Vol 16 No 4, April 2007
1009-1963/2007/16(04)/0897-4

Chinese Physics

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Exact solution to the one-dimensional Dirac equation of linear potential*

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(Received 5 April 2006; revised manuscript received 7 August 2006)

In this paper the one-dimensional Dirac equation with linear potential has been solved by the method of canonical transformation. The bound-state wavefunctions and the corresponding energy spectrum have been obtained for all bound states.

Keywords: Dirac equation，linear potential，exact solution
PACC: 0365

1. Introduction

It is well known that the problems which can be exactly solved in quantum mechanics are very rare. Therefore, it is very important to find more potentials which can be fully determined.[1−26] Recently, considerable interest has been devoted to the one-dimensional Schrödinger equation[27−29] and one-dimensional Dirac equation[30−32] for the time-dependent linear potential and the corresponding wavefunctions. The three-dimensional Dirac equation[33] and Klein–Gordon equation[34] with time-independent linear potential have been studied and the bound-state exact solutions have been given. However, in the one-dimensional case, the exact solution of Dirac equation with time-independent linear potential is still absent. In the present work, we will extend the method in Ref.[33] to the one-dimensional Dirac equation with the time-independent linear potential. The results show that by introducing a canonical transformation and operating the operator $\gamma_3\frac{\mathrm{d}}{\mathrm{d}z}$ on the Dirac equation, the equations satisfied by each component of the Dirac spinor wavefunction after the canonical transformation becomes the Schrödinger equation of harmonic-oscillator potential. For this reason, the wavefunctions and corresponding energy spectrum can be easily found by the standard method of solving the Schrödinger equation with harmonic-oscillator potential.

2. The exact solution of Dirac equation with linear potential

In the one-dimensional case, the Dirac equation[35] with potential $V(z)$ is ($\hbar = c = 1$)

$$\{\gamma_3\partial_z + \gamma_4\partial_t + m + V(z)\}\psi(z,t) = 0, \tag{1}$$

where γ_3 and γ_4 is a 4×4 Dirac matrix and $\psi(z,t) = (\psi_1, \psi_2, \psi_3, \psi_4)$ is a Dirac spinor four component wavefunction.

Letting

$$\psi(z,t) = \varphi(z)\mathrm{e}^{-\mathrm{i}Et}, \quad \varphi(z) = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{pmatrix} \tag{2}$$

and Substituting Eq.(2) into Eq.(1), we can obtain

$$\gamma_3\frac{\mathrm{d}}{\mathrm{d}z}\varphi(z) = \{\gamma_4 E - m - V(z)\}\varphi(z). \tag{3}$$

Operating on the both sides of Eq.(3) with $\gamma_3\frac{\mathrm{d}}{\mathrm{d}z}$ we get

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}\varphi(z) = \left\{ -E^2 + [m + V(z)]^2 - \gamma_3\frac{\mathrm{d}V(Z)}{\mathrm{d}z}\right\}\varphi(z). \tag{4}$$

*Project supported by the National Natural Science Foundation (Grant No 10347003, 50375031) and the Natural Science Foundation of Guizhou Province.
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Introducing the following canonical transformation:

$$\varphi(z) = U \begin{pmatrix} \bar{\varphi}_1(z) \\ \bar{\varphi}_2(z) \\ \bar{\varphi}_3(z) \\ \bar{\varphi}_4(z) \end{pmatrix}, \quad U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ \mathrm{i} & -\mathrm{i} & 0 & 0 \\ 0 & 0 & \mathrm{i} & -\mathrm{i} \end{pmatrix}, \quad (5)$$

we can rewrite Eq.(4) in the form

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}z^2}(\bar{\varphi}_1 + \bar{\varphi}_2) = & \{-E^2 + [m + V(z)]^2\}(\bar{\varphi}_1 + \bar{\varphi}_2) \\ & - \frac{\mathrm{d}V(z)}{\mathrm{d}z}(\bar{\varphi}_1 - \bar{\varphi}_2), \\ \frac{\mathrm{d}^2}{\mathrm{d}z^2}(\bar{\varphi}_3 + \bar{\varphi}_4) = & \{-E^2 + [m + V(z)]^2\}(\bar{\varphi}_3 + \bar{\varphi}_4) \\ & - \frac{\mathrm{d}V(z)}{\mathrm{d}z}(\bar{\varphi}_4 - \bar{\varphi}_3), \\ \frac{\mathrm{d}^2}{\mathrm{d}z^2}(\bar{\varphi}_1 - \bar{\varphi}_2) = & \{-E^2 + [m + V(z)]^2\}(\bar{\varphi}_1 - \bar{\varphi}_2) \\ & - \frac{\mathrm{d}V(z)}{\mathrm{d}z}(\bar{\varphi}_1 + \bar{\varphi}_2), \\ \frac{\mathrm{d}^2}{\mathrm{d}z^2}(\bar{\varphi}_3 - \bar{\varphi}_4) = & \{-E^2 + [m + V(z)]^2\}(\bar{\varphi}_3 - \bar{\varphi}_4) \\ & - \frac{\mathrm{d}V(z)}{\mathrm{d}z}(\bar{\varphi}_4 + \bar{\varphi}_3). \end{aligned} \quad (6)$$

