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Stability and vibration of a helical rod with circular cross section in a viscous medium*

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The stability and vibration of a thin elastic helical rod with circular cross section in a viscous medium are discussed. The dynamical equations of the rod in the viscous medium are established in the Frenet coordinates of the centreline with the Euler angles describing the attitudes of the cross section as variables. We have proved that the Lyapunov and Euler conditions of stability of a helical rod in the space domain are the necessary conditions for the asymptotic stability of the rod in the time domain. The free frequencies and damping coefficients of torsional and flexural vibrations of the helical rod in the viscous medium are calculated.

Keywords: thin elastic rod, viscous medium, torsional vibration, flexural vibration
PACC: 0320, 0340D

1. Introduction

The analysis of helical equilibrium and stability of a thin elastic rod plays an important role in studying DNA and other long polymer chains. The problems of equilibrium of the rod have been fully discussed on the basis of the Kirchhoff's kinetic analogy.[1] More recently, consideration has been given to the dynamics of elastic rod, and it is necessary to treat the rod as a system with dual arguments s and t. Klapper established dynamical equations of the rod, and presented a numerical algorithm to discuss the stability problem.[2] Goriely and Shipman discussed the stability of helical equilibrium of a rod numerically.[3] Liu *et al* discussed the dynamical stability and vibration of a helical rod with circular cross section.[4−7] In the investigation of dynamics of elastic rod in the background of DNA the environment of viscous medium should be taken into consideration.[8−10] The torsional vibration of an arbitrary planar rod and the planar flexural vibrations of a straight or annular rod in a viscous medium have been discussed.[11] In the present paper we discuss the dynamic stability and three-dimensional vibration of a helical rod with circular cross section in a viscous medium. The dynamical equations of the rod are established in the Frenet coordinates of the centreline with the Euler angles describing the attitudes of the cross section as variables. Even though the Lyapunov stability of helical equilibrium has been proved in static analysis, the rod is buckling when the axial load reaches an Euler's critical value.[6,7] This discrepancy results from a different understanding of the Lyapunov and Euler stabilities. It is shown that the Euler critical load is a particular value of the external force corresponding to the buckling and ensuring the satisfaction of boundary conditions on both ends of the rod. From the Lyapunov stability concepts for a discrete system with two arguments s and t,[4] the dynamical stability of the rod depends on the eigenvalues in time domain when the eigenvalues in space domain have been known in static analysis. We prove that the Lyapunov and Euler stability conditions of the rod in statics are the necessary conditions for the asymptotic stability in dynamics. When the asymptotic stability conditions are satisfied the free frequencies and the damping coefficients of torsional and flexural vibrations of the helical rod in a viscous medium can be determined by the eigenvalues of time domain.

2. Dynamical equations of elastic rod in viscous medium

Consider the motion of a thin elastic rod with circular cross section in a viscous medium. Under the Kirchhoff conditions the rod is homogeneous and

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isotropic, and its cross section is rigid and perpendicular to the centreline, which is not extensible and has no intrinsic curvature and twisting. Assume that the external forces acting on the end of the rod are simplified into a force screw composed of $\boldsymbol{F}_0$ and $\boldsymbol{M}_0$ along the ζ axis. Establish the inertial reference frame (O-$\xi\eta\zeta$) with a fixed point O as origin. Move the origin of (O-$\xi\eta\zeta$) to an arbitrary point P on the centreline, and denote a rotation ψ of (P-$\xi\eta\zeta$) about the ζ-axis by (P-$x_1y_1z_1$), and a rotation ϑ of (P-$x_1y_1z_1$) about the x_1-axis by (P-$x_2y_2z_2$), and a rotation φ of (P-$x_2y_2z_2$) about the z_2-axis by a body-fixed frame (P-xyz). Both (P-xyz) and (P-$x_2y_2z_2$) are the principal axes of the circular cross section. ψ, ϑ, φ are Euler's angles of the cross section relative to (P-$\xi\eta\zeta$) (Fig.1).

