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The effect of quantum noise on multiplayer quantum game*

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It has recently been realized that quantum strategies have a great advantage over classical ones in quantum games. However, quantum states are easily affected by the quantum noise, resulting in decoherence. In this paper, we investigate the effect of quantum noise on a multiplayer quantum game with a certain strategic space, with all players affected by the same quantum noise at the same time. Our results show that in a maximally entangled state, a special Nash equilibrium appears in the range of $0 \leq p \leq 0.622$ (p is the quantum noise parameter), and then disappears in the range of $0.622 < p \leq 1$. Increasing the amount of quantum noise leads to the reduction of the quantum player's payoff.

Keywords: entanglement, quantum game, quantum noise, Nash equilibrium
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1. Introduction

Game theory has been widely used in economics, biology and social sciences to model decision making in conflicting situations. Recent development in quantum computation and quantum information theory[1−5] prompts the scope of game theory to extend to the quantum world. Actually, a variety of problems in quantum computation and quantum information, such as quantum cryptography, quantum cloning and quantum algorithms, can be regarded as quantum games.[6] A quantum game model has been developed by Eisert[7] in which the paradox in the classical two-prisoner dilemma can be solved in a maximal entangled state. The properties of the two-prisoner dilemma game in different degrees of entanglement have been presented in Ref.[8]. The investigations of multi-player and multi-choice quantum games,[6,9,10] and continuous-variable quantum games[11] have been pursued recently. The quantum games are also realized on nuclear magnetic resonance (NMR) quantum computers.[12]

However, in any realistic quantum information processing system, the quantum noise from the environment cannot be entirely eliminated. In Ref.[13], Chen *et al* discussed quantum noise in quantum dilemma games. Quantum noise is found to have no effect on the Nash equilibrium in this model. In Ref.[14], Flitney and Abbott discussed the effect of quantum noise on the restricted quantum game and other types of quantum games. They have found that the loss of advantage to the quantum player is very similar to what occurs when the degree of entanglement between the players' qubits is reduced. However, all the above work focused on a 2-player quantum game(within a 2-qubit system). The effect of quantum noise on multiplayer quantum games is little studied. In this paper, we consider a 3-player quantum prisoner dilemma under quantum noise. We find that in a maximally entangled state, a special Nash equilibrium exists in the range of $0 \leq p \leq 0.622$ (p is the quantum noise parameter), and then disappears in the range of $0.622 < p \leq 1$. Increasing the amount of quantum noise leads to the reduction of the quantum player's payoff.

2. Quantum game under quantum noise

First, we recall the scheme of multiplayer quantum game presented in Ref.[9]. Classically, there are three players, and each player can choose two possible strategies: cooperate (C) and defect (D). The payoff

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table for the three players is shown in Table 1. From the table, one can see that strategy D is the dominant strategy for each player. Whatever other players do, each is better off defecting than cooperating. (D, D, D) is the unique Nash equilibrium of the game, with payoff (1,1,1). Unfortunately, this situation is worse than when they both choose C. (C, C, C) is a Pareto strategy optimal, which is regarded as the most efficient strategic profile of the game. The dilemma comes from the fact that Pareto optimal strategy is for both players to cooperate.

Table 1. The payoff matrix of the 3-player Prisoners' Dilemma.

		Colin C		Colin D	
		Bob		Bob	
		C	D	C	D
Alice	C	(3,3,3)	(2,5,2)	(2,2,5)	(0,4,4)
	D	(5,2,2)	(4,4,0)	(4,0,4)	(1,1,1)

Note: The first entry in the parenthesis denotes the payoff of Alice, the second number denotes the payoff of Bob, and the third number denotes the payoff of Colin.

In a quantum version, each player possesses a qubit $|C\rangle$ ($|D\rangle$). The qubits are prepared in the initial state $|C\rangle\,|C\rangle\,|C\rangle$ where the first, second and third entries refer to Alice's, Bob's and Colin's qubits, and then the entangling gate $\hat{J}$ acts on it. The initial state of the quantum game is $|\psi_i\rangle = \widehat{J}\,|CCC\rangle$, where $\hat{J}$ is $\exp(\mathrm{i}\frac{\gamma}{2}\hat{\sigma}_x \otimes \hat{\sigma}_x \otimes \hat{\sigma}_x)$, with $\gamma \in [0, \pi/2]$ measuring the entanglement of the state $|\psi_i\rangle$. Then the unitary $\hat{U}_A$, $\hat{U}_B$ and $\hat{U}_C$ are chosen from a 2-parameter set of unitary $2 \otimes 2$ matrices

$$\hat{U}(\theta, \phi) = \begin{pmatrix} \cos\theta/2 & \mathrm{e}^{\mathrm{i}\phi}\sin\theta/2 \\ -\mathrm{e}^{-\mathrm{i}\phi}\sin\theta/2 & \cos\theta/2 \end{pmatrix} \tag{1}$$

