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Statistical analysis of aggregation in freeway traffic*

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We restudy the master-equation approach to aggregation in freeway traffic based on the theory of birth–death process, in which the clustering behaviour in one-lane freeway traffic model is investigated. The transition probabilities for the jump processes are reconstructed by using Greenshields' model, and the equation of the mean size of the cluster at any time t is derived from the birth–death equation. Numerical experiments show the clustering behaviours varying with time very well.

Keywords: traffic flow, statistics mechanism, master equation, birth–death process
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1. Introduction

Traffic flow has attracted the attention of not only traffic engineers but also physicists.[1] Different from traffic engineers who concern the feasibility of attaining the limit capacity of traffic networks and its quantification, physicists are interested in investigating the existence of various physical states and transitions in traffic systems, and attempt understanding the fundamental principles of governing the flow of vehicular traffic by means of statistical physics, which is a novel branch of physics.[2]

Over the past almost half a century, theoretical analysis and computer simulations based on varieties of traffic flow models have been conducted, which provide deep insight into the properties of the models and help people in better understanding the complex phenomena observed in real traffic circumstances. In general, the approaches to studying traffic flow may be grouped into two kinds: one is to use the mathematics and physics to obtain theoretical approaches, such as fluid-dynamical and gas-kinetic approaches;[3−5] the other is to use microscopic simulations, in which cars are considered as interacting particles, such as car-follow models[6,7] and cellular automata models.[8,9]

Based on several approaches such as a cellular automata model for freeway traffic[8−10] and the others[11,12] the fundamental diagram (the flux-density diagram) shows the phase transition from the free-jet situation to the car cluster regime with start–stop waves. It has also been shown by Kerner and Kinhaüser[13] that for high densities a car cluster can spontaneously appear in which the average velocity of cars is considerably lower than in the initial flow.[13,14] The phase transition separates the low density situation (in which all cars move independently at their maximal speeds) from the high-density region in which the formation of car's aggregates as bound states reduce the velocities of cars.

The spontaneous formation of traffic jams on highways have been noticed as one of primary traffic questions. Many researchers have used theories to try to explain and describe it. Barlovic *et al*[15] studied the random walk theory of jamming in a single-lane traffic model based on a cellular automaton approach, which may predict the resolving probabilities and lifetimes of jam clusters. Kühne *et al*[2] and Mahnke and Pieret[11] concluded the clustering behaviour (known as congestion) in an initially homogeneous traffic stream by a master equation, and applied a probabilistic model to describing the traffic features. However, the results are obtained mostly by using computer simulations.

The theoretical description of the jam formation has not been well developed because the synchronized mode possesses extremely complex structure.[16] Thereby, the aim of the present paper is to obtain analytical result of the master equation based on the statistic theory of birth–death process. In addition, the transition probabilities for the jump processes are

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also reconstructed.

2. The master equation

The master equation is a partial differential equation describing the probability distribution with the time evolution and random variable variation for a stochastic dynamical system that follows a Markov process. It has been successfully used to describe the linear models such as one-dimensional birth–death equations, random walks and fatigue crack growth.[17,18] We will derive the master equation of aggregation in freeway traffic in the following.

Traffic flow on a single-lane road is considered, and there appears only the spontaneous formation of a jam regarded as a large car cluster arising on the road. A car cluster of size n is an aggregation of n vehicles whose individual speeds are zero and the minimal allowed distances between vehicles are zero.

Assume that there appears a size m of car cluster at time t, and consider that the probability of the attachment of a 'free' car unit to its upstream boundary appearing in $t\to t+\triangle t$ is $W_+(m,t)\triangle t$ and the rate of cars escaping from the cluster at its downstream front is $W_-(m,t)\triangle t$. Then the probability of no change in the unit number in $t\to t+\triangle t$ is $1-(W_+(m,t)+W_-(m,t))\triangle t$. The probability of more than one unit appearing in $t\to t+\triangle t$ is zero.