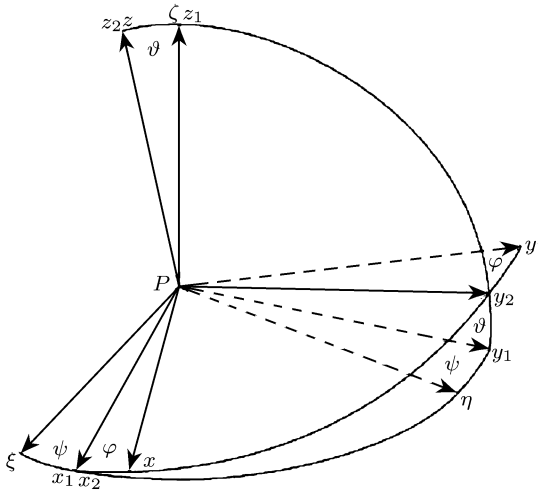


Fig.1. Euler's angles determining the attitude of cross section of the rod.

Define the arc-coordinate s along the centreline of the rod with the end point P_0 as origin. All the variables of the motion are functions of are-coordinate s and time t as dual arguments. Introduce $\boldsymbol{\omega}_{\mathrm{F}}$ and $\boldsymbol{\Omega}_{\mathrm{F}}$, $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$ as the derivatives of infinitesimal angular displacements of the frames (P-$x_2y_2z_2$) and (P-xyz) with respect to arc-coordinate s and time t, respectively, where ω is the curvature/twisting vector and Ω is the angular velocity of the cross section. Letting $\boldsymbol{r}$ be the position vector of point P relative to the fixed point O, $\boldsymbol{v} = \partial\boldsymbol{r}/\partial t$ be the velocity of P, $\boldsymbol{e}_3 = \partial\boldsymbol{r}/\partial s$ be the basis-vector along the tangent of centreline, and taking (P-$x_2y_2z_2$) as a reference frame of derivation, the kinematic relationship $\partial\boldsymbol{v}/\partial s = \partial\boldsymbol{e}_3/\partial t$ can be written as

$$\frac{\tilde{\partial}\boldsymbol{v}}{\partial s} + \boldsymbol{\omega}_{\mathrm{F}} \times \boldsymbol{v} = \boldsymbol{\Omega}_{\mathrm{F}} \times \boldsymbol{e}_3, \tag{1}$$

where the tilt denotes the local derivative with respect to (P-$x_2y_2z_2$). The dynamical equations of the rod can be derived from the theorems of linear and angular momenta as

$$\frac{\tilde{\partial}\boldsymbol{F}}{\partial s} + \boldsymbol{\omega}_{\mathrm{F}} \times \boldsymbol{F} + \boldsymbol{f} - \rho S\frac{\partial\boldsymbol{v}}{\partial t} = 0, \tag{2}$$

$$\frac{\tilde{\partial}\boldsymbol{M}}{\partial s} + \boldsymbol{\omega}_{\mathrm{F}} \times \boldsymbol{M} + \boldsymbol{e}_3 \times \boldsymbol{F} + \boldsymbol{m} - \frac{\tilde{\partial}}{\partial t}(\boldsymbol{J}\cdot\boldsymbol{\Omega}) - \boldsymbol{\Omega}_{\mathrm{F}} \times (\boldsymbol{J}\cdot\boldsymbol{\Omega}) = 0, \tag{3}$$

where $\boldsymbol{F}$ and $\boldsymbol{M}$ are the resultant vector and moment of the internal force on the cross section at point P, and ρ, S, $\boldsymbol{J}$ are the density, area of the cross section and the inertia tensor of the rod with unit length, respectively. Numbers 1, 2, 3 denote the components of each vectors $\boldsymbol{F}, \boldsymbol{M}, \boldsymbol{v}, \boldsymbol{\omega}, \boldsymbol{\Omega}, \boldsymbol{\omega}_{\mathrm{F}}, \boldsymbol{\Omega}_{\mathrm{F}}, \boldsymbol{f}, \boldsymbol{m}$ and the diagonal elements of the tensor $\boldsymbol{J}$ in (P-$x_2y_2z_2$), where $J_1 = J_2 = \rho I$, $J_3 = \rho I_0$, I and $I_0 = 2I$ are the inertial moment and polar inertial moment of the cross section, respectively. The components of $\boldsymbol{M}$ can be written by a linear constitutive relation as

$$M_1 = A\omega_1, \quad M_2 = A\omega_2, \quad M_3 = C\omega_3, \tag{4}$$

where $A = EI$ and $C = GI_0$ are, the bending and twisting stiffnesses of the cross section about the x- and z- axes; E and G are the Young's and shear moduli, respectively. Expressing $\omega_i, \Omega_i, \omega_{\mathrm{F}i}, \Omega_{\mathrm{F}i} (i = 1, 2, 3)$ by the Euler angles, denoting the partial derivative with respect to variables s and t by dash and dot, respectively, and introducing a quasi-coordinate ϕ to describe the absolute torsional angle by the following definition:

$$\phi' = \psi'\cos\vartheta + \varphi', \quad \dot{\phi} = \dot{\psi}\cos\vartheta + \dot{\varphi}, \tag{5}$$

we obtain

$$\begin{aligned} &\omega_{\mathrm{F}1} = \omega_1 = \vartheta', \quad \omega_{\mathrm{F}2} = \omega_2 = \psi'\sin\vartheta, \quad \omega_{\mathrm{F}3} = \psi'\cos\vartheta, \quad \omega_3 = \phi', \\ &\Omega_{\mathrm{F}1} = \Omega_1 = \dot{\vartheta}, \quad \Omega_{\mathrm{F}2} = \Omega_2 = \dot{\psi}\sin\vartheta, \quad \Omega_{\mathrm{F}3} = \dot{\psi}\cos\vartheta, \quad \Omega_3 = \dot{\phi}. \end{aligned} \tag{6}$$

When the rod moves slowly in the medium the viscous force and torque can be expressed by the following formulae:[10−12]

$$\begin{aligned} &f_1 = -c_1 v_1, \quad f_2 = -c_1 v_2, \quad f_3 = -c_3 v_3, \\ &m_1 = m_2 = 0, \quad m_3 = -c_0 \Omega_3, \end{aligned} \tag{7}$$

where c_1 and c_3 are the tangential and radial resistance coefficients of viscous force, respectively, and c_0 is the coefficient of viscous torque. For an extremely thin rod, when the length of the rod L is much larger than the radius of cross section a, the following approximate formulae may be applied in calculation:[12]

$$c_1 \approx 2c_3 \approx 4\pi\mu/\ln(L/a), \quad c_0 \approx 4\pi\mu a^2. \tag{8}$$

Substituting Eqs.(6) and (7) into the projections of vector equations (1) to (3) in (P-$x_2y_2z_2$), we obtain the dynamical equations of the rod with circular cross section in a viscous medium as follows:

$$A\vartheta'' + (C\phi' - A\psi'\cos\vartheta)\psi'\sin\vartheta - F_2 - J_1\ddot{\vartheta} - J_1(2\dot{\phi} - \dot{\psi}\cos\vartheta)\dot{\psi}\sin\vartheta = 0, \tag{9a}$$

$$A\psi''\sin\vartheta + (2A\psi'\cos\vartheta - C\phi')\vartheta' + F_1 - J_1\ddot{\psi}\sin\vartheta - 2J_1\dot{\vartheta}(\dot{\psi}\cos\vartheta - \dot{\phi}) = 0, \tag{9b}$$

$$C\phi'' - 2J_1\ddot{\phi} - c_0\dot{\phi} = 0, \tag{9c}$$

$$F_1' + \psi'(F_3\sin\vartheta - F_2\cos\vartheta) - \rho S\dot{v}_1 - c_1 v_1 = 0, \tag{9d}$$

$$F_2' + \psi' F_1\cos\vartheta - \vartheta' F_3 - \rho S\dot{v}_2 - c_1 v_2 = 0, \tag{9e}$$

$$F_3' - \psi' F_1\sin\vartheta + \vartheta' F_2 - \rho S\dot{v}_3 - c_3 v_3 = 0, \tag{9f}$$

$$v_1' + \psi'(v_3\sin\vartheta - v_2\cos\vartheta) - \dot{\psi}\sin\vartheta = 0, \tag{9g}$$

$$v_2' + \psi' v_1\cos\vartheta - \vartheta' v_3 + \dot{\vartheta} = 0, \tag{9h}$$

$$v_3' + \vartheta' v_2 - \psi' v_1\sin\vartheta = 0. \tag{9i}$$

3. Helical equilibrium

In dynamical equations (9), the variable $\phi(s,t)$ can be determined by Eq.(9c) independently. We will prove later that the torsional motion described by $\phi(s,t)$ is an attenuating vibration under the action of viscous dissipative force. When the torsional motion attenuates to zero, Eqs.(9) possess particular solutions as follows:

$$\begin{aligned} &\vartheta = \vartheta_0, \quad \psi' = \omega_0 = \sin\vartheta_0/R, \\ &\phi = 0, \quad v_i = 0 (i = 1,2,3), \\ &F_1 = 0, \quad F_2 = F_0\sin\vartheta_0, \\ &F_3 = F_0\cos\vartheta_0, \quad F_0 = -A\omega_0^2\cos\vartheta_0, \end{aligned} \tag{10}$$

which correspond to a helical equilibrium with radius R and pitch angle $\alpha = \pi/2 - \vartheta_0$ in the absence of torsion. F_0 is the modulus of the resultant internal force $\boldsymbol{F}$ along the ζ-axis, acting on each cross section. The negative value of F_0 means that the external forces acting on both ends are compressive. The resultant moment of external force screw on each cross section can be calculated as

$$\begin{aligned} &M_1 = 0, \quad M_2 = M_0\sin\vartheta_0 + F_0 R\cos\vartheta_0, \\ &M_3 = M_0\cos\vartheta_0 - F_0 R\sin\vartheta_0. \end{aligned} \tag{11}$$

In the helical equilibrium (P-$x_2y_2z_2$) is coincident with the Frenet frame of the centreline. The components of the curvature/twisting vector $\boldsymbol{\omega}$ in (P-$x_2y_2z_2$) can be written as

$$\omega_1 = 0, \quad \omega_2 = \omega_0\sin\vartheta_0, \quad \omega_3 = \omega_0\cos\vartheta_0. \tag{12}$$

Substituting Eq.(12) into Eq.(4), and considering the balance of internal torque (4) with external torque (11), we obtain the torsional moment M_0 as

$$M_0 = \omega_0[A - (A - C)\cos^2\vartheta_0]. \tag{13}$$

4. Stability and torsional vibration

Define the dimensionless arc-coordinate $\bar{s}$ and time $\bar{t}$ as

$$\bar{s} = \omega_0 s, \quad \bar{t} = \sqrt{\frac{A}{\rho S}}\omega_0^2 t, \tag{14}$$

let the dash and dot denote the partial derivatives with respect to dimensionless variables $\bar{s}$ and $\bar{t}$ respectively, and introduce the following dimensionless parameters:

$$\beta = \frac{I\omega_0^2}{S}, \quad \bar{c_0} = \frac{c_0}{\sqrt{\rho S A}}, \quad \nu = \frac{A - C}{C}, \tag{15}$$

where ν is the Poisson ratio of the rod with circular cross section, then the dimensionless equation of torsional vibration of the rod will be derived as

$$(1+\nu)^{-1}\phi'' - 2\beta\ddot{\phi} - \bar{c_0}\dot{\phi} = 0. \tag{16}$$

Substituting the following particular solution:

$$\phi(\bar{s},\bar{t}\,) = \phi_0 \exp(\lambda\bar{s} + w\bar{t}\,) \tag{17}$$

into Eq.(16), we obtain the characteristic equation

$$2\beta w^2 + \bar{c_0} w - \lambda^2(1+\nu)^{-1} = 0. \tag{18}$$

The asymptotic stability of the quasi-coordinate $\phi(s,t)$ in the time domain requires $\lambda^2 < 0$, which can be regarded as the necessary condition for the asymptotic stability of torsional motion of the rod in time domain. Let $\lambda = \pm \mathrm{i}k$, where k is the angular frequency of the perturbed configuration $\phi(s,t)$ with respect to the dimensionless arc-coordinate $\bar{s}$. The wave length of variation of the torsional angle after perturbation is $l = 2\pi/k\omega_0$, and the length of the helical rod with N cycles is $L = 2\pi N/\omega_0$. Therefore the total wave number is $L/l = kN$, and the dimensionless parameter k is equal to the wave number per cycle of the perturbed elastica. In the case of weak damping, where $\bar{c_0}^2 < 8\beta k^2/(1+\nu)$, the eigenvalue of Eq.(18) can be solved as

$$w = -\delta_{\mathrm{T}} + \mathrm{i}\mu_{\mathrm{T}}, \quad \delta_{\mathrm{T}} = \frac{\bar{c_0}}{4\beta},$$
$$\mu_{\mathrm{T}} = \frac{1}{4\beta}\sqrt{\frac{8\beta k^2}{(1+\nu)} - \bar{c_0}^2}. \tag{19}$$

