

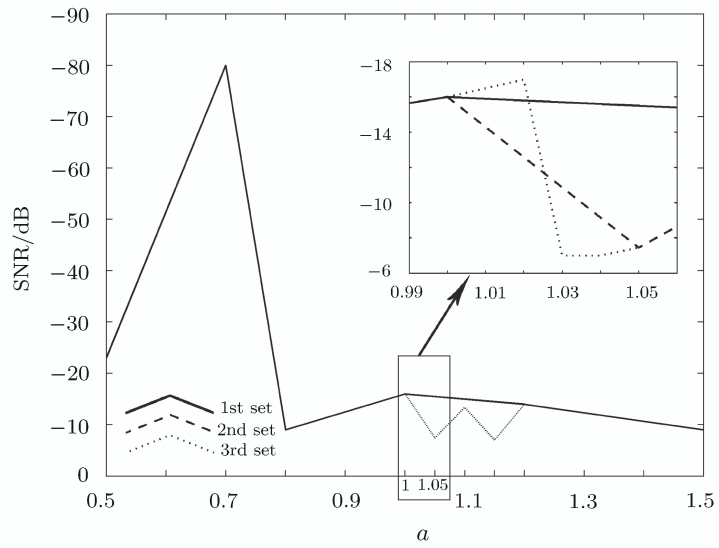


Fig.1. Relationship between the change of the coefficient a of the linear part of the restoring force and the results of the detection signal to noise ratio, S/N of the system ($b = 1$).

3.2. Simulation experiment on the change of b for $a = 0.7$

The results of the computation are shown in Table 2. When $0.5 \leq b \leq 2$, the detection $(S/N)_{\max}$ of the system appears near $b = 1$; it cannot be excluded that the change of S/N induced by the change of b also has self-organization property, but for the interval [0.5,2], $b = 1$ is acceptable. Considering the influence of the change of the coefficients a, b of the restoring force on the detection S/N of the system synthetically, one can set $a = 0.7$, $b = 1$, then it can be obtained that the restoring force function giving better detection effects is $(-0.7x + x^3)$.

Table 2. Relationship between the change of coefficient b of the nonlinear part of the restoring force and the detection S/N of the system ($a = 0.7$).

b	0.5	0.6	0.7	0.8	0.9	0.95	1	1.2	1.5	2
(S/N)/dB	−17	−7.44	−3.46	−3.5	−6.98	−17	−80	−4.9	−6.98	−6.9

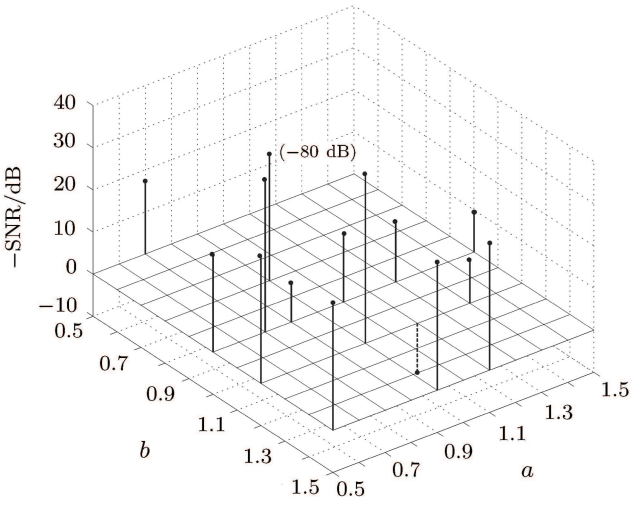


Fig.2. The complex figure about the influence of the changes of a and b on the detection SNR.

Figure 2 is plotted based on the two parts of results shown above. It can be seen that there are two high SNR points in the figure, namely ($a = 0.7$, $b = 1$) and ($a = 0.9$, $b = 1.2$), the corresponding SNR are −80dB and −40dB, respectively. When $b = 1.5$, for a general value of a, the SNR is also comparatively high. Nevertheless, the spatial distribution of SNR does not show a general regularity. Moreover, it cannot be excluded that there exists a more ideal pair of coefficients than (−0.7, 1).

4. Discussion and conclusions

There is no noise at the right side of the equation expressing the stability of the periodic solutions of the Duffing equation with x^3. If the action of noise is considered, such as in equation (6), the chaotic oscillator system constructed by the equation can also get into the large-scale periodic phase state conditionally, when the phase state is quasi-periodic.[4,5] The process

to determine the coefficients a, b of the restoring force $(ax + bx^3)$ is not consummate, namely the complete test of a, b cannot be done; but it can be certain that the restoring force function constructed by $a = 0.7$ and $b = 1$ is a comparatively good one. It is needed to be careful in choosing k, a and b at the same time when using the Equation (6) as the weak-signal detection system in practice.

In this paper, the stability of the periodic solutions of Duffing equation with x^3 is proved theoretically, which is the basis for using the chaotic system constructed by this equation to detect weak signals; at the same time，a better restoring force function is obtained through numerical simulation, thereby the weak-signal detection ability of the Duffing oscillator system is further improved.

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