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Variational iteration method for solving perturbed mechanism of western boundary undercurrents in the Pacific*

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A class of perturbed mechanisms for the western boundary undercurrents in the Pacific is considered. The model of generalized governing equations is studied. Employing the method of variational iteration, an approximate solution of corresponding model is obtained. It is proved from the results that the solution for the variational iteration method can be used for analysing operation of the perturbed mechanism of western boundary undercurrents in the Pacific.

Keywords: undercurrents, perturbation, approximate solution
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1. Introduction

The studies on western boundary currents (WBC), the counter-undercurrents observed in the thermocline, have been included in many papers.[1−3] The existence of the western boundary undercurrents suggests the baroclinicity of WBC, so the results of studies on their baroclinic structure and dynamics should have a strong bearing on the baroclinic theory of oceanic circulation. However, because of their recent discovery, baroclinic WBCs have not yet become an important part in the baroclinic theory of oceanic circulation.[3] The continuous solution of the baroclinic WBCs matching the interior has not been found.[4] It is noteworthy that a few models of the abyssal circulation include the compensatory action of the deep western boundary currents near the bottom.[5] Moreover, the above results are similarity solutions which could not satisfy all boundary conditions essential from the physics and thus impose strict restrictions on the validity. So far the barotropic theory of the WBC has been successfully tested, but the baroclinic theory is still far from perfection, as the present knowledge on the baroclinic structure of the WBC is still only based on observations.[6]

Recently, many scholars have investigated the relevant nonlinear perturbation problem. Approximate methods have been developed and improved, including the method of averaging, boundary layer method, matched asymptotic expansion method and multiple scales method.[7−16] Using the differential inequalities and other methods, Mo *et al* also considered a class of singularly perturbed nonlinear boundary value problems expressed in ordinary differential equations,[17] reaction diffusion equations,[18,19] the shock layer solution of nonlinear equations for singularly perturbed problems,[20,21] and the problems of atmospheric physics.[22−28] In this paper we consider the approximate solution for a perturbed mechanism of the western boundary undercurrents in the Pacific by using a simple and special method[29] for nonlinear problems.

2. The model of generalized governing equations

A two-dimensional ideal fluid dynamic model is presented in a vertical plane with x and z as the hori-

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zontal and vertical coordinates, respectively. Assume the flow is horizontal and independent of the downstream direction. Let the z-axis point upward with the origin on the undisturbed sea surface and the x-axis be cross-stream. In this paper we consider the following nonlinear problem of the generalized governing equations:[6]

$$\frac{1}{\rho_0}\frac{\partial p}{\partial x} - fv = F(x,z,v,p,\varepsilon), \tag{1}$$

$$\frac{\partial p}{\partial z} + \rho g = G(x,z,v,p,\varepsilon), \tag{2}$$

$$v|_{z=\eta(x)} = \bar{v}_0, \quad p|_{z=\eta(x)} = 0, \tag{3}$$

where v is the perpendicular velocity and p is the pressure to the bounded region of the model plane, ρ is density, ρ_0 is the mean value of ρ, f is Coriolis parameter, g is the gravity acceleration, the right-hand sides of Eqs.(1) and (2) are perturbed terms, $z = \eta(x)$ is the sea surface height and $\bar{v}_0 = \dfrac{g\rho(x,\eta(x))}{f\rho_0}\dfrac{\mathrm{d}\eta}{\mathrm{d}x}$; and F, G, ρ and η are sufficiently smooth functions for the above variables.

3. Variational iteration solution of the model

According to the variational iteration method proposed by He,[29] a correction functional can be constructed as follows.

Introducing a functional

$$\bar{F}(p,v,\lambda_1) = p - \int_z^{\eta(x)} \lambda\left(\frac{\partial p}{\partial z_1} + \rho g - G(x,z_1,\bar{v},\bar{p},\varepsilon)\right)\mathrm{d}z_1, \tag{4}$$

where $\bar{v}, \bar{p}$ are restricted variables[29] of v, p respectively, and λ is the Lagrange multiplicator.

Compute the variational $\delta\bar{F}$ of functional (4):

$$\delta\bar{F} = \delta p - (\lambda\delta p)|_{z_1=z} + \int_z^{\eta(x)} \frac{\partial\lambda}{\partial z_1}\delta p\mathrm{d}z_1.$$

Let $\delta\bar{F} = 0$. Thus we have

$$\frac{\mathrm{d}\lambda}{\mathrm{d}z_1} = 0, \tag{5}$$

$$\lambda|_{z_1=z} = 1. \tag{6}$$

From Eqs.(5) and (6), we have

$$\lambda = 1. \tag{7}$$

From Eqs.(1), (2) and (4), (7) we construct the following variational iteration:

$$p_{n+1} = p_n - \int_z^{\eta(x)} \Big(\frac{\partial p_n}{\partial z_1} + \rho g - G(x,z_1,v_n,p_n)\Big)\mathrm{d}z_1, \tag{8}$$

$$v_{n+1} = v_n - \int_z^{\eta(x)} \Big(v_n + \frac{1}{f}\Big[\frac{1}{\rho_0}\frac{\partial p_{n+1}}{\partial x} + F(x,z_1,v_n,p_{n+1})\Big]\Big)\mathrm{d}z_1. \tag{9}$$

Obviously, under the suitable conditions, we can obtain a convergent sequence[30]$(\{v_n(x,z)\},\{p_n(x,z)\})$ from Eqs.(8) and (9), then we have $v(x,z) = \lim\limits_{n\to\infty} v_n(x,z)$, and $p(x,z) = \lim\limits_{n\to\infty} p_n(x,z)$ is a set of solutions of the governing equations (1) and (2).

