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Energy spectrum and entanglement of two tunnel-coupled Bose–Einstein condensates*

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This paper obtains the energy-spectrum and eigenstate corrections of two-mode Bose–Einstein condensates (BECs) coupled by quantum tunnelling by perturbation method in both strong and weak tunnelling regions. The population imbalance between two BECs are then studied in terms of the low-lying eigenstates which also characterize the intrinsic entanglement between the two-mode BECs. The strong parity effect in the weak tunnelling region is also investigated.

Keywords: energy spectrum, entanglement, Bose–Einstein condensates
PACC: 7450, 0365, 0530J

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1. Introduction

Quantum tunnelling of macroscopically coherent systems is one of the most intriguing phenomena well known in condensed matter physics for long,[1,2] whose most fascinating feature is that a macroscopic object with many degrees of freedom manifests nonclassical behaviour calculable only by using quantum mechanics. Quantum tunnelling has been observed to play an important role in a great deal of physical processes, e.g. in superfluids and in superconductors as well as in magnetic systems. The experimental realization of Bose–Einstein condensation (BEC) in dilute alkali atomic gases opened up the exploration of quantum phenomena in a fire-new domain. Indeed, recent experiments have successfully demonstrated quantum tunnelling for condensates confined in an array of optical potentials[3] and observed the phase transition between superfluid and Mott-insulator.[4] The essential underlying physics can be understood with the study of a simple pair of tunnel-coupled BECs (single-component BECs trapped in a double-well potential with an adjustable barrier height[5] or atoms consisting of two internal hyperfine levels $|F = 1,\ m_F = -1\rangle$ and $|F = 2,\ m_F = 1\rangle$ in a single trap which are coupled by a radiation field.[6,7]) Condensates in such a tunable environment exhibit rich phenomena related to the macroscopic quantum coherence such as the atomic Josephson effect[8] and matter-wave interference,[6] which all arise from quantum tunnelling.

As an idealization the two linked BECs can be considered as a two-mode bosonic system, which has been studied extensively by using a variety of methods, such as the Gross-Pitaevskii approximation,[9] mean-field theory,[10,11] the Bethe ansatz method[12,13] and numerical diagonalization,[14] etc. Most of the investigations have been concentrated on the dynamics and little works have been done toward the energy spectrum and corresponding eigenstates of the BECs consisting of two coupled subsystems. In the present paper we study the eigen problem with perturbation method for both weak and strong tunnel couplings, from which some new features of the tunnel-coupled BECs are explored.

In the formalism of second quantization, the system can be described by the bimodal Hamiltonian (natural unit $\hbar = 1$ is used throughout)[15]

$$\widehat{H} = \sum_{l=1,2} \widehat{H}_l + \widehat{H}_{\rm i} + \widehat{H}_{\rm t},$$

with

$$\widehat{H}_l = \omega_l \hat{a}_l^+ \hat{a}_l + \frac{\eta_l}{2} \hat{a}_l^+ \hat{a}_l^+ \hat{a}_l \hat{a}_l,$$

$$\widehat{H}_{\rm i} = \chi \hat{a}_1^+ \hat{a}_1 \hat{a}_2^+ \hat{a}_2,$$

$$\widehat{H}_{\rm t} = G\left(\hat{a}_1^+ \hat{a}_2 + \hat{a}_2^+ \hat{a}_1\right),$$

where $\hat{a}_1, \hat{a}_2$ denote the single-particle annihilation op-

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erators of each mode obeying the usual boson commutation relations $[\hat{a}_{\rm i}, \hat{a}_j^+] = \delta_{ij}$. The parameters η_l, χ and G describe the nonlinear atom-interaction strength in each mode, the atom-interaction and tunnel-coupling strengths between two modes respectively. By introducing the pseudo-angular-momentum operators in terms of Schwinger relation the above model Hamiltonian reduces to

$$\widehat{H} = \omega \widehat{J}_z + q\widehat{J}_z^2 + G(\widehat{J}_+ + \widehat{J}_-), \tag{1}$$