Using the Markov theory,[17,18] we have the condition of probability

$$\begin{aligned}P_{1|1}(m,t|n,t+\triangle t) &= W_+(m,t)\delta_{n,m+1}\triangle t+W_-(m,t)\\ &\times\delta_{n,m-1}\triangle t+(1-(W_+(m,t)\\ &+W_-(m,t)))\delta_{n,m}\triangle t+O(\triangle t^2).\end{aligned}\tag{1}$$

The probability of taking n value at time t can be expressed as

$$P(n,t+\triangle t)=\sum_m P(m,t)P_{1|1}(m,t|n,t+\triangle t).\tag{2}$$

Substituting the expression (2) into the expression (1), we obtain a master equation of aggregation in freeway traffic as follows:

$$\begin{aligned}\frac{\partial P(n,t)}{\partial t} &= W_+(n-1,t)P(n-1,t)+W_-(n+1,t)\\ &\times P(n+1,t)-(W_+(n,t)+W_-(n,t))P(n,t),\end{aligned}\tag{3}$$

where $P(n,t)$ is the probability to find a size-n cluster at time t. Therefore we have obtained the same master equation as Mahnke and Pieret's.[11] Nevertheless, the transition rates are needed in order to deal with this equation.

3. The transition rates

A homogeneous flow and a single-lane road are considered, which is similar to the case in Ref.[11]. Because there exists only one jam on the road, the whole region of traffic flow may be separated into two regions: low-density region and congested region. We consider that cars will transfer from the low-density region to the congested region, at the same time, the other cars will escape from the congested region to the low-density region. The following assumptions are required:[11] 1) There exists a time constant of the relaxation for a driver to adapt himself or herself to the actual downstream state of the flow in free traffic, which is dependent on the driver's technical-psychological reaction to the situation; 2) Time constants for each car to speed up and to slow down are identical; 3) The braking time constant is much greater than the accelerating one. It is obvious that mean-field theory must be introduced.

The transition rates are defined as follows: considering the free flow upstream of the jam, we assume that it is homogeneous, and has an average speed u and headway h (bumper-to-bumper effective distance). To describe the homogeneous flow behaviour, it is necessary to choose a relation between the speed of a given vehicle and the distance to its leading vehicle. There are many expressions of the relation between speed and space density of vehicles. We choose a realistic and well-known model.[19] On the basis of a linear speed–density relationship, the following equation is obtained:[19]

$$u=u_{\rm f}\Big(1-\frac{k}{k_{\rm jam}}\Big),\tag{4}$$

where $u_{\rm f}$ is the allowed maximal speed, k is the space density of vehicle flow, and $k_{\rm jam}$ is the maximal space density of congested flow. Based on the assumptions above, we consider that the average speed of the free flow upstream of the jam is $u'=u_{\rm f}(1-k'/k_{\rm jam})$, where k' is the space density of flow of cars in free-flow region. At time t, the car directly behind the jam has to drive an average headway h, if the extreme assumption that the decelerating time constant is zero is made, and it needs to take an average h/u' time to get into the jam. Hence, the probability of the attachment of the 'free' cars to the jam at time $t+dt$ is $u'{\rm d}t/h$. The

transition rate per time unit is

$$W_+(n) = \frac{u'}{h}, \tag{5}$$

where u' and h are general functions of n. On the other hand, assume that the leading car of the jam can come freely again within the average time τ. So the transition rate of escaping from the jam is[11]

$$W_-(n) = \frac{1}{\tau}. \tag{6}$$

It is obvious that W_+ contains only a random variable n based on the mean-field theory. Now there is a circular road of length L with N cars moving on it to be considered, and all the cars are assumed to be identical vehicles of effective length l_0. When the road is crowded with cars, we obtain the space density of congested flow $k_{\rm jam} = 1/l_0$. Hence, the relation between the headway h and the size of the jam n is

$$h(n) = \frac{L - Nl_0}{N - n}. \tag{7}$$

With the help of the expressions (5) and (7), we rewrite the rate of the attachment as

$$W_+(n) = \frac{u'}{h} = u_{\rm f}\Big(1 - l_0\Big(\frac{N-n}{L-nl_0}\Big)\Big)\Big(\frac{N-n}{L-Nl_0}\Big). \tag{8}$$

It is evident that the forms of curve $W_+(n)$ for different densities $k = N/L$ are different(see Fig. 1). For $W_-(n)$, we have a straight line $W_- = 1/\tau = \text{const}$ in Fig.1.

The curves shown in Fig.1 allow us to describe the different behaviours at different densities $\rho = kl_0$. $1/\tau$ is not bigger than the maximum value of W_+, which leads to a stable homogeneous flow for the whole range of densities. We follow Mahnke's choice for realistic values of general parameters, $l_0 = 8$ m and $u_{\rm f} = 40$ m/s.[11] In a low density range, the free flow of nearly noninteracting particles is dominant and a jamming transition does not occur. We obtain the value of critical density $k^* = 1/\tau u_{\rm f}$. Below the critical value, the growing transition is always less probable than the decay transition, and hence any initial jam will disappear. When the density of cars exceeds the critical density k^*, there appears a jam. Therefore, an equilibrium size occurs as the growing transition balances with the decay transition, which is a stable stationary size of the jam. This means that, after a long time, the system will attain to equilibrium.

