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Chaotic synchronization via linear controller*

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A technical framework of constructing a linear controller for chaotic synchronization by utilizing the stability theory of cascade-connected system is presented. Based on the method developed in the paper, two simple and linear feedback controllers, as examples, are derived for the synchronization of Liu chaotic system and Duffing oscillator, respectively. This method is quite flexible in constructing a control law. Its effectiveness is also illustrated by the simulation results.

Keywords: linear controller, chaotic synchronization, Liu system, Duffing oscillator
PACC: 0545

1. Introduction

Chaos and its application have been of broad interest in nonlinear science. In 1963, Lorenz found the best-known canonical chaotic attractor in a simple three-dimensional autonomous system.[1] In 1999, Chen found a similar but nonequivalent chaotic attractor.[2] In 2002, Lu *et al*[3,4] found the Lu chaotic system and a unified chaotic system, which can connect the Lorenz, Chen and Lu systems together. Very recently, Liu has reported a new chaotic system, the Liu system, which is similar to the Lorenz chaotic attractor, but not an equivalent chaotic attractor.[5] The Liu chaotic system can be given by

$$\begin{aligned} \dot{x}_1 &= a(x_2 - x_1), \\ \dot{x}_2 &= bx_1 - kx_1x_3, \\ \dot{x}_3 &= -cx_3 + hx_1^2, \end{aligned} \tag{1}$$

where x_1, x_2, x_3 are state variables of the system, and a, b, c, k, h are parameters. The system exhibits chaos when $a = 10$, $b = 40$, $k = 1$, $c = 2.5$ and $h = 4$. Since the Liu chaotic system is a new chaotic system, investigating its property of dynamics and application is of great importance. The synchronization of this system was investigated by Chen *et al*[6] via a nonlinear state feedback. However, up to now, the linear controller for the synchronization of the Liu system has never been reported. In this paper, we treat the Liu system as an example to illustrate the effectiveness of the proposed method.

Since the work of Pecora and Carroll,[7] chaotic synchronization has been extensively investigated.[7−15] A great number of methods have been proposed to deal with the chaotic synchronization, such as the sampled-data feedback synchronization method,[8] impulsive control method,[9] adaptive design method,[10−12] invariant manifold method,[13] backstepping design method,[14] sliding mode control method,[15,16] and other control methods.[17−20] The controllers mentioned above are all nonlinear. As is well known, the feedback linear controller is easy to implement and also economic. The chaotic synchronization via linear controller was investigated in Refs.[21,22]. Jiang and Zheng[21] viewed the problem of chaotic synchronization as a special case of observer design, so it could be solved by the linear control theory. The methods they proposed require finding out Lipschitz constant for a nonlinear function. However, even if the Lipschitz constant is known, the large Lipschitz constant always results in a high gain that is not easy to realize in practice. Liu[22] gave out the linear controller on the assumption that the nonlinear function of the chaotic system satisfies upper triangle form (see assumptions in Ref.[22]). However, there exist some well-known chaotic systems such as Lorenz system, Chen system, Lu system and Liu system and so on which do not satisfy the form assumed in Ref.[22]. In order to overcome the above weakness, it is important and meaningful to provide a new method to design the linear controller for a chaotic system.

In this paper, based on the stability theory of cascade-connected system,[23−25] which has been fully

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developed in nonlinear differential geometry theory, and the state feedback technique, we give out a technical framework of constructing a linear controller for chaotic synchronization. Compared with existing results,[21,22] the proposed method can avoid finding the system Lipschitz constant and discard the assumption that nonlinear function in the system requires satisfying the upper triangle form assumed in Ref.[22]. Finally, two simple and linear feedback controllers, as examples, are derived for the Liu system and Duffing oscillator, respectively, which also serves to illustrate the effectiveness of the proposed method.