where the pseudo-angular-momentum operators take the form

$$\begin{aligned} \widehat{J}_x &= \frac{1}{2}\left(\hat{a}_1^+\hat{a}_2 + \hat{a}_2^+\hat{a}_1\right), \\ \widehat{J}_y &= \frac{1}{2{\rm i}}\left(\hat{a}_1^+\hat{a}_2 - \hat{a}_2^+\hat{a}_1\right), \\ \widehat{J}_z &= \frac{1}{2}\left(\hat{a}_1^+\hat{a}_1 - \hat{a}_2^+\hat{a}_2\right), \end{aligned} \tag{2}$$

and $\widehat{J}_\pm = \widehat{J}_x \pm {\rm i}\widehat{J}_y$. The Casimir invariant is

$$\widehat{J}^2 = \frac{\widehat{N}}{2}\Big(\frac{\widehat{N}}{2} + 1\Big),$$

where $\widehat{N} = \widehat{N}_1 + \widehat{N}_2 = \hat{a}_1^+\hat{a}_1 + \hat{a}_2^+\hat{a}_2$ is the total number operator, which is conserved and set to the total number of atoms $N = 2j$ with j being the quantum number of angular momentum. The physical meaning of the coupling parameters of Eq.(1) for different realizable systems may be found in Ref.[9]. To be brief,

$$\omega = \omega_1 - \omega_2 + (N-1)(\eta_1 - \eta_2)/2$$

is the difference between chemical potentials of the two modes. The parameter $q = (\eta_1 - \eta_2)/2 - \chi$ provides the strength of the scattering interaction between bosons and G still denotes the tunnel-coupling. Note that different coupling regimes of the model are characterized by the ratio q/G. It is useful to consider the following three regimes:[9] Rabi regime $\rightarrow q/G \ll 1/N$, Josephson regime $\rightarrow 1/N \ll q/G \ll N$ and Fock regime $\rightarrow q/G \gg N$. Note that the crossover between Rabi and Josephson regimes is in the strong tunnel-coupling region while the weak tunnel-coupling corresponds to the crossover between Josephson and Fock regimes. And it is well known that an analogy can be drawn between the dynamics of Eq.(1) and the motion of a pendulum.[8]

Previous studies for the general properties of two-mode BECs have been focused on the quantum dynamics of the system as well as the effects of the nonlinear interactions between atoms. Meanwhile we emphasize how the atom-interaction, particularly the strength of the tunnel-coupling, affects the quantum tunnelling behaviour and the quantum correlation. In this paper we investigate the energy spectra and the wavefunction of the ground-state in the limit of strong and weak tunnel-couplings by means of perturbation method. These results are used to extract the population imbalance between two modes and to characterize the intrinsic entanglement for the ground state in the different coupling regions of the system. Moreover, our approach also provides a useful tool in studying the behaviour of the crossover regions between the Rabi, Josephson and Fock regimes, which is an important problem in quantum tunnelling. We will mostly discuss the case $\omega = 0$ (corresponding to the case of symmetric trap potential $\omega_1 = \omega_2$) as achieved in Refs.[12,13,15].

2. The case of strong tunnel coupling

2.1. The energy gap

We start our analysis with strong tunnel-coupling region. Firstly, we determine the energy spectrum analytically by using second-order perturbation theory with the Hamiltonian rewritten as perturbable form $\widehat{H} = \widehat{H}^{(0)} + \widehat{W}$, where

$$\widehat{H}^{(0)} = G(\widehat{J}_+ + \widehat{J}_-), \quad W = q\widehat{J}_z^2. \tag{3}$$

Here we take advantage of unitary transformation obtained by means of the generator of SU(2) coherent states to solve exactly the first part of the Hamiltonian $\widehat{H}^{(0)}$ and to derive analytic formulas of both unperturbed energy and wavefunction. The generalized gauge transformation in terms of the unitary operator[16] is defined by

$$\widehat{U} = \exp\Big[\frac{\lambda}{2}(\widehat{J}_- - \widehat{J}_+)\Big], \tag{4}$$

where auxiliary parameter is seen to be $\lambda = -\pi/2$ when $\omega = 0$. The stationary-state Schrödinger equation is invariable under the gauge transformation such that

$$\widehat{H}'^{(0)}|\psi'\rangle^{(0)} = E^{(0)}|\psi'\rangle^{(0)},$$

where

$$\widehat{H}'^{(0)} = \widehat{U}\widehat{H}\widehat{U}^+, \qquad |\psi'\rangle^{(0)} = \widehat{U}|\psi\rangle^{(0)},$$