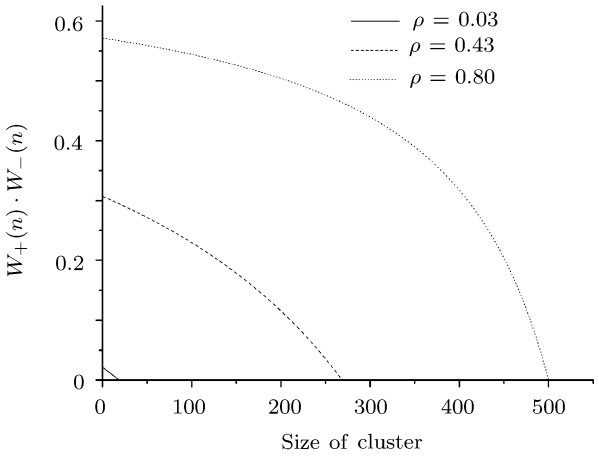


Fig.1. Analytical curves for the transition rates (the product of growth rate and decay rate) for different regimes (dimensionless densities). The parameters are $u_{\rm f} = 40$ m/s, $l_0 = 8$ m, and $\tau = 7$ s, $L = 5000$ m.

4. Statistical analysis of aggregation

We rewrite Eq.(8) for the convenience of discussion as follows:

$$W_+(n) = u_{\rm f}\Big(\frac{N-n}{L-Nl_0}\Big) - l_0u_{\rm f}\Big(\frac{(N-n)^2}{(L-Nl_0)(L-nl_0)}\Big) = u_{\rm f}\Big(\frac{N-n}{L-Nl_0}\Big) - l_0u_{\rm f}\Big(\frac{(1/L-n/NL)^2}{(1/N-l_0/L)(1/N-nl_0/NL)}\Big). \tag{9}$$

Considering n/NL as a small value (due to $n < N$) and using Maclaurin's expansion at $x_0 = n_0/NL$ (n_0 is the size of cluster) for Eq.(9), we obtain

$$\begin{aligned} W_+(n) = {} & \frac{Nu_{\rm f}}{L(1-l_0N/L)} - \frac{Nu_{\rm f}x_0}{1-l_0N/L} - \frac{l_0u_{\rm f}(1/L^2 - 2x_0/L + {x_0}^2)}{(1/N-l_0/L)(1/N-l_0x_0)} \\ & + \Bigg(-\frac{Nu_{\rm f}}{1-l_0N/L} + \frac{l_0u_{\rm f}(\frac{-2/L+2x_0}{1/N-l_0x_0} - \frac{l_0(1/L^2)-2x_0/L+{x_0}^2}{(1/N-l_0x_0)^2})}{1/N-l_0/L}\Bigg)\Big(\frac{n}{NL} - x_0\Big) + o\Big(\frac{n}{NL} - x_0\Big), \end{aligned} \tag{10}$$

where we expand only to the first-order n/NL for simplicity. Let

$$A = \frac{Nu_{\rm f}}{L(1-l_0N/L)} - \frac{Nu_{\rm f}x_0}{1-l_0N/L} - \frac{l_0u_{\rm f}(1/L^2 - 2x_0/L + {x_0}^2)}{(1/N-l_0/L)(1/N-l_0x_0)}$$

$$- x_0\Big(-\frac{Nu_{\rm f}}{1-l_0N/L}+\frac{l_0u_{\rm f}(\frac{-2/L+2x_0}{1/N-l_0x_0}-\frac{l_0(1/L^2)-2x_0/L+{x_0}^2}{(1/N-l_0x_0)^2})}{1/N-l_0/L}\Big),$$

and

$$B=\frac{1}{NL}\Big(-\frac{Nu_{\rm f}}{1-l_0N/L}+\frac{l_0u_{\rm f}(\frac{-2/L+2x_0}{1/N-l_0x_0}-\frac{l_0(1/L^2)-2x_0/L+{x_0}^2}{(1/N-l_0x_0)^2})}{1/N-l_0/L}\Big),$$

then we will rewrite Eq.(10) as

$$W_+(n)=A+Bn. \tag{11}$$

Substituting the expression (6) and Eq.(11) into Eq.(3) yields

$$\frac{\partial P(n,t)}{\partial t}=(A+B(n-1))P(n-1,t)+\frac{1}{\tau}P(n+1,t) - \Big(A+\frac{1}{\tau}+Bn\Big)P(n,t). \tag{12}$$

