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Massive particles' tunnelling radiation from the black hole with a mass-quadruple moment*

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In this paper, we extend Zhang and Zhao's recent work to the black hole with a mass-quadruple moment. The behaviour of the tunnelling massive particles is investigated, and the emission rate at which massive particles tunnel across the event horizon of the black hole is calculated. The result is consistent with an underlying unitary theory, and takes the same functional form as that of a massless particle.

Keywords: black hole, tunnelling radiation, quantum theory
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1. Introduction

In 2000, a method describing the Hawking radiation as a tunnelling process, where a particle moves in dynamic geometry, was developed by Parikh and Wilczek.[1−3] They found that the barrier is created by the outgoing particle itself, and their key insight is to find a well-behaved coordinate system to calculate the emission rate. Their result is consistent with an underlying unitary theory. Recently, Zhang and Zhao[4−6] attempt to extend Parikh's method to investigate the behaviour of a charged massive particle's emission rate. In 2005, following Parikh's method, the emission rate of the massless outgoing particle from the black hole was calculated,[7−9] only investigating the tunnelling of the massless particle. This paper, applies Zhang's method to the spacetime with a mass-quadruple moment to calculate the massive particle's emission rate. In this case, a massive particle travels on the geodesics in the spacetime, and the black hole's energy will change. Throughout the paper, the geometrized units ($G \equiv c \equiv \hbar = 1$) are used.

2. Painlevé–Gutsunaev–Manko coordinates

The metric of the black hole with a mass-quadruple moment can be written as[10]

$$\mathrm{d}s^2 = m^2\mathrm{e}^{-2\psi}\left\{\mathrm{e}^{2\gamma}(x^2-y^2)\left[\frac{\mathrm{d}x^2}{x^2-1}+\frac{\mathrm{d}y^2}{y^2-1}\right]+(x^2-1)(y^2-1)\mathrm{d}\varphi^2\right\}-\mathrm{e}^{2\psi}\mathrm{d}t^2. \tag{1}$$

Where m, A_2 correspond to the source mass of the gravitational field and the mass-quadruple factor respectively. $\psi(x,y)$ and $\gamma(x,y)$ are metric functions. By the coordinate transformation $x = r/m - 1, y = \cos\theta, \varphi = \varphi$, the line element (1) can be rewritten as[11]

$$\mathrm{d}s^2 = \mathrm{e}^{2(\gamma^0-\psi^0)}\left[\left(1-\frac{2m}{r}\right)^{-1}\mathrm{d}r^2+r^2\mathrm{d}\varphi^2\right]+\mathrm{e}^{-2\psi^0}r^2\sin^2\theta\mathrm{d}\theta^2-\mathrm{e}^{2\psi^0}\left(1-\frac{2m}{r}\right)\mathrm{d}t^2. \tag{2}$$

Eq.(2) is called the Gutsunaev–Manko metric.

In spacetime for a certain hyper-surface $f(x) = 0$ the hyper-surface's normal vector $n_\mu = \dfrac{\partial f}{\partial x^\mu}$ is a non-spacelike one, moreover, it is light-like, and the hyper-surface which acts as the starting location of the spacetime one-way membrane is called the event horizon,

which needs to meet the following conditions:[12]

$$n^\mu n_\mu = g^{\mu\nu}\frac{\partial f}{\partial x^\mu}\frac{\partial f}{\partial x^\nu} = 0. \tag{3}$$

In the spacetime depicted by Eq.(2), the location of the horizon is decided by $f(r,\theta)=0$, therefore

$$g^{rr}\left(\frac{\partial f}{\partial r}\right)^2 + g^{\theta\theta}\left(\frac{\partial f}{\partial \theta}\right)^2 = 0, \tag{4}$$

from Eqs.(2) and (4) we have

$$\mathrm{e}^{-2(\gamma^0-\psi^0)}\left[\left(1-\frac{2m}{r}\right)\left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2\right] = 0. \tag{5}$$

Within the numeric area of r and θ the exponential factor $-2(\gamma^0-\psi^0)$ is not divergent at any other positions except for the divergence of two poles $\theta=0$ and π when $r=2m$. Hence, at any other positions out of the two poles, Eq.(5) serves as

$$\left(1-\frac{2m}{r}\right)\left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2 = 0. \tag{6}$$

From the separation of variables $f(r,\theta)=R(r)\Theta(\theta)$, we obtain

$$r(r-2m)\left[\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)/R\right]^2 = -\left[\left(\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right)/\Theta\right]^2, \tag{7}$$

When the polar angle transforms $\theta \to \pi-\theta$, the invariance of Gutsunaev–Manko metric (2) educes $\Theta(\pi-\theta)=\Theta(\theta)$, Eq.(7) elicits

$$r(r-2m)\left[\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)/R\right]^2 = 0. \tag{8}$$

The exclusive rational solution is given by $r(r-2m)=0$, namely $r=0$ or $r=2m$. Thereby, the hypersurface $r=2m$ determines the horizon in Gutsunaev–Manko spacetime.