and the state $|\psi\rangle^{(0)}$ is assumed to be the solution of original Schrödinger equation

$$\widehat{H}^{(0)}|\psi\rangle^{(0)} = E^{(0)}|\psi\rangle^{(0)}.$$

In terms of the generalized gauge transformation, $\widehat{H}'^{(0)}$ is diagonal in the $\widehat{J}_z$ representation

$$\widehat{H}'^{(0)} = -2G\widehat{J}_z.$$

It is easy to obtain the exact general solution of the original stationary-state Schrödinger equation

$$|\psi\rangle^{(0)} = \widehat{U}^{+}|\psi'\rangle^{(0)} = \widehat{U}^{+}|j,m\rangle,$$

and the zero-order energy of the state with exact magnetic quantum number m ($m = -j, ...j$) is

$$E_m^{(0)} = -2Gm.$$

So the zero-order ground-state energy and wavefunction are given respectively by

$$\begin{aligned} E_g^{(0)} &= -2Gj = -GN, \\ |\psi\rangle_g^{(0)} &= \widehat{U}^{+}|\psi'\rangle_g^{(0)} = \widehat{U}^{+}|j,j\rangle = |\boldsymbol{n}_x\rangle, \end{aligned} \tag{5}$$

where $|\boldsymbol{n}_x\rangle$ is also known as SU(2) coherent state which is the eigenstate of the operator $\widehat{J}\cdot\boldsymbol{n}_x$ such that $\widehat{J}\cdot\boldsymbol{n}_x|\boldsymbol{n}_x\rangle = |\boldsymbol{n}_x\rangle$ and $\boldsymbol{n}_x$ is the unit vector along x-axis. The first- and second-order corrections to the ground-state energy are given respectively by the well-known expression

$$\begin{aligned} E_g^{(1)} &= {}_g^{(0)}\langle\psi|W|\psi\rangle_g^{(0)} \\ &= q_g^{(0)}\langle\psi'|\widehat{U}\widehat{J}_z^2\widehat{U}^{+}|\psi'\rangle_g^{(0)}, \\ E_g^{(2)} &= \sum_{m\neq g}\frac{|W_{gm}|^2}{E_g^{(0)} - E_m^{(0)}} \\ &= \sum_{m\neq g}\frac{q^2|_g^{(0)}\langle\psi'|\widehat{U}\widehat{J}_z^2\widehat{U}^{+}|\psi'\rangle_m^{(0)}|^2}{E_g^{(0)} - E_m^{(0)}}, \end{aligned}$$

where $|\psi'\rangle_m^{(0)}$ denotes the zero-order wavefunction with magnetic quantum number m. We finally find that the ground-state energy including the first- and second-order corrections is

$$E_g(q,G) = -GN + \frac{q}{4}N + \frac{q^2}{32G}(N - N^2). \tag{6}$$

In the same way we obtain the corrections to the energy of the first-excited state with magnetic quantum number $(j-1)$

$$\begin{aligned} E_{\rm e}(q,G) = &-GN + 2G + \frac{q}{4}(3N-2) \\ &+ \frac{3q^2}{32G}(-N^2 + 3N - 2). \end{aligned} \tag{7}$$

So the energy gap between the ground and the first-excited states is

$$\begin{aligned} \Delta &= E_{\rm e}(q,G) - E_g(q,G) \\ &= 2G + \frac{q}{2}(N-1) + \frac{3q^2}{16G}(N-1), \end{aligned} \tag{8}$$

which is consistent with the result obtained by the algebraic Bethe ansatz method.[13]

2.2. Entanglement associated with the ground state and expectation value of the correlation operators

The first-order correction wavefunction of the ground-state is given by

$$\begin{aligned} |\psi\rangle_g &= |\boldsymbol{n}_x\rangle + |\psi\rangle_g^{(1)} \\ &= \sum_{m=-j}^{j}(C_{2j}^{j+m})^{\frac{1}{2}}\frac{1}{2^j}|j,m\rangle \\ &\quad + \frac{q}{4}\Big[-\frac{\pi^2\sqrt{6j(j-1)(2j-1)(2j-3)}}{32G}\,|j,j-4\rangle \\ &\quad + \frac{\pi\sqrt{6j(j-1)(2j-1)}}{8G}\,|j,j-3\rangle \\ &\quad - \frac{\sqrt{j(2j-1)}}{2G}\,|j,j-2\rangle \\ &\quad + \frac{\pi(2j-1)\sqrt{2j}}{8G}\,|j,j-1\rangle \\ &\quad - \frac{\pi^2 j(2j-1)}{32G}\,|j,j\rangle\Big], \end{aligned} \tag{9}$$