Evidently, Eq.(6) is a coupled differential equation for the four component wavefunction $\bar{\varphi}_1, \bar{\varphi}_2, \bar{\varphi}_3$ and $\bar{\varphi}_4$. Uncoupling Eq.(6), we obtain

$$\begin{aligned} \frac{\mathrm{d}^2\bar{\varphi}_1}{\mathrm{d}z^2} &= \left\{-E^2 + [m + V(z)]^2 - \frac{\mathrm{d}V(z)}{\mathrm{d}z}\right\}\bar{\varphi}_1, \\ \frac{\mathrm{d}^2\bar{\varphi}_2}{\mathrm{d}z^2} &= \left\{-E^2 + [m + V(z)]^2 + \frac{\mathrm{d}V(z)}{\mathrm{d}z}\right\}\bar{\varphi}_2, \\ \frac{\mathrm{d}^2\bar{\varphi}_3}{\mathrm{d}z^2} &= \left\{-E^2 + [m + V(z)]^2 + \frac{\mathrm{d}V(z)}{\mathrm{d}z}\right\}\bar{\varphi}_3, \\ \frac{\mathrm{d}^2\bar{\varphi}_4}{\mathrm{d}z^2} &= \left\{-E^2 + [m + V(z)]^2 - \frac{\mathrm{d}V(z)}{\mathrm{d}z}\right\}\bar{\varphi}_4. \end{aligned} \quad (7)$$

Now we consider a model where the potential is the time-independent linear potential $V(z) = kz$ (k is a constant), and from Eq.(7) one can easily get

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + \alpha_1 - \xi^2\right\} f_1 = 0, \left\{\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + \alpha_2 - \xi^2\right\} f_2 = 0, \quad (8)$$

where $\xi = \sqrt{k}\left(z + \frac{m}{k}\right), \alpha_1 = \dfrac{E^2 + k}{k}$, $\alpha_2 = \dfrac{E^2 - k}{k}$, f_1 stands for $\bar{\varphi}_1$ or $\bar{\varphi}_4$, and f_2 stands for $\bar{\varphi}_2$ or $\bar{\varphi}_3$. It is very clear that Eq.(8) is just the same as the Schrödinger equation of harmonic oscillator. Comparing with the well-known result of non-relativistic harmonic oscillator, one immediately obtain the energy spectrum

$$E_n = \pm\sqrt{2(n+1)k}\,, \quad n = 0, 1, 2, \ldots \quad (9)$$

and the two functions (f_1 and f_2) expressed in the form

$$\begin{aligned} f_1(\xi) &= H_{n+1}(\xi)\exp\left(-\frac{1}{2}\xi^2\right), \\ f_2(\xi) &= H_n(\xi)\exp\left(-\frac{1}{2}\xi^2\right). \end{aligned} \quad (10)$$

Where $H_n(\xi)$ is the Hermite polynomial. With the help of Eq.(4) and Eq.(10), we can write $\varphi(\xi)$ in the following form:

$$\begin{aligned} \varphi(\xi) = & \frac{1}{\sqrt{2}} \begin{pmatrix} c_1 H_{n+1}(\xi) + c_2 H_n(\xi) \\ c_3 H_n(\xi) + c_4 H_{n+1}(\xi) \\ \mathrm{i}c_1 H_{n+1}(\xi) - \mathrm{i}c_2 H_n(\xi) \\ \mathrm{i}c_3 H_n(\xi) - \mathrm{i}c_4 H_{n+1}(\xi) \end{pmatrix} \\ & \times \exp\left(-\frac{1}{2}\xi^2\right), \end{aligned} \quad (11)$$

where c_1, c_2, c_3 and c_4 are four parameters.

