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Fast scheme for generating quantum-interference states and GHZ state of N trapped ions*

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We propose a fast scheme to generate the quantum-interference states of N trapped ions. In the scheme the ions are driven by a standing-wave laser beam whose carrier frequency is tuned such that the ion transition can take place. We also propose a simple and fast scheme to produce the GHZ state of N hot trapped ions and this scheme is insensitive to the heating of vibrational motion, which is important from the viewpoint of decoherence.

Keywords: quantum-interference states, trapped ions, the GHZ state
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The quantum superposition principle[1] plays a fundamental role in quantum mechanics,[2] Its striking consequences are particularly embodied in the superposition of coherence states. Although these superposition states are quantum states closest to classical ones, they show highly nonclassical properties, such as squeezing,[3] sub-Poissonian[4] and oscillating number statistics, due to the quantum interference between the coherent components.[5]

The advances in ion cooling and trapping have opened new prospects for studying the superposition of coherence states. Thus the generation of macroscopic quantum interference states(the so-called the Schrödinger-cat states[6]) via trapped ions attracts extensive attention. Single-mode Schrödinger-cat states of single ion or more ions have been theoretically investigated.[7−11] Various schemes for producing two-mode Schrödinger-cat states of single ionic vibrational motion have been proposed.[11−15]

In this paper, we propose a simple and efficient scheme for generating single-mode Schrödinger-cat states for the motion of N trapped ions in a fast and controllable way. The distinct feature of the scheme is that it is easy to manipulate experimentally. Firstly, the number N of the trapped ions in principle may be an arbitrary integer, which enables us to generate various superpositions of coherent states, while of previous contributions, the majority are devoted only to a single trapped ion. Secondly, in our scheme, the ions do not need to be exactly located at the node of the standing wave,[16,17] which is easy to control experimentally, while in the Ref.[12], the centre of the trap must be located at the node of the standing wave. Thirdly, the time required to generate the desired state is determined only by the laser intensity but not by off-resonant excitations, which is very significative to generating bigger superposition states and enhancing the operation speed, while in Ref.[12] the operation time is very long because of the high detuning limit to the ion-laser interaction. In addition, in this paper, we can also prepare GHZ state and our scheme is insensitive to the heating of the vibrational mode.

We consider N two-level ions confined in a linear trap. We simultaneously drive the N ions with a standing-wave laser field, tuned to the ionic carrier frequency ω_0. The total Hamiltonian of the system may be written as

$$H = H_0 + H_{\rm int}, \tag{1}$$

with

$$H_0 = \nu b^+ b + \omega_0 \sum_{i=1}^{N} \frac{\sigma_{z,i}}{2} + \sum_{\lambda=1}^{N-1} \nu_\lambda a_\lambda^+ a_\lambda, \tag{2}$$

and

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$$H_{\rm int} = \sum_{i=1}^{N} \Omega_i \sin(kx_i + \theta_i) \times (\mathrm{e}^{-\mathrm{i}\omega_0 t}\sigma_{+,i} + \mathrm{e}^{\mathrm{i}\omega_0 t}\sigma_{-,i})], \quad (3)$$

where operators b and a_λ (b^+ and a_λ^+) are the annihilation (creation) operators associated with the centre-of-mass mode of frequency ν and with the $N-1$ other vibrational modes of frequency ν_λ, respectively. The operators $\sigma_{z,i} = |e\rangle_i\langle e| - |g\rangle_i\langle g|$; $\sigma_{+,i} = |e\rangle_i\langle g|$ and $\sigma_{-,i} = |g\rangle_i\langle e|$, with $|e\rangle_i$ and $|g\rangle_i$ being the excited and ground states of the ith ion.

For simplicity, we assume that Rabi frequency Ω_i characterizing ion-laser interaction strength is the same for all ions, i.e. $\Omega_i = \Omega$, $i = 1, 2, \ldots, N$. We assume that the positions of the centres of all ions relative to a node of the laser field are the same, which are characterized by θ, i.e. $\theta_i = \theta$, $i = 1, 2, \ldots, N$. We may also assume that only the centre-of-mass motion will be excited, given that the next eigenfrequency is $\nu_{\rm r} = \sqrt{3}\nu$, corresponding to the stretch mode. The following frequencies($\geq \sqrt{29/5}\nu$) depend on the number of ions and have been calculated in Ref.[18]. Thus in a frame rotation at the laser frequency and in the Lamb–Dicke limit, the Hamiltonian for this system is given by[19]