where C_{2j}^{j+m} is a binomial coefficient. For the pair of tunnel-coupled BECs described by Eq.(1), a general state $|\psi\rangle$ can be written in terms of the Fock states as

$$|\psi\rangle = \sum_{n=0}^{N} c_n|n\rangle_1|N-n\rangle_2,$$

where c_n are complex coefficients with n bosons in mode '1' it means $N-n$ bosons in mode '2'. There is an obvious relation that $|j,m\rangle \equiv |N+2m\rangle_1|N-2m\rangle_2$ and we then have

$$|\psi\rangle = \sum_{n=0}^{N} c_n\Big|j, n - \frac{N}{2}\Big\rangle.$$

So the entropy of entanglement between two modes can be evaluated as

$$\begin{aligned} E(\rho) &= -\sum_{n=0}^{N}|c_n|^2\ln|c_n|^2 \\ &= -\sum_{m=-j}^{j}|C_m|^2\ln|C_m|^2, \end{aligned} \tag{10}$$

which has been proved in Ref.[11]. So the entropy of entanglement associated with the ground state is

$$
\begin{aligned}
E(\rho) = & -\sum_{m=-N/2}^{N/2-5}\left|\left(C_N^{N/2+m}\right)^{1/2}\left(\frac{\sqrt{2}}{2}\right)^N\right|^2\ln\left|\left(C_N^{N/2+m}\right)^{1/2}\left(\frac{\sqrt{2}}{2}\right)^N\right|^2 \\
& -\left|(C_N^{N-4})^{1/2}\left(\frac{\sqrt{2}}{2}\right)^N-\frac{q\pi^2\sqrt{3N(N-1)(N-2)(N-3)}}{128\sqrt{2}G}\right|^2 \\
& \times\ln\left|(C_N^{N-4})^{1/2}\left(\frac{\sqrt{2}}{2}\right)^N-\frac{q\pi^2\sqrt{3N(N-1)(N-2)(N-3)}}{128\sqrt{2}G}\right|^2 \\
& -\left|(C_N^{N-3})^{1/2}\left(\frac{\sqrt{2}}{2}\right)^N+\frac{q\pi\sqrt{3N(N-1)(N-2)}}{32\sqrt{2}G}\right|^2 \\
& \times\ln\left|(C_N^{N-3})^{1/2}\left(\frac{\sqrt{2}}{2}\right)^N+\frac{q\pi\sqrt{3N(N-1)(N-2)}}{32\sqrt{2}G}\right|^2 \\
& -\left|(C_N^{N-2})^{1/2}\left(\frac{\sqrt{2}}{2}\right)^N-\frac{q\sqrt{N(N-1)}}{8\sqrt{2}G}\right|^2\times\ln\left|(C_N^{N-2})^{1/2}\left(\frac{\sqrt{2}}{2}\right)^N-\frac{q\sqrt{N(N-1)}}{8\sqrt{2}G}\right|^2 \\
& -\left|(C_N^{N-1})^{1/2}\left(\frac{\sqrt{2}}{2}\right)^N+\frac{q\pi(N-1)\sqrt{N}}{32G}\right|^2\times\ln\left|(C_N^{N-1})^{1/2}\left(\frac{\sqrt{2}}{2}\right)^N+\frac{q\pi(N-1)\sqrt{N}}{32G}\right|^2 \\
& -\left|\left(\frac{\sqrt{2}}{2}\right)^N-\frac{q\pi^2N(N-1)}{256G}\right|^2\times\ln\left|\left(\frac{\sqrt{2}}{2}\right)^N-\frac{q\pi^2N(N-1)}{256G}\right|^2. \qquad (11)
\end{aligned}
$$

Figure 1 shows the entanglement as a function of q/G. In the case of $q = 0$, it is easy to show that the ground state is the two-site analogue of the superfluid phase of Bose–Hubbard model. Even if q is small the occupation number fluctuations are large due to the strong tunnel-coupling , which results in phase coherence between two modes characterized by the high entanglement. As the ratio q/G continues to increase, the entanglement asymptotically approaches a maximum value which is still less than the maximal entanglement of the system, $\ln(N+1)$. Then the entanglement gradually decreases with the increase of q/G.