However, it must be pointed out that the $\varphi(\xi)$ in Eq.(11) is the solution of Eq.(4), but not of Eq.(3), because the two sides of Eq.(3) have been operated by the operator $\gamma_3\dfrac{\mathrm{d}}{\mathrm{d}z}$. In order to obtain the solutions of Eq.(3), let us substitute Eq.(11) into Eq.(3) and make use of the property of Hermite polynomial, then we can obtain the following equations:

$$\begin{aligned} 2c_1(n+1)H_n(\xi) + c_2 H_{n+1}(\xi) &= \frac{E}{\sqrt{k}}c_1 H_{n+1}(\xi) + \frac{E}{\sqrt{k}}c_2 H_n(\xi), \\ 2c_4(n+1)H_n(\xi) + c_3 H_{n+1}(\xi) &= \frac{E}{\sqrt{k}}c_4 H_{n+1}(\xi) + \frac{E}{\sqrt{k}}c_3 H_n(\xi), \\ 2c_1(n+1)H_n(\xi) - c_2 H_{n+1}(\xi) &= -\frac{E}{\sqrt{k}}c_1 H_{n+1}(\xi) + \frac{E}{\sqrt{k}}c_2 H_n(\xi), \\ 2c_4(n+1)H_n(\xi) - c_3 H_{n+1}(\xi) &= -\frac{E}{\sqrt{k}}c_4 H_{n+1}(\xi) + \frac{E}{\sqrt{k}}c_3 H_n(\xi), \end{aligned} \quad (12)$$

Inserting E of Eq.(9) into Eq.(12) we get

$$\begin{aligned} c_2 &= \frac{2\sqrt{2}(n+1)}{E}c_1 = \sqrt{2(n+1)}c_1, \\ c_3 &= \frac{2\sqrt{2}(n+1)}{E}c_4 = \sqrt{2(n+1)}c_4, \text{ if } E>0, \\ c_2 &= \frac{2\sqrt{2}(n+1)}{E}c_1 = -\sqrt{2(n+1)}c_1, \text{ if } E<0. \\ c_3 &= \frac{2\sqrt{2}(n+1)}{E}c_4 = -\sqrt{2(n+1)}c_4, \end{aligned} \tag{13}$$

In other words, when the coefficients c_1, c_2, c_3 and c_4 satisfy Eq.(13), Eq.(11) must be the solution of Eq.(3). According to above analysis, the solutions of Eq.(3) can be written in the following forms.

When $c_2 = \sqrt{2(n+1)}c_1$ and $c_3 = \sqrt{2(n+1)}c_4$ (namely $E>0$),

$$\varphi(\xi) = \frac{1}{\sqrt{2}}\begin{pmatrix} c_1H_{n+1}(\xi) + c_1\sqrt{2(n+1)}H_n(\xi) \\ c_4\sqrt{2(n+1)}H_n(\xi) + c_4H_{n+1}(\xi) \\ \mathrm{i}c_1H_{n+1}(\xi) - \mathrm{i}c_1\sqrt{2(n+1)}H_n(\xi) \\ \mathrm{i}c_4\sqrt{2(n+1)}H_n(\xi) - \mathrm{i}c_4H_{n+1}(\xi) \end{pmatrix} \times \exp\left(-\frac{1}{2}\xi^2\right). \tag{14}$$

When $c_2 = -\sqrt{2(n+1)}c_1$ and $c_3 = -\sqrt{2(n+1)}c_4$ (namely $E<0$),

$$\varphi(\xi) = \frac{1}{\sqrt{2}}\begin{pmatrix} c_1H_{n+1}(\xi) - c_1\sqrt{2(n+1)}H_n(\xi) \\ -c_4\sqrt{2(n+1)}H_n(\xi) + c_4H_{n+1}(\xi) \\ \mathrm{i}c_1H_{n+1}(\xi) + \mathrm{i}c_1\sqrt{2(n+1)}H_n(\xi) \\ -\mathrm{i}c_4\sqrt{2(n+1)}H_n(\xi) - \mathrm{i}c_4H_{n+1}(\xi) \end{pmatrix} \times \exp\left(-\frac{1}{2}\xi^2\right). \tag{15}$$

In the Eqs.(14) and (15), there are two unknown coefficients c_1 and c_4, but we only have the normalization condition ($\int \varphi^+(\xi,t)\varphi(\xi,t)\mathrm{d}\xi = 1$). Therefore one of the two coefficients c_1 and c_4 can, in general, be arbitrarily chosen. For an example, if we choose one of the two coefficients c_1 and c_4 to be zero respectively, finally we will get the exact solutions of Dirac equation with time-independent linear potential as follows.