When $r\approx 2m$, via round the horizon surface there exists the following approximation:

$$\mathrm{e}^{2(\gamma^0-2\psi^0)} = \mathrm{e}^{2(-A_2/2-3A_2^2/8)}. \tag{9}$$

Equation (9) is strictly tenable, and then there is

$$\begin{aligned}\mathrm{d}s^2 = & -\mathrm{e}^{3A_2^2/8}\left(1-\frac{2m}{r}\right)\mathrm{d}t^2 \\ & +\mathrm{e}^{-A_2-3A_2^2/8}\left(1-\frac{2m}{r}\right)^{-1}\mathrm{d}t^2 \\ & +\mathrm{e}^{3A_2^2/8}r^2\sin^2\theta\mathrm{d}\theta^2 \\ & +\mathrm{e}^{-A_2-3A_2^2/8}r^2\mathrm{d}\varphi^2.\end{aligned} \tag{10}$$

To obtain the Painlevé spacetime coordinates, let us do the following transformation:

$$t_\mathrm{s} = t + f(r). \tag{11}$$

This satisfies

$$\frac{1}{1-g(r)} - \frac{1-g(r)}{\mathrm{e}^{-A_2}}(f'(r))^2 = 1, \tag{12}$$

where

$$1-g(r) = \mathrm{e}^{3A_2^2/8}(1-2m/r). \tag{13}$$

Equation (10) becomes

$$\begin{aligned}\mathrm{d}s^2 = & -(1-g(r))\mathrm{d}t_\mathrm{s}^2 + \frac{\mathrm{e}^{-A_2}}{1-g(r)}\mathrm{d}r^2 \\ & +\mathrm{e}^{3A_2^2/8}r^2\sin^2\theta\mathrm{d}\theta^2 \\ & +\mathrm{e}^{-A_2-3A_2^2/8}r^2\mathrm{d}\varphi^2.\end{aligned} \tag{14}$$

At the position $r=2m$ in the horizon surface, metric (14) is no longer singular, but is still the inverse invariable of inherent time. So, a spectator falling plumb down will not see anything non-normal when transiting the horizon surface.

From Eq.(11), we have

$$\mathrm{d}t_\mathrm{s} = \mathrm{d}t + f'(r)\mathrm{d}r. \tag{15}$$

Substituting Eqs.(12) and (15) into Eq. (14), we can read off the Painlevé line element

$$\begin{aligned}\mathrm{d}s^2 = & -\mathrm{e}^{3A_2^2/8}\left(1-\frac{2m}{r}\right)\mathrm{d}t^2 \pm 2\mathrm{e}^{-A_2/2+3A_2^2/16}\sqrt{\frac{2m}{r}}\mathrm{d}t\mathrm{d}r + \mathrm{e}^{-A_2}\mathrm{d}r^2 + \mathrm{e}^{3A_2^2/8}r^2\sin^2\theta\mathrm{d}\theta^2 + \mathrm{e}^{-A_2-3A_2^2/8}r^2\mathrm{d}\varphi^2 \\ = & \ g_{00}\mathrm{d}t^2 + 2g_{01}\mathrm{d}t\mathrm{d}r + \mathrm{e}^{-A_2}\mathrm{d}r^2 + \mathrm{e}^{3A_2^2/8}r^2\sin^2\theta\mathrm{d}\theta^2 + \mathrm{e}^{-A_2-3A_2^2/8}r^2\mathrm{d}\varphi^2.\end{aligned} \tag{16}$$

3. Behaviour of the massive tunnelling particle

Now there is no singularity at the event horizon $r = 2m$, and the true character of the spacetime, as being stationary but not static, is manifest. Note that we must choose the positive (negative) sign in Eq.(16) when we study the tunnelling process taking place at the event horizon. Moreover, another important feature is found which is rarely stressed in the relevant literatures. We describe this as follows.

According to Landau's theory of the coordinate clock synchronization,[13] in a spacetime decomposed in (3+1), the coordinate time difference of two events, which take place simultaneously in different places, is

$$\Delta t = -\int \frac{g_{0i}}{g_{00}} \mathrm{d}x^i \quad (i = 1, 2, 3). \tag{17}$$

If the simultaneity of coordinate clocks can be transmitted from one place to another and has nothing to do with the integration path, components of the metric should satisfy[14]

$$\frac{\partial}{\partial x^i}\left(-\frac{g_{oj}}{g_{00}}\right) = \frac{\partial}{\partial x^j}\left(-\frac{g_{oi}}{g_{00}}\right) \quad (i, j = 1, 2, 3). \tag{18}$$

Obviously, the line element (16) satisfies condition (18), that is, the coordinate clock synchronization in the Painlevé coordinate can be transmitted from one to another though the line element is not diagonal. In quantum mechanics, particle tunnelling a barrier is an instantaneous process. Thus this feature is necessary for us to discuss the tunnelling process.