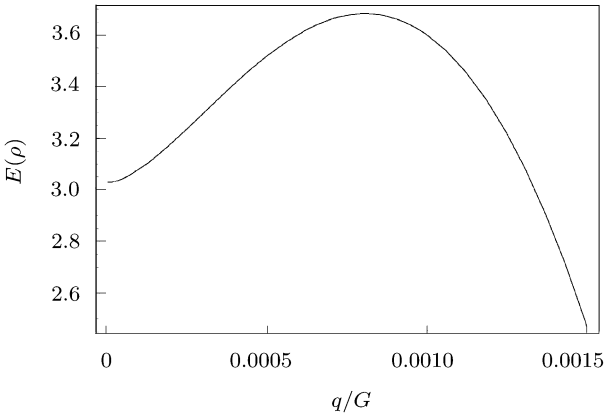


Fig.1. The entropy of entanglement via the ratio q/G for $N = 100$.

The expectation value of the correlation operators $\widehat{J}_z$ characterizes the population imbalance between the coupled two-mode BECs. It is well known that the relative phase is conjugate to the population imbalance $\Delta N = N_1 - N_2$, which are useful in understanding the dynamics and quantum tunnelling effect of the bimodal model. For the ground state in the limit of strong tunnelling, we have

$$
\begin{aligned}
\Delta N = & 2\langle\psi|\widehat{J}_z|\psi\rangle_g \\
= & -(C_N^{N-4})^{1/2}\left(\frac{\sqrt{2}}{2}\right)^N \\
& \times\frac{q\pi^2\sqrt{3N(N-1)(N-2)(N-3)}}{32\sqrt{2}G} \\
& +(C_N^{N-3})^{1/2}\left(\frac{\sqrt{2}}{2}\right)^N \\
& \times\frac{q\pi\sqrt{3N(N-1)(N-2)}}{8\sqrt{2}G} \\
& -(C_N^{N-2})^{1/2}\left(\frac{\sqrt{2}}{2}\right)^N\times\frac{q\sqrt{N(N-1)}}{2\sqrt{2}G} \\
& +(C_N^{N-1})^{1/2}\left(\frac{\sqrt{2}}{2}\right)^N\times\frac{q\pi(N-1)\sqrt{N}}{8G} \\
& -\left(\frac{\sqrt{2}}{2}\right)^N\times\frac{q\pi^2N(N-1)}{64G}, \qquad (12)
\end{aligned}
$$

where we retain only terms up to and including those

of $O(q)$.

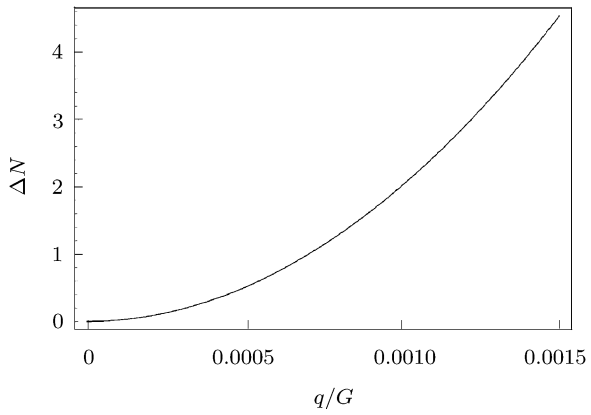


Fig.2. The population imbalance via the ratio q/G for $N = 100$.

In Fig.2 we plot the population imbalance of Eq.(12) as a function of the ratio q/G and indicate that the population imbalance between the coupled two modes increases with q/G.

