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A new type of conserved quantity of Mei symmetry for Lagrange system

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This paper studies a new type of conserved quantity which is directly induced by Mei symmetry of the Lagrange system. Firstly, the definition and criterion of Mei symmetry for the Lagrange system are given. Secondly, a coordination function is introduced, and the conditions of existence of the new conserved quantity as well as its forms are proposed. Lastly, an illustrated example is given. The result indicates that the coordination function can be selected properly according to the demand for finding the gauge function, and thereby the gauge function can be found more easily. Furthermore, since the choice of the coordination function has multiformity, many more conserved quantities of Mei symmetry for the Lagrange system can be obtained.

Keywords: Lagrange system, Mei symmetry, new conserved quantity
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1. Introduction

There are intimate relations between symmetries and conserved quantities. The modern methods of finding conserved quantities by using symmetries are mainly the Noether symmetry,[1] Lie symmetry[2] and Mei symmetry.[3,4] In recent years, more and more attention has been paid to the research on the main three symmetries and conserved quantities of mechanical systems. Many achievements have been obtained in this popular research field.[5−18] Conserved quantities can be induced by the main three symmetries both directly and indirectly. There are mainly three kinds of conserved quantities induced by symmetries: the Noether conserved quantity, the Hojman conserved quantity and the Mei conserved quantity (it is also called a new type of conserved quantity).[18] Noether symmetry can induce the Noether conserved quantity directly and the Hojman conserved quantity as well as the Mei conserved quantity indirectly. Also, Lie symmetry can lead to the Hojman conserved quantity directly and the other two conserved quantities indirectly. The Mei conserved quantity can be induced directly by Mei symmetry and the other two conserved quantities can be induced indirectly by it.

In this paper, we study a new type of conserved quantity which is directly induced by Mei symmetry of the Lagrange system, and propose the conditions of existence of the new conserved quantity as well as its forms. An illustrated example is given. The Mei conserved quantity of the Lagrange system is a special case of the new conserved quantity given in this paper.

2. Mei symmetry of the system and its criterion equations

Consider a mechanical system composed of N particles, and its configuration is determined by n generalized coordinates $q_s(s = 1, 2, \ldots, n)$. The motion of the system is subjected to the ideal holonomic constraints, then the equations of motion for the system are

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = 0, \quad (s = 1, 2, \ldots, n), \tag{1}$$

where $L = L(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ is the Lagrangian of the system. Equation (1) can be written succinctly as

$$E_s(L) = 0, \tag{2}$$

where

$$E_s = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}_s} - \frac{\partial}{\partial q_s}. \tag{3}$$

Introduce the infinitesimal transformations

$$\begin{aligned} t^* &= t + \varepsilon\xi_0(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \\ q_s^*(t^*) &= q_s(t) + \varepsilon\xi_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \end{aligned} \tag{4}$$

where ε is an infinitesimal parameter and ξ_0, ξ_s are infinitesimal generators. Under the infinitesimal

transformations (4), $L = L(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ becomes $L^* = L(t^*, \boldsymbol{q}^*, \dot{\boldsymbol{q}}^*)$.

Definition After the Lagrangian L being replaced by the transformed Lagrangian L^*, if the form of Eq.(2) is invariant, that is

$$E_s(L^*) = 0, \tag{5}$$

then the invariance is called Mei symmetry of the Lagrange system.

Expanding L^*, we have

$$\begin{aligned} L^* = & \ L(t^*, \boldsymbol{q}^*, \dot{\boldsymbol{q}}^*) \\ = & \ L(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) + \varepsilon \tilde{X}^{(1)}(L) + O(\varepsilon^2), \end{aligned} \tag{6}$$

where

$$\tilde{X}^{(1)} = \xi_0 \frac{\partial}{\partial t} + \xi_s \frac{\partial}{\partial q_s} + \left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t} \xi_s - \dot{q}_s \frac{\bar{\mathrm{d}}}{\mathrm{d}t} \xi_0 \right) \frac{\partial}{\partial \dot{q}_s}, \tag{7}$$

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \dot{q}_s \frac{\partial}{\partial q_s} + \ddot{q}_s \frac{\partial}{\partial \dot{q}_s}. \tag{8}$$

Substituting Eq.(6) into Eq.(5), ignoring ε^2 and the higher-order infinitesimal terms, we obtain

$$E_s[\tilde{X}^{(1)}(L)] = 0. \tag{9}$$

Then, we have

Criterion For Lagrange system (2), if the infinitesimal generators ξ_0 and ξ_s satisfy Eq.(9), the corresponding invariance is the Mei symmetry of the system.