The radial null geodesics are given by[1]

$$\dot{r} \equiv \frac{\mathrm{d}r}{\mathrm{d}t} = \mathrm{e}^{A_2/2+3A_2^2/16}\left(\pm 1 - \sqrt{\frac{2m}{r}}\right), \tag{19}$$

where the upper (lower) sign in Eq.(19) corresponds to outgoing (ingoing) geodesics, under the implicit assumption that t increases towards the future. These equations are modified when the particles self-gravitation is taken into account.

In this picture, we investigate the tunnelling of the massive particles. Since the world line of massive quanta is not light-like, it does not follow radial-lightlike geodesics (19) when it tunnels across the horizon. For the sake of simplicity, we consider the outgoing massive particle as a massive shell (de Broglie s-wave). Using the Wentzel–Kramers–Brillouin approximation, we get the wave equation

$$\psi(r, t) = C\mathrm{e}^{\mathrm{i}\left(\int_{\gamma_{i-\varepsilon}}^{\gamma_i} p_r \mathrm{d}r - \omega t\right)}, \tag{20}$$

where $r_i - \varepsilon$ indicates the initial location of the particle. If let

$$\int_{\gamma_i-\varepsilon}^{\gamma_i} p_r \mathrm{d}r - \omega t = \phi_0, \tag{21}$$

then, we have

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \dot{r} = \frac{\omega}{k}, \tag{22}$$

where k is the de Broglie wave number. Comparing Eq.(22) with the definition of the phase velocity, we see that $\dot{r}$ is the phase velocity of the de Broglie wave. Unlike the electromagnetic wave, the phase velocity ν_p of the de Broglie wave is not equal to the group velocity ν_g, the definitions and relationship between them are

$$\nu_\mathrm{p} = \frac{\mathrm{d}r}{\mathrm{d}t} = \dot{r} = \frac{\omega}{k}, \tag{23}$$

$$\nu_\mathrm{g} = \frac{\mathrm{d}r_\mathrm{s}}{\mathrm{d}t} = \frac{\mathrm{d}\omega}{\mathrm{d}k}, \tag{24}$$

$$\nu_\mathrm{p} = \frac{1}{2}\nu_\mathrm{g}. \tag{25}$$

To obtain the formula of the phase velocity $\dot{r}$, now we investigate the behaviour of a massive particle tunnelling across the horizon.

Since tunnelling across the barrier is an instantaneous process, there are two simultaneous events during the process of emission. One event is particle tunnelling into the barrier; another is particle tunnelling out of the barrier. In terms of Landau's theory of the coordinate clock synchronization, the difference of coordinate times of these two simultaneous events is

$$\mathrm{d}t = -\frac{g_{0i}}{g_{00}}\mathrm{d}x^i = -\frac{g_{01}}{g_{00}}\mathrm{d}r_s \quad (\mathrm{d}\theta = \mathrm{d}\varphi = 0), \tag{26}$$

where r_s is the location of the particle. The group velocity is

$$\nu_\mathrm{g} = \frac{\mathrm{d}r_\mathrm{s}}{\mathrm{d}t} = -\frac{g_{00}}{g_{01}}, \tag{27}$$

and the phase velocity is therefore

$$\nu_\mathrm{p} = \dot{r} = \frac{1}{2}\nu_\mathrm{g} = -\frac{1}{2}\frac{g_{00}}{g_{01}}. \tag{28}$$

Substituting g_{00} and g_{01} into Eq.(20), we obtain

$$\dot{r} = \mp\frac{1}{2r}\exp\left(\frac{A_2}{2} + \frac{3A_2^2}{16}\right)\frac{r - 2m}{\sqrt{2m/r}}. \tag{29}$$