Case 1 When setting $c_4 = 0$ and $c_2 = \sqrt{2(n+1)}c_1$,

$$\psi(\xi,t) = \frac{1}{\sqrt{2}}c_1\begin{pmatrix} H_{n+1}(\xi) + \sqrt{2(n+1)}H_n(\xi) \\ 0 \\ \mathrm{i}H_{n+1}(\xi) - \mathrm{i}\sqrt{2(n+1)}H_n(\xi) \\ 0 \end{pmatrix} \times \exp\left(-\frac{1}{2}\xi^2\right)\exp(-\mathrm{i}Et). \tag{16}$$

Case 2 When setting $c_1 = 0$ and $c_3 = \sqrt{2(n+1)}c_4$,

$$\psi(\xi,t) = \frac{1}{\sqrt{2}}c_4\begin{pmatrix} 0 \\ \sqrt{2(n+1)}H_n(\xi) + H_{n+1}(\xi) \\ 0 \\ \mathrm{i}\sqrt{2(n+1)}H_n(\xi) - \mathrm{i}H_{n+1}(\xi) \end{pmatrix} \times \exp\left(-\frac{1}{2}\xi^2\right)\exp(-\mathrm{i}Et). \tag{17}$$

Case 3 When setting $c_4 = 0$ and $c_2 = -\sqrt{2(n+1)}c_1$,

$$\psi(\xi,t) = \frac{1}{\sqrt{2}}c_1\begin{pmatrix} H_{n+1}(\xi) - \sqrt{2(n+1)}H_n(\xi) \\ 0 \\ \mathrm{i}H_{n+1}(\xi) + \mathrm{i}\sqrt{2(n+1)}H_n(\xi) \\ 0 \end{pmatrix} \times \exp\left(-\frac{1}{2}\xi^2\right)\exp(-\mathrm{i}Et). \tag{18}$$

Case 4 When setting $c_1 = 0$ and $c_3 = -\sqrt{2(n+1)}c_4$,

$$\psi(\xi,t) = \frac{1}{\sqrt{2}}c_4\begin{pmatrix} 0 \\ -\sqrt{2(n+1)}H_n(\xi) + H_{n+1}(\xi) \\ 0 \\ -\mathrm{i}\sqrt{2(n+1)}H_n(\xi) - \mathrm{i}H_{n+1}(\xi) \end{pmatrix} \times \exp\left(-\frac{1}{2}\xi^2\right)\exp(-\mathrm{i}Et). \tag{19}$$

The two constants (c_1 and c_4) in Eqs.(16)–(19) can be determined by the normalization condition.

Finally, let us take the nonrelativistic limit of the solution of Dirac equation with linear potential. In the nonrelativistic limit ($\frac{\nu}{c} \ll 1$), $\alpha_1 = \alpha_2$ can be found from Eq.(8) if we do not use $c=1$, in the previous part of this work. So we have $f_1(\xi) = f_2(\xi) = H_n(\xi)\exp(-\xi^2/2)$, and Eq.(11) becomes

$$\varphi(\xi) = \frac{1}{\sqrt{2}}\begin{pmatrix} c_1 + c_2 \\ c_3 + c_4 \\ \mathrm{i}(c_1 - c_2) \\ \mathrm{i}(c_3 - c_4) \end{pmatrix} H_n(\xi)\exp(-\xi^2/2). \tag{20}$$

Substituting Eq.(20) into Eq.(3), we can obtain $c_2 = c_3 = 0$. Finally, the solution of nonrelativistic limit of Dirac equation with linear potential can be

written as

$$\varphi(\xi) = \frac{1}{\sqrt{2}} \begin{pmatrix} c_1 \\ c_4 \\ \mathrm{i}c_1 \\ -\mathrm{i}c_4 \end{pmatrix} H_n(\xi) \exp(-\xi^2/2). \quad (21)$$

It is well known that the solution of Schrödinger Equation is the Airy function for one-dimensional linear potential model. It is noted that the Dirac equation differs from the Schrödinger equation in nature though the nonrelativistic limit of the Dirac equation is the Schrödinger equation.[36,37] For example, in the three-dimensional case, the Schrödinger equation of linear potential leads to the bound-state solution, while the Dirac equation with vectorlike linear potential cannot give the bound-state solution due to the Klein paradox. Thus, the solution of the Dirac equation of linear potential, in the nonrelativistic limit case, cannot be directly reduced to the Airy function. Moreover, we have $n \to \infty(\alpha_1 = \alpha_2 = 2n+1)$ because of $\alpha_1 = \alpha_2 \to \infty$, in the nonrelativistic limit case. Hence from Eq.(9), we easily get

$$\frac{\mathrm{d}E_n}{\mathrm{d}n} = \pm\frac{k}{\sqrt{2(n+1)k}} \xrightarrow{n\to\infty} 0. \quad (22)$$

Equation (22) means that the energy spectrum is continuous.