Equation (9) is called the criterion equation of Mei symmetry for the Lagrange system.

3. The new conserved quantity of Mei symmetry

The conditions of existence of the new conserved quantity as well as its forms, which is induced directly by Mei symmetry of the Lagrange system, are given in the following theorem.

Theorem For the Lagrange system (2), if the infinitesimal generators ξ_0, ξ_s of Mei symmetry and a gauge function $G_{\mathrm{M}} = G_{\mathrm{M}}(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ satisfy

$$\tilde{X}^{(1)}(L) \frac{\bar{\mathrm{d}} f}{\mathrm{d}t} + \tilde{X}^{(1)}[\tilde{X}^{(1)}(L)] + \frac{\partial \tilde{X}^{(1)}(L)}{\partial t}(f - \xi_0) + \dot{q}_s \frac{\partial \tilde{X}^{(1)}(L)}{\partial \dot{q}_s} \left(\frac{\bar{\mathrm{d}} \xi_0}{\mathrm{d}t} - \frac{\bar{\mathrm{d}} f}{\mathrm{d}t} \right) + \frac{\bar{\mathrm{d}} G_{\mathrm{M}}}{\mathrm{d}t} = 0, \tag{10}$$

then Mei symmetry of the system will lead to the following new conserved quantity:

$$I_{\mathrm{M}} = \tilde{X}^{(1)}(L) f + \frac{\partial \tilde{X}^{(1)}(L)}{\partial \dot{q}_s}(\xi_s - \dot{q}_s f) + G_{\mathrm{M}} = \mathrm{const}, \tag{11}$$

where $f = f(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ is a random function to make G_{M} exist. We call f a coordination function.

Proof Taking the derivative of I_{M} to t, and using Eq.(10), we have

$$\begin{aligned} \frac{\bar{\mathrm{d}} I_{\mathrm{M}}}{\mathrm{d}t} = & \left(\frac{\partial \tilde{X}^{(1)}(L)}{\partial t} + \dot{q}_s \frac{\partial \tilde{X}^{(1)}(L)}{\partial q_s} + \ddot{q}_s \frac{\partial \tilde{X}^{(1)}(L)}{\partial \dot{q}_s} \right) f + \tilde{X}^{(1)}(L) \frac{\bar{\mathrm{d}} f}{\mathrm{d}t} + \frac{\bar{\mathrm{d}}}{\mathrm{d}t} \left(\frac{\partial \tilde{X}^{(1)}(L)}{\partial \dot{q}_s} \right)(\xi_s - \dot{q}_s f) \\ & + \frac{\partial \tilde{X}^{(1)}(L)}{\partial \dot{q}_s} \left(\frac{\bar{\mathrm{d}} \xi_s}{\mathrm{d}t} - \ddot{q}_s f - \dot{q}_s \frac{\bar{\mathrm{d}} f}{\mathrm{d}t} \right) - \tilde{X}^{(1)}(L) \frac{\bar{\mathrm{d}} f}{\mathrm{d}t} - \frac{\partial \tilde{X}^{(1)}(L)}{\partial t} \xi_0 - \frac{\partial \tilde{X}^{(1)}(L)}{\partial q_s} \xi_s \\ & - \frac{\partial \tilde{X}^{(1)}(L)}{\partial \dot{q}_s} \left(\frac{\bar{\mathrm{d}} \xi_s}{\mathrm{d}t} - \dot{q}_s \frac{\bar{\mathrm{d}} \xi_0}{\mathrm{d}t} \right) - \frac{\partial \tilde{X}^{(1)}(L)}{\partial t}(f - \xi_0) - \dot{q}_s \frac{\partial \tilde{X}^{(1)}(L)}{\partial \dot{q}_s} \left(\frac{\bar{\mathrm{d}} \xi_0}{\mathrm{d}t} - \frac{\bar{\mathrm{d}} f}{\mathrm{d}t} \right) \\ = & \left[\frac{\bar{\mathrm{d}}}{\mathrm{d}t} \left(\frac{\partial \tilde{X}^{(1)}(L)}{\partial \dot{q}_s} \right) - \frac{\partial \tilde{X}^{(1)}(L)}{\partial q_s} \right](\xi_s - \dot{q}_s f) \\ = & \ 0. \end{aligned} \tag{12}$$