$$H^{\rm L} = \nu b^+ b + \Omega[\eta(b + b^+)\cos\theta + \sin\theta]J_x, \quad (4)$$

where $J_x = \sum_{i=1}^{N} \sigma_{x,i}$ is the collective spin operator, and η is the Lamb–Dicke parameter. We define[17]

$$\begin{aligned} H^{\rm L} &= H_0^{\rm L} + H_{\rm int}^{\rm L}, \\ H_0^{\rm L} &= \nu b^+ b + \Omega \sin\theta J_x, \\ H_{\rm int}^{\rm L} &= \Omega\eta(b^+ + b)\cos\theta J_x. \end{aligned} \quad (5)$$

In an interaction picture the interaction Hamiltonian is described by

$$H^{\rm I} = \Omega\eta\cos\theta(b\mathrm{e}^{-\mathrm{i}\nu t} + b^+\mathrm{e}^{\mathrm{i}\nu t})J_x. \quad (6)$$

The evolution operator for Hamiltonian $H^{\rm I}$ can be written in the form of[20,21]

$$U(t) = \mathrm{e}^{A(t)J_x^2}\mathrm{e}^{B(t)J_x b}\mathrm{e}^{C(t)J_x b^+}. \quad (7)$$

Using the Schrödinger equation

$$\mathrm{i}\frac{\mathrm{d}U(t)}{\mathrm{d}t} = H^{\rm I}U(t), \quad (8)$$

we obtain

$$\begin{aligned} B(t) &= -\mathrm{i}\eta\Omega\cos\theta\int_0^t \mathrm{e}^{-\mathrm{i}\nu t'}\mathrm{d}t' = \frac{\Omega\eta\cos\theta}{\nu}(\mathrm{e}^{-\mathrm{i}\nu t} - 1), \\ C(t) &= -\mathrm{i}\eta\Omega\cos\theta\int_0^t \mathrm{e}^{\mathrm{i}\nu t'}\mathrm{d}t' = \frac{-\Omega\eta\cos\theta}{\nu}(\mathrm{e}^{\mathrm{i}\nu t} - 1), \\ A(t) &= \mathrm{i}\eta\Omega\cos\theta\int_0^t \mathrm{e}^{\mathrm{i}\nu t'}B(t')\mathrm{d}t' = \mathrm{i}\Omega^2\eta^2\cos^2\theta\left[\frac{t}{\nu} - \frac{1}{\mathrm{i}\nu^2}(\mathrm{e}^{\mathrm{i}\nu t} - 1)\right]. \end{aligned} \quad (9)$$

Apply the Baker–Campbell–Hausdorff relation $\mathrm{e}^x\mathrm{e}^y = \mathrm{e}^{x+y}\mathrm{e}^{\frac{[x,y]}{2}}$ to Eq.(7), and we will obtain another form of $U(t)$:

$$U(t) = \mathrm{e}^{\left[A(t) - \frac{|\alpha|^2}{2}\right]J_x^2}\mathrm{e}^{(\alpha b^+ - \alpha^* b)J_x}, \quad (10)$$

where $\alpha = C(t)$.

From Eq.(10), we easily show that in the interaction picture, the time evolution operator at time t is a sum of products of unitary operators on the motional states and projection operators on the ion internal states

$$U(t) = \sum_{j,m} \mathrm{e}^{\left[A(t) - \frac{|\alpha|^2}{2}\right]4m^2} \times D(2m\alpha)|j,m\rangle_x\langle j,m|, \quad (11)$$

where $D(\alpha) = \mathrm{e}^{\alpha b^+ - \alpha^* b}$ is the displacement operator which generates , when acting on the ground state $|0\rangle$, coherent states of the vibrational motion of the centre of mass[22]

$$|\alpha\rangle = D(\alpha)|0\rangle = \mathrm{e}^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle. \quad (12)$$

$|j,m\rangle_x$ are the simultaneous eigenvectors of the operators J_x and $J^2 = J_x^2 + J_y^2 + J_z^2$. And the corresponding eigenvalues of two operators are $m = -j, -(j-1), \cdots, j$ and $j(j+1)$, respectively. j varies from 0 $\left(\frac{1}{2}\right)$ to $\frac{N}{2}$ in steps of 1, if N is even (odd).