3. Conclusion

In summary, in this paper, by introducing a canonical transformation and operating the operator $\gamma_3 \frac{\mathrm{d}}{\mathrm{d}z}$ on Eq.(3), we find that the equation satisfied by each component of Dirac spinor wavefunction after the canonical transformation is identical with the Schrödinger equation of harmonic-oscillator potential. So the exact solution of the one-dimensional Dirac equation with linear potential can be found by the standard procedure of solving the Schrödinger equation of harmonic oscillator. Its bound-state wavefunctions can be expressed by Hermite polynomial $H_n(\xi)$ and the corresponding energy spectrum has the form $E_n = \pm\sqrt{2(n+1)k}$ $(n=0, 1, 2, \ldots)$.

References

[1] Cftic H, Hall R L and Saad N 2005 *Phys. Rev.* A **72** 022101
[2] Alhaidari A D 2001 *Phys. Rev. Lett.* **87** 210405
[3] Alhaidari A D 2002 *Phys. Rev. Lett.* **88** 189901
[4] Hou C F, Li Y Z and Zhou Z X 1999 *Acta Phys. Sin.* **48** 1999 (in Chinese)
[5] Chen G and Lou Z M 2003 *Acta Phys. Sin.* **52** 1071(in Chinese)
[6] Chen C Y, Liu C L, Lu F L and Su D S 2003 *Acta Phys. Sin.* **52** 1579 (in Chinese)
[7] Chen G 2004 *Acta Phys. Sin.* **53** 0680 (in Chinese)
[8] Chen G 2004 *Acta Phys. Sin.* **53** 0684 (in Chinese)
[9] Lu F L and Chen C Y 2004 *Acta Phys. Sin.* **5**3 1652 (in Chinese)
[10] Li N, Ju D X and Ren Z Z 2005 *Acta Phys. Sin.* **54** 2520 (in Chinese)
[11] Chen Z D and Chen G 2005 *Acta Phys. Sin.* **54** 2524 (in Chinese)
[12] Zhang M G and Wang Z B 2006 *Acta Phys. Sin.* **55** 521 (in Chinese)
[13] Guo J Y 2002 *Acta Phys. Sin.* **51** 1453 (in Chinese)
[14] Hou C F, Zhou Z Q and Li Y 1999 *Chin. Phys.* **8** 561
[15] Liu D Y 1999 *Chin. Phys.* **8** 1
[16] Li L Q and Guo G C 1998 *Chin. Phys.* **7** 727
[17] Zhang T, Zhu A D, Jin Z and Zhao Y F 2003 *Chin. Phys.* **12** 1360
[18] Lu J 2004 *Chin. Phys.* **13** 811
[19] Liang M L and Sun Y J 2004 *Acta Phys. Sin.* **53** 3663 (in Chinese)
[20] Ran Y Q , Xue L H and Hu S. Z 2002 *Acta Phys. Sin.* **51** 2435 (in Chinese)
[21] Ma Y Q, Zhang J, Chen Y K and Dai H 2005 *Chin. Phys.* **14** 2170
[22] Zhang X A, Chen K and Duan Z L 2005 *Chin. Phys.* **14** 42
[23] Lu F L, Chen C Y and Sun D S 2005 *Chin. Phys.* **14** 463
[24] Chen G 2005 *Chin. Phys.* **14** 460
[25] Chen G 2005 *Chin. Phys.* **14** 1075
[26] Zhang M C and Wan Z B 2006 *Acta Phys. Sin.* **55** 525 (in Chinese)
[27] Guedes I 2001 *Phys. Rev.* A **63** 034102
[28] Feng M 2001 *Phys. Rev.* A **64** 034101
[29] Bekkar H, Benamira F and Maamache M 2002 *Phys. Rev.* A **68** 016101
[30] Maamache M, Bekkar H and Benamira F 2004 *Europhys. Lett.* **67**(5) 695
[31] Castro A S and Dutra S A 2003 *Phys. Rev.* A **67** 054101
[32] Landdim R R and Guedes I 2000 *Phys. Rev.* A **61** 054101
[33] Su R Kand Zhang Y H 1984 *J. Phys.* A **17** 851
[34] Chen G and Zhao D F 2003 *Acta Phys. Sin.* **52** 2954 (in Chinese)
[35] Baltin R and March N H 1987 *J. Phys.* A **20** 5517
[36] Gunion J F and Li L F 1975 *Phys. Rev.* D **27** 2910
[37] Critchfield C L 1976 *J. Math. Phys.* **17** 261