3. The case of weak tunnel-coupling

3.1. The energy gap

In the weak tunnel-coupling region, we treat the tunnel part of the Hamiltonian (1) with coupling constant G as perturbation. Similarly, we obtain the corrections to the energy spectrum (for the case that N is even number) in terms of the degenerate perturbation theory:

$$E_g(q,G) = -\frac{G^2(N^2+2N)}{2q},$$

$$E_{\rm e}(q,G) = q + \frac{2G^2(j^2+j+1)}{3q} = q + \frac{G^2(N^2+2N+4)}{6q},$$

So the energy gap between the ground and the first excited states is

$$\Delta = E_{\rm e}(q,G) - E_g(q,G) = q + \frac{2G^2(N+1)^2}{3q}, \quad (13)$$

which is also consistent with the result obtained by the algebraic Bethe ansatz method.[13]

3.2. Entanglement and expectation value of the correlation operators

The first-order correction wavefunction of the ground state is obtained as

$$|\psi\rangle_g = |j,0\rangle - \frac{G\sqrt{j(j+1)}}{q}(|j,-1\rangle + |j,1\rangle).$$

Thus the entropy of entanglement, the population imbalance between the two modes in weak tunnelling are respectively given by

$$E(\rho) = -\sum_{m=-j}^{j} |c_n|^2 \ln |c_n|^2 = -\frac{1}{2}\Big(\frac{G}{q}\Big)^2 \times N(N+2) \times \ln\Big[\frac{1}{4}\Big(\frac{G}{q}\Big)^2 \times N(N+2)\Big], \quad (14)$$

$$\Delta N = 2\langle\psi|\widehat{J}_z|\psi\rangle_g = 0\,. \quad (15)$$

3.3. Parity effect

Now we perform a similar analysis for odd N. The energy gap between the ground and the first excited states is

$$\Delta = 2q - \frac{G}{2}(N+1) + \frac{3G^2(N+1)^2}{16q}. \quad (16)$$

The entropy of entanglement and the population imbalance between the two modes are respectively

$$E(\rho) = -\Big(\ln\frac{1}{2} + \frac{1}{16}\Big(\frac{G}{q}\Big)^2 \times (N-1)(N+3) \times \ln\Big[\frac{1}{32}\Big(\frac{G}{q}\Big)^2 \times (N-1)(N+3)\Big]\Big), \quad (17)$$

$$\Delta N = 2\langle\psi|\widehat{J}_z|\psi\rangle_g = 0. \quad (18)$$

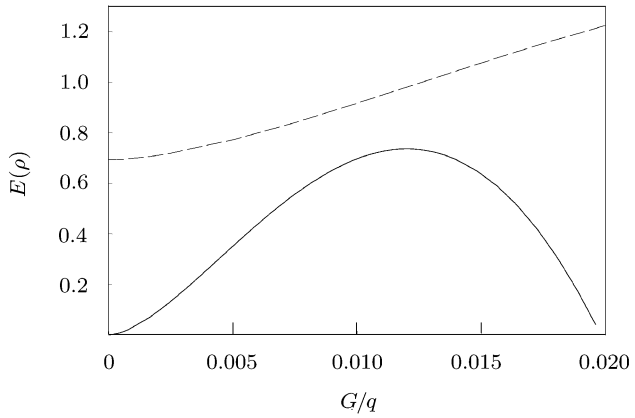


Fig.3. The entropy of entanglement via the ratio G/q (solid line for $N = 100$ and dashed line for $N = 101$).

In Fig.3 we show in detail the entropy of entanglement via the ratio G/q in two different cases (even and odd N) : $N = 100$ (solid line) and $N = 101$ (dashed line). Since G is the tunnelling parameter, the larger its value the stronger the coupling between two modes of the system. So for the case $G = 0$ with even N the bimodal system has zero entanglement. But the situation is greatly different when N is odd where the entanglement does not vanish even if $G = 0$. This indicates a novel parity effect in the weak tunnelling region similar to the spin parity effect in the macroscopic quantum coherence of spin systems.

Furthermore, comparing Fig.3 with Fig.1, we can see that the entropy of entanglement in weak tunnel-coupling region is much smaller than that of the strong tunnel-coupling region. That is to say, the crossover region between Rabi and Josephson regimes indicates stronger coherence in the ground state. In the opposite crossover region between Josephson and Fock regimes, the system indicates weaker coherence.

4. Conclusion

To summarize, we have investigated the energy spectra of a tunnel-coupled pair of BECs in both strong and weak tunnelling regions by using perturbation method and obtained the explicit formula of population imbalance between two modes as well as the entropy of entanglement. Our approach makes it feasible to undertake an analysis of the crossover regions between the Rabi, Josephson and Fock regimes. In the weak tunnelling region quantum tunnelling displays a novel parity effect, whose possible physical implications are worth exploring.

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