The perturbed torsional motion is an attenuating vibration with a frequency μ_{T} and damping coefficient δ_{T}. In the strong damping medium, where $\bar{c_0}^2 \gg 8\beta k^2/(1+\nu)$, eliminating the terms with the orders more than the second order of $\beta k^2/(1+\nu)\bar{c_0}^2$, Eq.(18) has two negative real roots:

$$w = -\delta_{\mathrm{T}i}(i=1,2), \quad \text{where} \quad \delta_{\mathrm{T}1} = \frac{\bar{c_0}}{2\beta}$$
$$\text{and} \quad \delta_{\mathrm{T}2} = \frac{k^2}{(1+\nu)\bar{c_0}}. \tag{20}$$

Then the perturbed torsional angle brings about an attenuating non-reciprocate motion with damping coefficients $\delta_{\mathrm{T}1}$ and $\delta_{\mathrm{T}2}$.

5. Static stability of flexural deformation

In a static equilibrium state since there are no distributed forces the resultant force on each cross section is constant, and its components may be expressed as

$$F_1 = 0, \quad F_2 = F_0 \sin\vartheta, \quad F_3 = F_0\cos\vartheta. \tag{21}$$

Introducing the perturbation variables $x_i (i = 1,2)$ as

$$x_1 = \vartheta - \vartheta_0, \quad x_2 = \psi - \bar{s}, \tag{22}$$

substituting (22) into Eqs.(9a) and (9b), and neglecting the terms with the orders higher than the second order of perturbation variables lead to linearized perturbation equations as follows:

$$x_1'' + \sin^2\vartheta_0 x_1 - \sin 2\vartheta_0 x_2' = 0, \tag{23a}$$
$$2\cos\vartheta_0 x_1' + \sin\vartheta_0 x_2'' = 0. \tag{23b}$$

Applying the particular solution $x_i(\bar{s}) = x_{i0}\exp(\lambda\bar{s})(i = 1,2)$, we obtain the characteristic equation of Eqs.(23)

$$\lambda^2 \sin\vartheta_0(\lambda^2 + 1 + 3\cos^2\vartheta_0) = 0. \tag{24}$$

Besides the zero root, the eigenvalues of Eq.(24) are purely imaginary, $\lambda = \pm \mathrm{i}k$, thus the Lyapunov stability in space domain is satisfied. The parameter k is the angular frequency of the perturbed elastica relative to the dimensionless arc-coordinate $\bar{s}$, and expressed as

$$k^2 = 1 + 3\cos^2\vartheta_0. \tag{25}$$

The Lyapunov stability is a concept of stability with respect to the initial perturbation, occurring at one end of the rod. It is necessary to add a complementary condition to the perturbed elastica to satisfy the boundary condition at another end. When both ends have the same constraint, the condition $x_i(L) = x_i(0)(i = 1,\ldots,6)$ leads to a restriction of the parameter k as

$$k\bar{L} = 2n\pi \quad (n = 1,2,\ldots), \tag{26}$$

where n is an arbitrary integer, and $\bar{L} = \omega_0 L$ with $L = 2\pi RN/\sin\vartheta_0$ being the length of a helical rod with N cycles. It follows from the above that the dimensionless parameter k is equal to the wave number per cycle of the perturbed elastica, and expressed as

$$k = \frac{2n\pi}{\omega_0 L} = \frac{n}{N}. \tag{27}$$

Substitute Eq.(27) into Eq.(25) to determine ϑ_0, then we will obtain the particular values of external force screw satisfying the boundary conditions at both ends from Eqs.(10) and (13). From the viewpoint of Euler's stability the particular values can be regarded as

Euler's critical loads of a helical rod under axial compression and torsion:

$$\frac{|F_0|_{\rm cr}}{A\omega_0^2}=\frac{1}{\sqrt{3}}\sqrt{\left(\frac{n}{N}\right)^2-3},$$
$$\frac{M_{0,\rm cr}}{A\omega_0}=1-\frac{v}{3(1+\nu)}\left[\left(\frac{n}{N}\right)^2-1\right]. \quad (28)$$

6. Dynamic stability and flexural vibration

In the discussion of dynamic stability and flexural vibration we introduce dimensionless viscous resistance coefficients $\bar{c}_i(i=1,3)$ and perturbation variables $x_i(i=3,\ldots,8)$ as follows:

$$\bar{c}_i=\frac{c_i}{\omega_0^2\sqrt{\rho SA}}\ \ (i=1,3),\quad x_3=\frac{F_1}{A\omega_0^2},$$
$$x_4=\frac{F_2}{A\omega_0^2}+\cos\vartheta_0\sin\vartheta, x_5=\frac{F_3}{A\omega_0^2}+\cos\vartheta_0\cos\vartheta,$$
$$x_{5+i}=\frac{v_i}{\omega_0}\sqrt{\frac{\rho S}{A}}\ \ (i=1,2,3). \quad (29)$$