for all $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq \pi/2$, the strategies C and D can be associated with $U(0, 0)$ and $U(\pi, 0)$ respectively. Finally the disentangling gate $\widehat{J}^+$ is carried out and the system is measured in the computational basis. The game's final state can be written as

$$\begin{aligned} |\psi_{\mathrm{f}}\rangle =& |\psi_{\mathrm{f}}(\hat{U}_A, \hat{U}_B, \hat{U}_C)\rangle \\ =& \hat{J}^+(\hat{U}_A \otimes \hat{U}_B \otimes \hat{U}_C)\hat{J}|000\rangle. \end{aligned} \tag{2}$$

Therefore the payoff for Alice should be the expected payoff

$$\begin{aligned} \$_A = & 5P_{DCC} + 4(P_{DDC} + P_{DCD}) + 3P_{CCC} \\ & + 2(P_{CCD} + P_{CDC}) + 1P_{DDD} + 0P_{CDD}, \end{aligned} \tag{3}$$

where $P_{\sigma\sigma'\sigma''} = |\langle\sigma\sigma'\sigma''|\psi_{\mathrm{f}}\rangle|^2$ is the probability at which $|\psi_{\mathrm{f}}\rangle$ collapses into $|\sigma\sigma'\sigma''\rangle$. Payoff functions of Bob and Colin can be obtained through a similar analysis.

In order to consider the effect of noise in this quantum game, the noise is introduced before and after the players have adopted an appropriate strategy, which is shown in Fig.1. We suppose that all the players are affected by the same quantum noise at the same time.

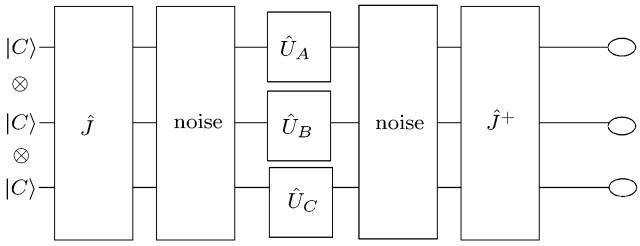


Fig.1. The model of multiplayer quantum game with quantum noise.

3. Calculation and discussion

It is convenient to use the density matrix notation for the state of the system and the operator-sum representation for the quantum operators. Phase damping is the ideal model of quantum noise that captures many of most important features of quantum noise occurring in quantum systems. Dephase can result in decoherence, which randomizes the relative phases of the quantum states, and dissipation that modifies the populations of the quantum states. The pure dephasing of a qubit can be expressed as

$$a|0\rangle + b|1\rangle \rightarrow a|0\rangle + b\ \mathrm{e}^{\mathrm{i}\phi}|1\rangle. \tag{4}$$

If we assume that the phase kick ϕ is a random variable with a Gaussian distribution of mean zero and variance 2λ, then the density matrix obtained after averaging over all values of ϕ is[1]

$$\begin{pmatrix} |a|^2 & a\bar{b} \\ \bar{a}b & |b|^2 \end{pmatrix} \rightarrow \begin{pmatrix} |a|^2 & a\bar{b}\,\mathrm{e}^{-\lambda} \\ \bar{a}b\,\mathrm{e}^{-\lambda} & |b|^2 \end{pmatrix}. \tag{5}$$

Over time, the random phase kicks cause an exponential decay of the off-diagonal elements of the density matrix. We use the operator-sum representation to model the decoherence as follows:

$$\rho' = \sum_{j=0}^{2} \mathcal{E}_j\, \rho\, \mathcal{E}_j^{\dagger}, \tag{6}$$

where $\mathcal{E}_0 = \sqrt{p}\ |0\rangle\langle 0|$, $\mathcal{E}_1 = \sqrt{p}\ |1\rangle\langle 1|$ and $\mathcal{E}_2 = \sqrt{1-p}\ \hat{I}$, in which p is the quantum noise parame-

ter, ρ is the density matrix of the system, and I is the unity matrix.

So the quantum game with quantum noise can be described, with the initial state specified by the density matrix $\rho_0 = |\psi_0\rangle\langle\psi_0|$, in the following manner:

$$\begin{aligned}
&\rho_1 = \hat{J}\rho_0\hat{J}^\dagger, && \text{(entanglement)},\\
&\rho_2 = D(\rho_1, p_1), && \text{(partial decoherence)},\\
&\rho_3 = (\hat{U}_A \otimes \hat{U}_B \otimes \hat{U}_C)\,\rho_2\,(\hat{U}_A \otimes \hat{U}_B \otimes \hat{U}_C)^\dagger, && \text{(players' moves)},\\
&\rho_4 = D(\rho_3, p_2), && \text{(partial decoherence)},\\
&\rho_5 = \hat{J}^\dagger\rho_4\hat{J}, && \text{(dis-entanglement)};
\end{aligned} \tag{7}$$

and the payoff to Alice is determined by Eq.(3).