2. Chaotic synchronization

Consider a master chaotic system given by

$$\begin{aligned} \dot{\varepsilon} &= G(\varepsilon, \eta), \\ \dot{\eta} &= A\eta + B\varepsilon, \end{aligned} \tag{2}$$

where $\varepsilon \in R^m$ and $\eta \in R^{n-m}$ are the state vectors of the system (2), A and B are the known constant matrices with an appropriate dimension, and $G : R^n \to R^m$ defines vector fields in appropriate dimensional space, and its slave system is given by

$$\begin{aligned} \dot{\hat{\varepsilon}} &= G(\hat{\varepsilon}, \hat{\eta}) + L_1 \begin{bmatrix} \varepsilon - \hat{\varepsilon} \\ \eta - \hat{\eta} \end{bmatrix}, \\ \dot{\hat{\eta}} &= A\hat{\eta} + B\hat{\varepsilon} + L_2 \begin{bmatrix} \varepsilon - \hat{\varepsilon} \\ \eta - \hat{\eta} \end{bmatrix}, \end{aligned} \tag{3}$$

where $\hat{\varepsilon} \in R^m$ and $\hat{\eta} \in R^{n-m}$ are the state vectors of the system (3), and $L_1 \in R^{m\times n}$, $L_2 \in R^{(n-m)\times n}$ are the feedback constant matrices. Defining $e_1 = \varepsilon - \hat{\varepsilon}$, and $e_2 = \eta - \hat{\eta}$, and subtracting the expression (3) from the expression (2) yield a dynamical system as

$$\begin{aligned} \dot{e}_1 &= H(\varepsilon, \eta, e_1, e_2) - L_1 \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \\ \dot{e_2} &= Ae_2 + Be_1 - L_2 \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \end{aligned} \tag{4}$$

where $H(\varepsilon, \eta, e_1, e_2) = G(\varepsilon, \eta) - G(\varepsilon - e_1, \eta - e_2)$.

Remark 1 Lots of chaotic systems, such as Lorenz system, Chen system, Lu system, Liu system, Rossler system, Chua system, have the form of the expression (2).

In order to make further discussion, two useful lemmas are presented in the following. Consider the cascade-connected system

$$\begin{aligned} \dot{x} &= f(x, z), \\ \dot{z} &= g(z), \end{aligned} \tag{5}$$

where $x \in R^m$, $z \in R^n$, $f(0, 0) = 0$, $g(0) = 0$; $f(x, z)$ and $g(z)$ satisfy the local Lipschitz condition.

Lemma 1[24] If the subsystem equations $\dot{x} = f(x, 0)$ and $\dot{z} = g(z)$ are both globally asymptotically stable at $x = 0$ and $z = 0$, respectively, and each solution of the system (5) is bounded, then the system (5) is also globally asymptotically stable at equilibrium point $(x, z) = (0, 0)$.

Lemma 2[25] Consider the nonlinear time-varying system $\dot{x} = f(x, t)$, if there exists a differential function $V : U \times J \to R$ satisfying the following conditions:

1) There exist a positive constant λ_0 and a scalar function $\bar{\lambda} : U \to R$ such that

$$\lambda_0 ||x||^2 \le ||V(x, t)|| \le \bar{\lambda}(x)||x||^2, \quad \forall (x, t) \in U \times J, \tag{6}$$

2) There exist positive constants $\lambda_V > 0$ and $\varepsilon \ge 0$ such that

$$\dot{V}(x, t)|_{\dot{x}=f(x,t)} \le -\lambda_V \bar{\lambda}(x)||x||^2 + \varepsilon, \quad \forall (x, t) \in U \times J, \tag{7}$$

then the solution of the system $\dot{x} = f(x, t)$ is bounded by

$$\begin{aligned} ||x|| \le\ & \frac{1}{\lambda_0} V(x_0, t_0) \mathrm{e}^{-\lambda_V (t-t_0)} \\ & + \frac{\varepsilon}{\lambda_0 \lambda_V} \left(1 - \mathrm{e}^{-\lambda_V (t-t_0)}\right) \le M, \end{aligned} \tag{8}$$

where $M = \dfrac{1}{\lambda_0} V(x_0, t_0) + \dfrac{\varepsilon}{\lambda_0 \lambda_V}$.