Here, the upper sign corresponds to the equation of motion of the outgoing particle near the event horizon, and the lower sign corresponds to that of the ingoing particle near the event horizon. Moreover, if the

self-gravitation is included, Eqs.(16), (27), (28) should be replaced by $m \to m-\omega$, where ω is the particles energy and the upper (lower) sign corresponds to outgoing (ingoing) motion of the massive particles. When the massive particle tunnels out, the line element (16) should be rewritten as

$$\begin{aligned} \mathrm{d}s^2 = & -\mathrm{e}^{3A_2^2/8}\left[1-\frac{2(m-\omega)}{r}-\frac{\lambda}{3}r^2\right]\mathrm{d}t^2 \pm 2\mathrm{e}^{-A_2/2+3A_2^2/16}\sqrt{\frac{2(m-\omega)}{r}+\frac{\lambda}{3}r^2}\mathrm{d}t\mathrm{d}r \\ & +\mathrm{e}^{-A_2}\mathrm{d}r^2+\mathrm{e}^{3A_2^2/8}r^2\sin^2\theta\mathrm{d}\theta+\mathrm{e}^{-A_2-3A_2^2/8}r^2\mathrm{d}\varphi^2. \end{aligned} \tag{30}$$

4. Tunnelling rate

Let us calculate the emission rate at which a massive particle tunnels across the event horizon of the black hole with a mass-quadruple moment. Because of the infinite blue shift near the horizon, the characteristic wavelength of any wave packet is always arbitrarily small there; in the semi-classical limit, the radial wave-number approaches infinity and the point particle or Wentzel–Kramers–Brillouin approximation is justified[15]

$$\Gamma \approx \exp(-2\mathrm{Im}Z). \tag{31}$$

For a positive-energy s-wave, the imaginary part of the action can be expressed as

$$\mathrm{Im}Z = \int_{r_\mathrm{i}}^{r_\mathrm{f}} p_\mathrm{r}\mathrm{d}r = \int_{r_\mathrm{i}}^{r_\mathrm{f}}\int_0^{p_\mathrm{r}} p_\mathrm{r}'\mathrm{d}r. \tag{32}$$

where p_r is the canonical momentum conjugate to $r\cdot r_\mathrm{i}$ is the initial radius of the black hole, and r_f is the final radius of the hole. To proceed with an explicit calculation, it is useful to apply the Hamilton equation

$$\dot{r} = \frac{\mathrm{d}H}{\mathrm{d}p_\mathrm{r}} = \frac{\mathrm{d}(E-\omega)}{\mathrm{d}p_\mathrm{r}} = -\frac{\mathrm{d}\omega}{\mathrm{d}p_\mathrm{r}}. \tag{33}$$

where E represents the total energy of the black hole system, whereas $E-\omega$ can be regarded as the gravitational energy stored in the black hole. Energy conservation will be enforced in a natural way by keeping E fixed.

Substituting Eq.(33) into Eq.(32) we have

$$\mathrm{Im}Z = -\mathrm{Im}\int_{r_\mathrm{in}}^{r_\mathrm{out}}\int_0^{\omega}\frac{\mathrm{d}\omega'}{\dot{r}}\mathrm{d}r, \tag{34}$$

Before evaluating the above integral, we must obtain the expression of $\dot{r}$, which is a function of ω'. If the self-gravitation is taken into account, the form of this expression becomes

$$\dot{r} = -\frac{1}{2r}\exp\left(\frac{A_2}{2}+\frac{3A_2^2}{16}\right)\frac{r-2(m-\omega')}{\sqrt{2(m-\omega')/r}}. \tag{35}$$

Substituting Eq.(35) into Eq.(34) we get

$$\begin{aligned} \mathrm{Im}Z = & \ \mathrm{Im}\int_{r_\mathrm{in}}^{r_\mathrm{out}}\int_0^{\omega}\frac{1}{2r}\mathrm{e}^{A_2/2+3A_2^2/16} \\ & \times\frac{r-2(m-\omega')}{\sqrt{2(m-\omega')/r}}\mathrm{d}\omega'\mathrm{d}r. \end{aligned} \tag{36}$$

Finishing the integral we obtain

$$\mathrm{Im}Z = -8\pi m\omega\mathrm{e}^{-A_2/2-3A_2^2/16}(1-\omega/2m). \tag{37}$$

Substituting Eq.(37) into Eq.(31), we get the tunnelling rate

$$\Gamma \approx \exp(-2\mathrm{Im}Z) = \mathrm{e}^{\Delta S_{B-H}}, \tag{38}$$

where ΔS_{B-H} is the difference of the entropies of the black hole with a mass-quadruple moment before and after the emission. The spectrum is consistent with an underlying unitary theory, and supports the information conservation.

5. Conclusions

In summary, we have considered radiation of the black hole with a mass-quadruple moment as a tunnelling process. Adopting the methodology of Wilczek and Parikh[1−3] and Zhang,[4−6] we have calculated the rate of massive particle tunnelling across a event horizon. Equation (38) is consistent with an underlying unitary theory. These frequency-dependent corrections indicate that the emission spectrum of Hawking-like radiation deviates from perfect thermal spectrum, but supports Parikh's conclusion.

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