We choose that the initial state of each ion is $|g\rangle_j$ and the external state of the ions is $|0\rangle$, then the initial

state of the system is

$$|\varphi(0)\rangle = |gg\cdots g\rangle\otimes|0\rangle = \left|\frac{N}{2},-\frac{N}{2}\right\rangle_z \otimes |0\rangle. \quad (13)$$

After an interaction time t with the laser field, the state of the system evolves into

$$|\varphi(t)\rangle = \sum_{m,m'} \mathrm{e}^{\left[A(t)-\frac{|\alpha|^2}{2}\right]4m^2} d^{\frac{N}{2}}_{m',m}\left(\frac{\pi}{2}\right) d^{\frac{N}{2}}_{m,-\frac{N}{2}} \times\left(-\frac{\pi}{2}\right)\left|\frac{N}{2},m'\right\rangle_z \otimes |2m\alpha\rangle, \quad (14)$$

where $d^j_{m',m}(\phi) =_z \langle j,m'|\mathrm{e}^{-\mathrm{i}\phi J_y}|j,m\rangle_z$ are matrix elements of the rotation operator along the y-axis in the $|j,m\rangle_z$ basis of the eigenstates of J^2 and J_z.[23]

From Eq.(14), we easily prepare a mesoscopic superposition of vibrational quantum states for N trapped ions. Now we measure the internal state of the ions, if the ions are in the internal state $|ee\ldots\rangle$, then the motional state of the ions is[24]

$$|\psi(t)\rangle = \sum_{m=-\frac{N}{2}}^{\frac{N}{2}} \mathrm{e}^{\left[A(t)-\frac{|\alpha|^2}{2}\right]4m^2} \times\frac{(-1)^{\frac{N}{2}-m}}{\left(\frac{N}{2}-m\right)!\left(\frac{N}{2}+m\right)!}|2m\alpha\rangle. \quad (15)$$

If N is even, this state is a superposition of the vacuum and a series of coherent states. For example, we choose $N=2$, then we obtain

$$|\psi(t) = \frac{1}{4}\left[\mathrm{e}^{4A(t)-2|\alpha|^2}(|2\alpha\rangle + |-2\alpha\rangle) - 2|0\rangle\right], \quad (16)$$

which is a triple mesoscopic mode superposition state. If N is odd, there is no vacuum state in the superposition. For example, we choose $N=1$, then we obtain

$$|\psi(t)\rangle = N_e(|\alpha\rangle - |-\alpha\rangle), \quad (17)$$

where N_e is a normalization factor. This is a single odd coherent state in the vibrational mode. When $N \geq 3$, from the time evolution operator of Eq.(11), we can obtain more sophisticated and bigger microscopic-mesoscopic entangled states and the vibrational mode superposition states.

Next we prepare a Greenberger–Horne–Zeilinger state.[25] We define the new atomic basis[26,27] as

$$|+\rangle_j = \frac{1}{\sqrt{2}}(|e\rangle_j + |g\rangle_j),$$
$$|-\rangle_j = \frac{1}{\sqrt{2}}(|e\rangle_j - |g\rangle_j). \quad (18)$$

Then we can rewrite $U(t)$ as

$$U(t) = \mathrm{e}^{4\left[A(t)-\frac{|\alpha|^2}{2}\right]S_z^2}\mathrm{e}^{2(\alpha b^+ - \alpha^* b)S_z}, \quad (19)$$

where $S_z = \frac{1}{2}\sum_{j=1}^{N}(|+\rangle_j\langle+| - |-\rangle_j\langle-|)$. If we choose the interaction time t to satisfy the condition $t=\tau=\frac{2m\pi}{\nu}$, where m is a positive integer, then we have $\alpha(\tau)=0$, and $A(\tau) = \frac{\mathrm{i}\eta^2\Omega^2\cos^2\theta 2m\pi}{\nu^2}$. Thus the time evolution operator reduces to

$$U(\tau) = \mathrm{e}^{\mathrm{i}\chi S_z^2}, \quad (20)$$

where $\chi = -\mathrm{i}4A(\tau)$. From Eq.(20), it is easy to see that, at the time τ, the internal states and the collective vibrational mode for N ions are disentangled. Thus the scheme allows the vibrational mode to be in any state, for example, a thermal state.