Remark 2 In Ref.[25], the system $\dot{x} = f(x, t)$ in Lemma 2 is subject to $f(0, t) = 0$, but it can be cancelled if we just want to draw the conclusion that the solution of the system $\dot{x} = f(x, t)$ is bounded.

From Lemmas 1 and 2, we can divide the procedure of controller design into 3 steps.

Step 1 Decompose the system and work out its error dynamical system (4). Set $L_2 = [B, K]$, where $(A - K)$ is asymptotically stable.

Step 2 Construct L_1 such that the first subsystem of the system (4) $\dot{e}_1 = H(\varepsilon, \eta, e_1, 0) - L_1 \begin{bmatrix} e_1, \\ 0 \end{bmatrix}$ is globally asymptotically stable at $e_1 = 0$.

Step 3 Construct a Lyapunov function to prove that each solution of the subsystem $\dot{e}_1 = H(\varepsilon, \eta, e_1, e_2) - L_1 \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$ is bounded via Lemma 2.

According to Lemma 1, if L_1 and L_2 are derived from the above algorithm, then the system (4) must be globally asymptotically stable at $(e_1, e_2) = 0$, which implies that the synchronization is achieved. In the next section, we will give two examples to show the technique of how to prove each solution of the subsystem $\dot{e}_1 = H(\varepsilon, \eta, e_1, e_2) - L_1 \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$ to be bounded via Lemma 2.

3. Application to examples

3.1. Application to Liu system

Consider the Liu system described by Eq.(1). In this example we will apply only one state variable to synchronizing the Liu chaotic system. Moreover, the similar processing can be also used to synchronize other well-known chaotic systems, such as, Lorenz, Chen, and Lu systems. Consider the slave system with controllers, given by

$$\begin{aligned} \dot{\hat{x}}_1 &= a(\hat{x}_2 - \hat{x}_1) + L_1(x_2 - \hat{x}_2), \\ \dot{\hat{x}}_2 &= b\hat{x}_1 - k\hat{x}_1\hat{x}_3 + L_2(x_2 - \hat{x}_2), \\ \dot{\hat{x}}_3 &= -c\hat{x}_3 + h\hat{x}_1^2 + L_3(x_2 - \hat{x}_2), \end{aligned} \tag{9}$$

where $\hat{x}_1, \hat{x}_2, \hat{x}_3$ are state variables of the system, and $L_i (i = 1, 2, 3)$ are coefficients which we will work out later.

Step 1 From Eqs.(1)–(9), we have

$$\begin{aligned} \dot{e}_1 &= a(e_2 - e_1) - L_1 e_2, \\ \dot{e}_2 &= (b + ke_3 - kx_3)e_1 - kx_1 e_3 - L_2 e_2, \\ \dot{e}_3 &= -ce_3 - he_1^2 + 2he_1 x_1 - L_3 e_2. \end{aligned} \tag{10}$$

Let $L_1 = a$ and consider the following subsystem:

$$\dot{e}_1 = -ae_1. \tag{11}$$

Obviously, the subsystem (11) is globally asymptotically stable at $e_1 = 0$.

Step 2 Consider the remaining subsystem of the system (10), described as

$$\begin{aligned} \dot{e}_2 &= (b + ke_3 - kx_3)e_1 - kx_1 e_3 - L_2 e_2, \\ \dot{e}_3 &= -ce_3 - he_1^2 + 2he_1 x_1 - L_3 e_2, \end{aligned} \tag{12}$$

and substitute $e_1 = 0$ into the subsystem (12). According to the linear system theory, if $L_2 > 0$ and $L_3 = 0$ (we prefer to select the controller as simple as possible), then the system (12) with $e_1 = 0$ is globally asymptotically stable at $e_3 = 0$.