Substituting Eqs.(29) into Eqs.(9) and neglecting the terms with the orders higher than the second order of perturbation variables lead to linearized perturbation equations as follows:

$$x_1''-\varepsilon\ddot{x}_1+\sin^2\vartheta_0 x_1-\sin 2\vartheta_0 x_2'-x_4=0, \quad (30a)$$
$$2\cos\vartheta_0 x_1'+\sin\vartheta_0 x_2''-\varepsilon\ddot{x}_2\sin\vartheta_0+x_3=0, \quad (30b)$$
$$x_3'-\cos\vartheta_0 x_4+\sin\vartheta_0 x_5-\dot{x}_6-\bar{c}_1x_6=0, \quad (30c)$$
$$\cos\vartheta_0 x_3+x_4'-\dot{x}_7-\bar{c}_1x_7=0, \quad (30d)$$
$$-\sin\vartheta_0 x_3+x_5'-\dot{x}_8-\bar{c}_3x_8=0, \quad (30e)$$
$$-\sin\vartheta_0\dot{x}_2+x_6'-\cos\vartheta_0 x_7+\sin\vartheta_0 x_8=0, \quad (30f)$$
$$\dot{x}_1+\cos\vartheta_0 x_6+x_7'=0, \quad (30g)$$
$$-\sin\vartheta_0 x_6+x_8'=0. \quad (30h)$$

All the variables are functions of $\bar{s}$ and $\bar{t}$. The following particular solution can be used to analyse the dynamic stability of helical equilibrium:

$$x_i(\bar{s},\bar{t})=x_{i0}\exp(\mathrm{i}k\bar{s}+w\bar{t}),\quad (i=1,\ldots,8), \quad (31)$$

where the parameter k is known as $k=n/N$, since the static stability of helical equilibrium in spatial domain is satisfied, and the boundary conditions at both ends of the rod are the same in static or dynamic analysis.

As a special case of helical rod, we discuss a planar annular rod when $\vartheta=\pi/2$, $N=1$, $k=n$, and no deformation occurring along the normal of the ring plane. Let the perturbations $x_{i0}(i=1,4,7)$ of ϑ, F_2, v_2 be all equal to zero, then we will obtain the characteristic equation in time domain as

$$w^2+a_1w+a_2=0, \quad (32)$$

where coefficients a_1 and a_2 are determined by

$$a_1=\frac{\bar{c}_1n^2+\bar{c}_3}{n^2+1+\beta(n^2-1)^2},$$
$$a_2=\frac{n^2(n^2-1)^2}{n^2+1+\beta(n^2-1)^2},\quad (n=1,2,\ldots). \quad (33)$$

As a_1 and a_2 are positive, which means that the annular equilibrium of the rod is asymptotically stable. The perturbed motion is an attenuating flexural vibration, of which the parameters can be obtained from the characteristic equation (32) as follows:

$$w=-\delta\pm\mathrm{i}\mu,\quad \delta=\frac{a_1}{2},$$
$$\mu=\sqrt{\mu_0^2-\delta^2},\quad \mu_0=\sqrt{a_2}, \quad (34)$$

where μ and μ_0 are the free frequencies with or without damping, respectively, and δ is the damping coefficient. These results are identical with those obtained from the analysis in Ref.[11].

When the helical rod is thin enough, and the radius of helix is much larger than that of cross section, i.e. $R\gg a$, considering $\omega_0=\sin\vartheta_0/R$, $I\sim a^4$, $S\sim a^2$, and $\beta\sim(a/R)^2$ being sufficiently small, neglecting β and letting $c_1=2c_3$ yield the characteristic equation as follows:

$$a_0w^4+a_1w^3+a_2w^2+a_3w+a_4=0, \quad (35)$$

where coefficients $a_i(i=0,\ldots,4)$ are determined by

$$\begin{aligned}
a_0&=k^2+\sin^2\vartheta_0,\\
a_1&=\bar{c}_3(4k^2+3\sin^2\vartheta_0),\\
a_2&=2[k^4(k^2-1+6\cos^2\vartheta_0)\\
&\quad+\bar{c}_3^2(2k^2+\sin^2\vartheta_0)],\\
a_3&=\bar{c}_3k^2\{1+k^2[4k^2-5(1-5\cos^2\vartheta_0)]\\
&\quad+\sin^2\vartheta_0(1-6\cos^2\vartheta_0)\},\\
a_4&=k^4(k^2-1)^2(k^2-1-3\cos^2\vartheta_0).
\end{aligned} \quad (36)$$