In Ref.[9], Du *et al* have shown that in the maximally entangled game, the strategic profile $\mathrm{i}\hat{\sigma}_x \otimes \mathrm{i}\hat{\sigma}_x \otimes \mathrm{i}\hat{\sigma}_x$ is no longer the Nash equilibrium, but a special Nash equilibrium $\mathrm{i}\hat{\sigma}_y \otimes \mathrm{i}\hat{\sigma}_y \otimes \mathrm{i}\hat{\sigma}_y$ appears, and the strategic profile $\mathrm{i}\hat{\sigma}_y \otimes \mathrm{i}\hat{\sigma}_y \otimes \mathrm{i}\hat{\sigma}_y$ has the property of Pareto optimal strategy. The dilemma in the classical game is completely removed.

After introducing quantum noise in this quantum game, we consider a maximally entangled situation. First, we obtain the Alice's payoff as

$$\$_A(\hat{U}(\theta,\phi), \mathrm{i}\hat{\sigma}_y, \mathrm{i}\hat{\sigma}_y) = 2 + \frac{1}{2}\cos^2\frac{\theta_1}{2} - \frac{5}{2}(1-p_1)(1-p_2)\cdot\cos^2\frac{\theta_1}{2} + (1-p_1)(1-p_2)\sin^2\frac{\theta_1}{2}\cos 2\phi. \tag{8}$$

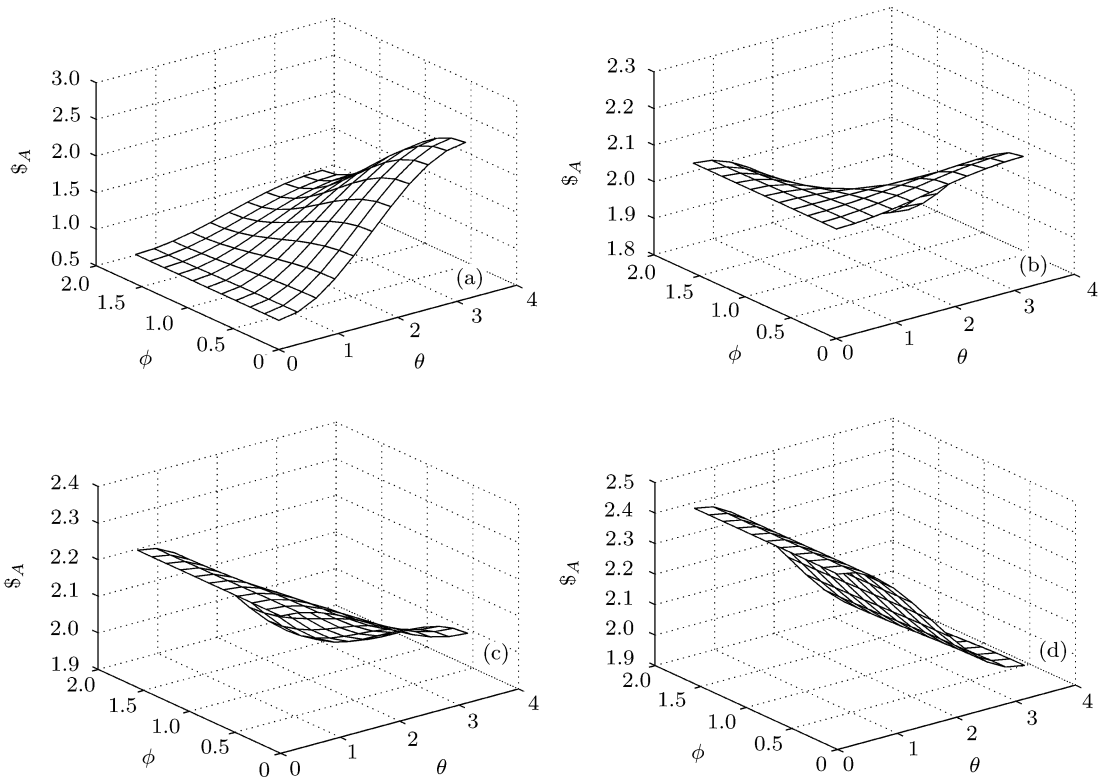


Fig.2. Alice's payoff as a function of the strategy $\theta_A(\theta_A \in [0,\pi])$ and ϕ_A $(\phi \in [0,\pi])$ at ascending values of the quantum noise parameter p of a dephase channel (a) $p = 0.2$, (b) $p = 0.6$, (c) $p = 0.7$, (d) $p = 0.9$.