Step 3 Select the Lyapunov function $V = pe_2^2/2 + qe_3^2/2$, where $p > 0$ and $q > 0$. The derivative of V is

$$\begin{aligned} \dot{V} = \; & -pL_2 e_2^2 - cqe_3^2 + p(b - kx_3)e_1 e_2 \\ & + pk(e_1 - x_1)e_2 e_3 + (2hqx_1 e_1 - hqe_1^2)e_3. \end{aligned}$$

Since $2ab \le \varepsilon a^2 + \varepsilon^{-1} b^2$ for $\varepsilon > 0$, we have

$$\begin{aligned} \dot{V} \le \; & -(pL_2 - \varepsilon_1 - \varepsilon_2)e_2^2 \\ & -(cq - \varepsilon_3 - N)e_3^2 + M, \end{aligned} \tag{13}$$

where

$$N = \varepsilon_2^{-1} \frac{(pke_1 - pkx_1)^2}{4},$$

$$M = \varepsilon_1^{-1} \frac{(pbe_1 - pkx_3 e_1)^2}{4} + \varepsilon_3^{-1} \frac{(2hqx_1 e_1 - hqe_1^2)^2}{4}$$

and ε_i, $(i = 1, 2, 3)$ are positive constants which can be selected arbitrarily.

Note that $|x_i| < S$ (the master system is a chaotic system) and $|e_1| < |e_1(0)|$. Therefore, M and N must be bounded. From Lemma 2 and the inequality (13), if $p > (\varepsilon_1 + \varepsilon_2)/L_2$ and $q > (\varepsilon_3 + N)/c$, then each solution of the system (12) is bounded. Thus, the system (10) is globally asymptotically stable at $(e_1, e_2, e_3) = (0, 0, 0)$.

To verify the effectiveness of the proposed controller, we perform the numerical simulation for the results that we obtained. Assume the initial value of the master system and slave system to be $x_1(0) = 1$, $x_2(0) = 2$, $x_3(0) = 3$ and $\hat{x}_1(0) = 5$, $\hat{x}_2(0) = 6$, $\hat{x}_3(0) = -4$, respectively, and let $L_1 = a, L_2 = 1, L_3 = 0$. As is expected, Fig.1 shows that the states of two Liu systems are asymptotically synchronized with each other.

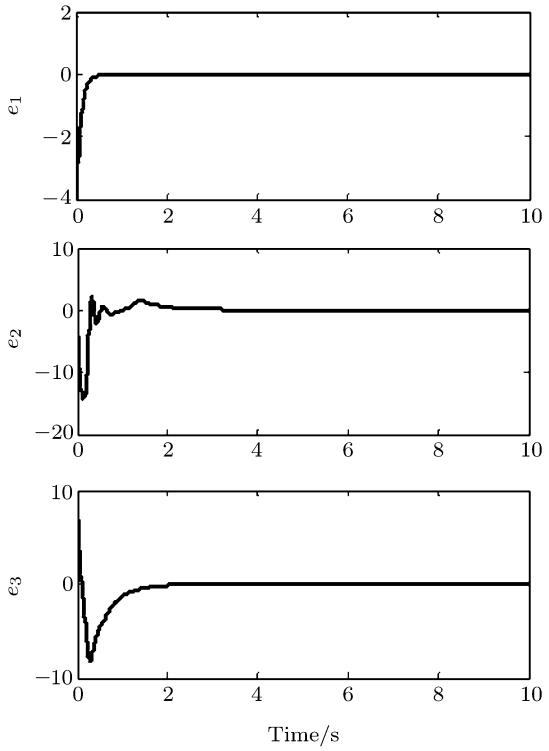


Fig.1. The time response errors between master and slave Liu systems.

3.2. Application to Duffing oscillator

Liu system discussed above can exhibit chaos under no external excitation condition. In the following we consider the Duffing oscillator, which exhibits chaotic phenomenon only under external excitation.