When the following condition is satisfied:

$$k^2>1+3\cos^2\vartheta_0, \quad (37)$$

all the conditions $a_i>0(i=0,\ldots,4)$ as part of Hurwitz's criteria can be satisfied. Thus the inequality (37) serves as a necessary condition for the asymptotic stability of helical rod in the time domain. The condition (37) is similar to Eq.(25) in form, except for the greater-than sign replacing the equality sign

in Eq.(25). Consequently the inequality (37) leads to a conclusion that the external force screw should be smaller than the Euler's critical loads:

$$|F_0| < |F_0|_{\rm cr}, \quad M_0 < M_{0,\rm cr}. \tag{38}$$

This means that Lyapunov's and Euler's stability conditions of the helical rod in space domain are the necessary conditions for the Lyapunov's asymptotic stability in time domain. Analogous results have been obtained for a helical rod without damping[6] and a straight or annular rod in a viscous medium[11] as special cases.

When the stability conditions are satisfied the parameters of flexural vibration of the helical rod can be calculated from the eigenvalues of Eq.(35) numerically. For a given $\bar{c}_3 = 2$, the free frequency μ and damping coefficient δ of flexural vibration versus pitching angle $\alpha = \pi/2 - \vartheta_0$ of helix for different values of wave number k are shown in Fig.2 and Fig.3, respectively.

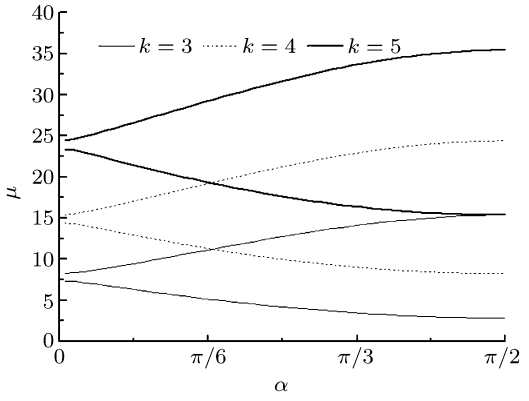


Fig.2. Free frequency μ of flexural vibration versus α for different values of k.

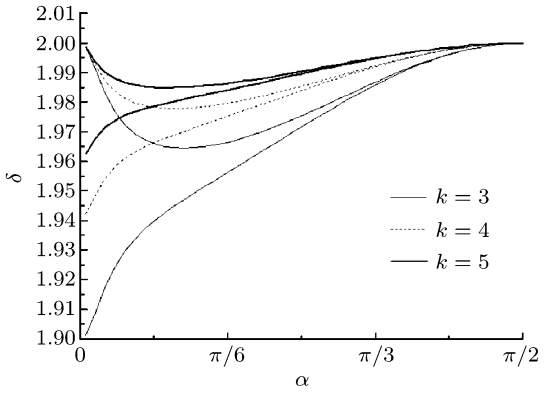


Fig.3. Damping coefficient δ of flexural vibration versus α for different values of k.

7. Conclusions

1. The different concepts of Lyapunov stability and Euler buckling in static analysis of equilibrium of elastic rod are closely related. A stable helical equilibrium from Lyapunov's concept in the space domain can be unstable from Euler's concept and the buckling occurs when the load reaches an Euler critical value. The Euler critical load of the helical rod is a special value of the external force when Lyapunov's stability conditions and the boundary conditions at both ends are satisfied.

2. The torsional deformation of the helical rod in a viscous medium is always asymptotically stable, and the rod suffers an attenuating torsional vibration.

3. In the stability analysis of flexural deformation of a helical rod in a viscous medium, the Lyapunov and Euler stability conditions in space domain are the necessary conditions for the Lyapunov asymptotic stability in time domain.

4. When the stability conditions are satisfied the free frequency and damping coefficient of flexural vibration of the helical rod in a viscous medium can be calculated from the eigenvalues of time domain.

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