We define the symmetrical states (the Dicke state[28] of the ionic system) as follows:[29]

$$|k\rangle = \binom{N}{k}^{-1/2}\sum\prod_{l=1}^{k}|+\rangle_{j_l}\otimes\prod_{j\neq j_1,\ldots,j_k}|-\rangle_j. \quad (21)$$

Here k denotes the number of ions in the states $|+\rangle_j$ and the sum is taken over permutations of the ions. Apply the collective operator S_z to the Dicke state $|k\rangle$, then we will obtain[29,30]

$$S_z|k\rangle = \left(k-\frac{N}{2}\right)|k\rangle, \quad (22)$$

where $k=0,1,\ldots,N$.

We now assume the each ion to be initially in state $|e\rangle_j$. $|e\rangle_j$ can be rewritten as

$$|e\rangle_j = \frac{1}{\sqrt{2}}(|+\rangle_j + |-\rangle_j). \quad (23)$$

Then the initial internal state for the N ions can be written as a Bloch state[31]

$$|\varphi(0)\rangle = \frac{1}{\sqrt{2^N}}\sum_{k=0}^{N}\binom{k}{N}^{1/2}|k\rangle. \quad (24)$$

Using Eq.(20), we obtain the evolution state of the system at an interaction time τ

$$|\varphi(\tau)\rangle = \frac{1}{\sqrt{2^N}}\sum_{k=0}^{N}\mathrm{e}^{\mathrm{i}\chi\left(k^2+\frac{N^2}{4}-kN\right)}\binom{k}{N}^{1/2}|k\rangle. \quad (25)$$

If we choose $\chi=\frac{\pi}{2}$, such that the parameters Ω, η, θ and ν satisfy $\cos\theta = \frac{\nu}{4\Omega\eta\sqrt{m}}$, then we can obtain the state[32,33]

$$|\varphi(\tau)\rangle = \frac{1}{\sqrt{2^{N+1}}}\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{8}N^2+\frac{\pi}{4}\right)}\left\{\prod_{j=1}^{N}[(-\mathrm{i})^N|+\rangle_j + |-\rangle_j] -\mathrm{i}\prod_{j=1}^{N}[-(-\mathrm{i})^N|+\rangle_j + |-\rangle_j]\right\}. \quad (26)$$

Since $[(-\mathrm{i})^N|+\rangle_j + |-\rangle_j]$ is orthogonal to $[-(-\mathrm{i})^N|+\rangle_j + |-\rangle_j]$, $|\varphi(\tau)\rangle$ is an N-particle maximally entangled state or a GHZ state.

We will discuss the experimental feasibility of the proposed scheme. Let $\Omega = 0.8\nu$ and $\eta = 0.07$, then $\delta = \nu - \Omega = 0.2\nu$.[34] The next eigenfrequency corresponding to the stretch mode is $\nu_r = \sqrt{3}\nu$. Then the detuning between the Rabi frequency and this vibrational frequency is $|\delta_r| = |\Omega - \nu_r| \approx 0.93\nu >> \delta$. Thus it is reasonable to neglect the stretch mode. In this paper, the amplitude of the coherent states in the obtainable superpositions is restricted by the Lamb–Dicke limitation $\eta|2m\alpha| << 1$.[8] The amplitude of the coherent states is $|(4m\eta\Omega/\nu)\cos\theta\sin(\nu t/2)|$, where $m = \{-N/2, N/2\}$. However, we can obtain a superposition of the coherent states with a large mean phonon number if a laser with an appropriate intensity is available and the Lamb–Dicke parameter is small enough. Suppose that we want to generate a superposition of the coherent states with a maximal amplitude $N|\alpha| = 2$, where N is the number of the ions. In this case $\eta N|\alpha| = 0.14$ and thus the Lamb–Dicke limitation is satisfied. We can produce the desired superposition state by choosing $|\cos\theta\sin(\nu t/2)| = 1$, and $N = 18$. The interaction time is determined by the value of $\sin(\nu t/2)$. We can choose an appropriate frequency ν to obtain a shorter interaction time.

In conclusion, we have presented a fast procedure to produce the quantum-interference states for the collective motion of N trapped ions using a single standing-wave field resonant with the ionic carrier frequency. The distinct feature of the scheme is that the number N of the trapped ions in principle may be an arbitrary integer, the required time is only limited by the available laser intensity. In addition, note that as long as the field is spatially homogeneous over the trapped ions, the only parameter that depends on N, in our model, is the time when we apply the laser pulse. Thus our scheme should help build a larger mesoscopic quantum superposition in trapped ions, and study experimentally decoherence processes and the applications to quantum information processing.

Furthermore we have proposed an alternative fast scheme to prepare the GHZ state including N ions and our scheme is insensitive to the heating of the vibrational mode.

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