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= ax + by - x^3 + c\cos(0.4t) \end{aligned} \tag{14}$$

is a master system, and

$$\begin{aligned} \dot{\hat{x}} &= \hat{y} + u, \\ \dot{\hat{y}} &= a\hat{x} + b\hat{y} - \hat{x}^3 + c\cos(0.4t) \end{aligned} \tag{15}$$

is a slave system, where $\cos(0.4t)$ is the external excitation, u is a controller and $a = 1.8, b = -0.1$, and $c = -1.1$.

Step 1 From Eqs.(14),(15), the error dynamical system is obtained as

$$\begin{aligned} \dot{e}_x &= e_y - u, \\ \dot{e}_y &= ae_x + be_y - e_x^3 + 3xe_x^2 - 3x^2e_x. \end{aligned} \tag{16}$$

The first equation of the system (16) gives $u = e_y + ke_x$,
where

$$\dot{e}_x = -ke_x, \quad k > 0. \tag{17}$$

Obviously, the subsystem (17) is globally asymptotically stable at $e_x = 0$.

Step 2 Consider the remaining subsystem of system (16)

$$\dot{e}_y = ae_x + be_y - e_x^3 + 3xe_x^2 - 3x^2e_x. \tag{18}$$

It is clear that the subsystem (18) with $e_x = 0$ is globally asymptotically stable at $e_y = 0$.

Step 3 Select Lyapunov function as $V = e_y^2/2$, then the derivative of V is

$$\begin{aligned} \dot{V} = &\ e_y[ae_x + be_y - e_x^3 + 3xe_x^2 - 3x^2e_x] \\ \leq &\ (b + \varepsilon)e_y^2 + M, \end{aligned}$$

where $M = \varepsilon^{-1}\dfrac{(ae_x - e_x^3 + 3xe_x^2 - 3x^2e_x)^2}{4}$, with $\varepsilon > 0$.

Since the master system is a chaotic system, there must exist a positive constant S such that $|x| \leq S$. From the expression (17), we obtain $|e_x(t)| \leq |e_x(0)|$, i.e. e_x is also bounded. Thus, M is bounded. From Lemma 2, we obtain that each solution e_y is bounded. Thus, the system (16) is globally asymptotically stable at $(e_x, e_y) = (0, 0)$. To verify the effectiveness of the proposed controller, we carry out the numerical simulation for the results that we obtained. Assume the initial values of the master system and slave system to be $x(0) = 1$, $y(0) = 2$ and $\hat{x}(0) = 3$, $\hat{y}(0) = 4$, respectively, and let $u = e_y + 2e_x$. As is expected, Fig.2 shows that the states of two Duffing oscillators are asymptotically synchronized with each other.

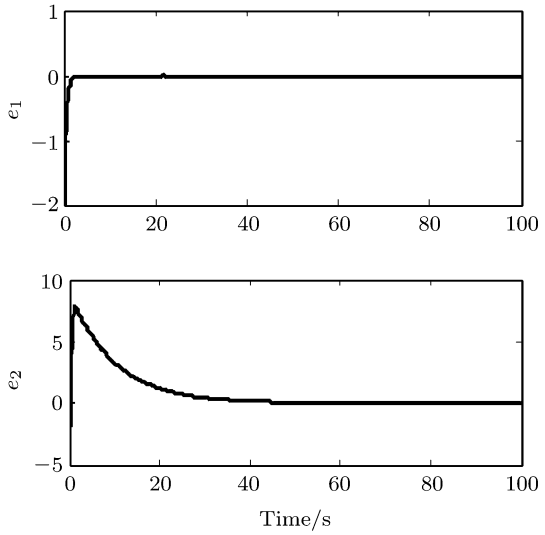


Fig.2. The time response errors between master and slave Duffing oscillators.

4. Conclusion

Based on the stability theory of the cascade-connected system, a novel method is developed to synchronize a general class of chaotic systems. Two simple and linear feedback controllers are derived, as examples, to synchronize the Liu chaotic system and Duffing oscillators, respectively, via the proposed method, and the numerical simulations show the effectiveness of the proposed controller.

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