Special case If $f = \xi_0$, Eqs.(10) and (11) respectively become

$$\tilde{X}^{(1)}(L)\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t} + \tilde{X}^{(1)}[\tilde{X}^{(1)}(L)] + \frac{\bar{\mathrm{d}}G_{\mathrm{M}}}{\mathrm{d}t} = 0, \quad (13)$$

$$I_{\mathrm{M}} = \tilde{X}^{(1)}(L)\xi_0 + \frac{\partial \tilde{X}^{(1)}(L)}{\partial \dot{q}_s}(\xi_s - \dot{q}_s\xi_0) + G_{\mathrm{M}} = \mathrm{const.} \quad (14)$$

Equation (13) is the condition from which Mei symmetry of Lagrange system can lead to the Mei conserved quantity. Equation (14) is just the Mei conserved quantity induced by Mei symmetry of the Lagrange system.[18]

We call the new conserved quantity (11) a generalized Mei conserved quantity of the Lagrange system.

4. Example

For the Lagrangian of a mechanical system

$$L = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) - q_2, \quad (15)$$

we study the Mei symmetry and the generalized Mei conserved quantity of the system.

The equations of motion (2) give

$$\ddot{q}_1 = 0, \quad \ddot{q}_2 = -1. \quad (16)$$

Making calculation, we have

$$\tilde{X}^{(1)}(L) = \dot{q}_1\left(\frac{\bar{\mathrm{d}}\xi_1}{\mathrm{d}t} - \dot{q}_1\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\right) + \dot{q}_2\left(\frac{\bar{\mathrm{d}}\xi_2}{\mathrm{d}t} - \dot{q}_2\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\right) - \xi_2. \quad (17)$$

Choosing the infinitesimal generators as

$$\xi_0 = 1, \quad \xi_1 = 1, \quad \xi_2 = \dot{q}_2 + t, \quad (18)$$

we obtain

$$\tilde{X}^{(1)}(L) = -\dot{q}_2 - t, \quad (19)$$

then we get

$$E_s[\tilde{X}^{(1)}L] = 0. \quad (20)$$

It is obvious that for the infinitesimal generators (18), Mei symmetry exists in the system.

Using Eq.(10), we obtain

$$\frac{\bar{\mathrm{d}}G_{\mathrm{M}}}{\mathrm{d}t} = f + t\frac{\bar{\mathrm{d}}f}{\mathrm{d}t}. \quad (21)$$

Select $f = \dot{q}_2$, then

$$G_{\mathrm{M}} = q_2 - \frac{1}{2}t^2, \quad (22)$$

Eq.(11) gives

$$I_{\mathrm{M}} = q_2 - \dot{q}_2 - \dot{q}_2 t - \frac{1}{2}t^2 - t = \mathrm{const.} \quad (23)$$

Select $f = \dot{q}_1$, then

$$G_{\mathrm{M}} = q_1, \quad (24)$$

Eq.(11) gives

$$I_{\mathrm{M}} = q_1 - \dot{q}_1 t - \dot{q}_2 - t = \mathrm{const.} \quad (25)$$

Especially when $f = \xi_0$, we have $G_{\mathrm{M}} = t$, then

$$I_{\mathrm{M}} = -\dot{q}_2 - t = \mathrm{const.} \quad (26)$$

5. Conclusion

The new conserved quantity is induced directly by Mei symmetry of the Lagrange system given in this paper. The generalized Mei conserved quantity has more general significance. The Mei conserved quantity of the system is a special case of the generalized Mei conserved quantity when $f = \xi_0$. The coordination function f in the result of this paper can be selected properly according to the demand for finding the gauge function, thereby the gauge function can be found more easily. Furthermore, since the choice of the coordination function has multiformity, many more conserved quantities of Mei symmetry for the Lagrange system can be obtained.

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