Now we first select an initial approximate solution $(v_0(x,z), p_0(x,z))$, that is, a set of solutions for the linear system expressed by Eqs.(1)–(3):

$$\frac{1}{\rho_0}\frac{\partial p}{\partial x} - fv = 0, \tag{10}$$

$$\frac{\partial p}{\partial z} + \rho g = 0, \tag{11}$$

$$v|_{z=\eta(x)} = \bar{v}_0, \quad p|_{z=\eta(x)} = 0. \tag{12}$$

From Eqs.(10)–(12), we have

$$p_0(x,z) = g\int_z^{\eta(x)} \rho(x,z_1)\mathrm{d}z_1, \tag{13}$$

$$v_0(x,z) = \frac{g}{f\rho_0}\Big[\int_z^{\eta(x)} \frac{\partial}{\partial x}\rho(x,z_1)\mathrm{d}z_1 + \rho(x,\eta(x))\frac{\mathrm{d}\eta}{\mathrm{d}x}\Big]. \tag{14}$$

Substituting the initial approximate solution (13) and (14) into Eqs.(8) and (9), we can obtain the first-order approximate solution of the system (1)–(3):

$$p_1(x,z) = \int_z^{\eta(x)} [g\rho(x,z_1) + G(x,z_1,v_0,p_0)]\mathrm{d}z_1, \tag{15}$$

$$v_1(x,z) = \frac{g}{f\rho_0}\Big[\int_z^{\eta(x)} \frac{\partial}{\partial x}\rho(x,z_1)\mathrm{d}z_1 + \rho(x,\eta(x))\frac{\mathrm{d}\eta}{\mathrm{d}x}\Big] - \frac{1}{f}\int_z^{\eta(x)} \Big[\int_{z_1}^{\eta(x)} \frac{\partial G}{\partial x}(x,z_2,v_0,p_0)\mathrm{d}z_2 + F(x,z_1,v_0,p_1)\Big]\mathrm{d}z_1, \tag{16}$$

where p_0, v_0 and p_1 are expressed as Eqs.(13), (14) and (15).

Using the same method, we can obtain higher-order approximate solutions of the generalized governing equations (1)–(3).

4. Example

We now consider a specific case of the model. The perturbed terms of the generalized governing equations (1) and (2) are $F = 0, G = \exp(-p)$, and $\rho = \rho_0$. Using the above method of variational iteration, it is easy to obtain the first-order approximate solution (p_1, v_1) of Eqs.(10)–(12)

$$p_1(x,z) = g\rho_0(\eta(x) - z) - \frac{1}{\rho_0 g}[1 - \exp(-\rho_9 g(\eta(x) - z)],$$

$$v_1(x,z) = \frac{\eta'(x)}{f}\Big[(g + \eta(x) - z) - \frac{1}{g\rho_0}(1 - \exp(-\rho_0 g(\eta(x) - z)))\Big].$$

Using the method of variational iteration, we can also obtain higher-order approximate solutions $(p_i, v_i),\ i = 2, 3, \ldots$.

In order to show the accuracy of the above result, we consider a special case, and give a set of parameters as follows: $\rho_0 = 1.04\,\mathrm{kgm}^{-3}$, $f = 2\times10^{-5}\,\mathrm{s}^{-1}$, $g = 9.8\,\mathrm{ms}^{-1}$, $\eta(x) \equiv 0$, $z \in [-2.0, -0.0]\,(\mathrm{km})$.

For example, we establish a comparison of the values between p_{num} for the numerical calculation and p_{app} for the approximate expansion of the variational iteration method. We obtain the following results (see Table 1).

Table 1. A comparison of the values of between p_{num} and p_{app} ($\mathrm{kg/cm}^2$).

z	-0.0	-0.2	-0.4	-0.6	-0.8	-1.0	-1.2	-1.4	-1.6	-1.8	-2.0
p_{num}	0.00	1.95	3.98	6.01	8.05	10.99	12.13	14.17	16.20	18.24	20.28
p_{app}	0.00	2.04	4.08	6.11	8.15	10.19	12.23	14.27	16.30	18.34	20.38

From the result of the table, we know that the values between p_{num} for the numerical calculation and p_{app} of the approximate expansion are less than 0.10 (kg). Thus using the variational iteration method will certainly attain a good accuracy for the computed value of the original problem.

5. Conclusions

1. The western boundary undercurrents in the Pacific are a complicated natural phenomenon. Hence we need to reduce basic models and solve them by using the approximate method. The variational iteration method is a simple and valid one.

2. The variational iteration method is an approximate method, which differs from general numerical method. The expansions of solution through the variational iteration method can be kept in the analytic operation. Thus, from Eqs.(15) and (16), we can further study the fixed quality and quantitative behaviours of the perpendicular velocity and the pressure of the western boundary undercurrents in the Pacific, which we do not discuss further in this paper.

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