Setting $p_1 = p_2 = 0$ yields the results of quantum game presented by Ref.[9]. Setting $p_1 = p_2 = p$, Fig.2 shows Alice's payoff as a function of θ and φ at ascending values of the decoherence parameter p. These figures show that as p increases (more noise), the quantum player's payoff becomes small. From Fig.2(a) and Fig.2(b), we find that Alice can obtain his maximally payoff at the point $\theta = \pi$, and $\varphi = 0$, which is corresponding to the operation $\mathrm{i}\sigma_y$. Then we also find that in the range of $0 \leq p \leq 0.622$, the best strategy for Alice is still $\mathrm{i}\sigma_y$. Because

$$\begin{aligned} \$_A(\hat{U}(\theta,\phi), \mathrm{i}\hat{\sigma}_y, \mathrm{i}\hat{\sigma}_y) &\leq \$_A(\mathrm{i}\hat{\sigma}_y, \mathrm{i}\hat{\sigma}_y, \mathrm{i}\hat{\sigma}_y) \\ &= 2 + (1-p_1)(1-P_2), \end{aligned} \tag{9}$$

for all $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq \pi/2$. Similarly,

$$\begin{aligned} &\$_B(\mathrm{i}\hat{\sigma}_y, \hat{U}(\theta,\phi), \mathrm{i}\hat{\sigma}_y) \leq \$_B(\mathrm{i}\hat{\sigma}_y, \mathrm{i}\hat{\sigma}_y, \mathrm{i}\hat{\sigma}_y); \\ &\$_C(\mathrm{i}\hat{\sigma}_y, \mathrm{i}\hat{\sigma}_y, \hat{U}(\theta,\phi)) \leq \$_C(\mathrm{i}\hat{\sigma}_y, \mathrm{i}\hat{\sigma}_y, \mathrm{i}\hat{\sigma}_y). \end{aligned} \tag{10}$$

Hence, no player can improve his individual payoff by unilaterally deviating from strategy $\mathrm{i}\hat{\sigma}_y \otimes \mathrm{i}\hat{\sigma}_y \otimes \mathrm{i}\hat{\sigma}_y$, which is a Nash equilibrium. However in the range of $0.622 < p \leq 1$, we find that

$$\$_A(\hat{U}(\theta,\phi), \mathrm{i}\hat{\sigma}_y, \mathrm{i}\hat{\sigma}_y) \leq \$_A(I, \mathrm{i}\hat{\sigma}_y, \mathrm{i}\hat{\sigma}_y), \tag{11}$$

as shown in Figs.2(c) and 2(d). The best strategy for Alice is I. The special Nash equilibrium disappears.

4. Conclusion

As multipartite physical systems tend to be complex, multiplayer quantum games may be more complicated and interesting than 2-player games.[6] In this paper, we have investigated the effect of quantum noise on a multiplayer quantum game with a certain strategic space. In a maximally entangle state, we find that a special Nash equilibrium appears in the range of $0 \leq p \leq 0.622$ (p is the quantum noise parameter), and then disappears in the range of $0.622 < p \leq 1$. Increasing the amount of quantum noise leads to the reduction of the quantum player's payoff.

References

[1] Nielsen M and Chuang I L 2000 *Quantum Computation and Quantum Information* (Cambridge: Cambridge University Press)

[2] Wu Y and Yang X 2005 *Phys. Rev.* A **71** 053806

[3] Wu Y, Payne M G, Hagley E W *et al* 2004 *Phys. Rev.* A **70** 063812
Yang X and Wu Y 2005 *J. Opt.* B: *Quantum Semiclass. Opt.* **7** 54

[4] Cao S and Fang M F 2006 *Chin. Phys.* **15** 276
Cao S and Fang M F 2006 *Chin. Phys.* **15** 60

[5] Li G X and Wu S P 2005 *Phys. Rev.* A **72** 064304

[6] Benjamn S C, Hayden P M 2001 *Phys. Rev.* A **64** 030301

[7] Eisert J, Wilkens M and Lewenstein M 1999 *Phys. Rev. Lett.* **83** 3077

[8] Du J F, Li H, Xu X *et al* 2001 *Phys. Lett.* A **289** 9

[9] Du J F, Li H, Xu X *et al* 2002 *Phys. Lett.* A **302** 229

[10] Du J F, Li H, Xu X *et al* 2002 *Chin. Phys. Lett.* **19** 1221

[11] Li H, Du J F and Massar S 2002 *Phys. Lett.* A **306** 73

[12] Du J F, Li H, Xu X, Shi M, Wu J and Han R 2002 *Phys. Rev. Lett.* **88** 137902

[13] Chen K L, Ang H, Kiang D *et al* 2003 *Phys. Lett.* A **316** 317

[14] Flitney A P and Abbott D 2005 *J. Phys.* A: *Math. Gen.* **38** 449