Using a generating function, we have

$$H(t,x)=\sum_{n=0}^{\infty}P(n,t)x^n. \tag{13}$$

Thus we can obtain a partial differential equation for the generating function H

$$x\frac{\partial H}{\partial t}=\Big[Ax^2+\frac{1}{\tau}-\Big(A+\frac{1}{\tau}\Big)x\Big]H + Bx^2(x-1)\frac{\partial H}{\partial x}. \tag{14}$$

Assume that the size of the cluster is n_0 at the beginning, then we will have

$$P(n,0)=\delta_{n,n_0}. \tag{15}$$

With the help of Eq.(13) and Eq.(15), we achieve

$$H(0,x)=x^{n_0}. \tag{16}$$

When $x=1$, using the normalized condition of $P(n,t)$, we obtain

$$H(t,1)=1. \tag{17}$$

Solving Eq.(14) by the use of Eq.(16), we find the analytical solution as follows:

$$H=x^{n_0}{\rm e}^{(x-1)/x\cdot(1-{\rm e}^{Bt})/B\tau} \times[x-(x-1){\rm e}^{Bt}]^{-(n_0+A/B)}. \tag{18}$$

Therefore, using Eq.(18), we can derive the values of mean $\langle n\rangle$ and the second degree moment $\langle n^2\rangle$ at any time t as follow:

$$\langle n(t)\rangle=x\frac{\partial H}{\partial x}\Big|_{x=1}=n_0{\rm e}^{Bt}+\frac{1-A\tau}{B\tau}(1-{\rm e}^{Bt}), \tag{19}$$

$$\langle n^2(t)\rangle=x\frac{\partial}{\partial x}\Big(x\frac{\partial H}{\partial x}\Big)\Big|_{x=1} =\Big[n_0{\rm e}^{Bt}+\frac{1-A\tau}{B\tau}+(1-{\rm e}^{Bt})\Big]^2 \times\Big[-\frac{1}{B\tau}-\Big(n_0+\frac{A}{B}\Big){\rm e}^{Bt}\Big](1-{\rm e}^{Bt}). \tag{20}$$

Using Eq.(19) and Eq.(20), we acquire the variance error and the relative error of the cluster size at any time t

$$\sigma_n(t)=\langle n^2(t)\rangle-\langle n(t)\rangle^2 =-\Big[\frac{1}{B\tau}+\Big(n_0+\frac{A}{B}\Big)\Big]{\rm e}^{Bt}(1-{\rm e}^{Bt}), \tag{21}$$

$$\alpha_n(t)=\sqrt{-\frac{[1/B\tau+(n_0+A/B)]{\rm e}^{Bt}(1-{\rm e}^{Bt})}{[n_0{\rm e}^{Bt}+(1-A\tau)(1-{\rm e}^{Bt})/B\tau]^2}}. \tag{22}$$

Equation (19) describes the dynamic behaviour of the system at far-from-critical densities. Since the values of A and B depend on x_0, i.e. n_0, Eq.(19) holds in some sections of time. Accordingly, we divide the measuring time into some sections and obtain the curves of the mean cluster size versus time t at different values of parameters A and B (see Fig.2). For different values of n_0, the system finally attains the same steady state (see Fig.3).

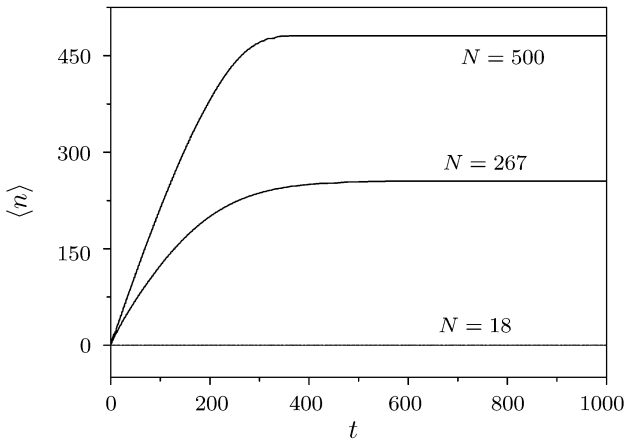


Fig.2. The curves of the mean cluster size $\langle n\rangle$ versus time t in an initially homogeneous traffic steam at different values of N(The parameters used are $u_{\rm f}=40$ m/s, $l_0=8$ m, $\tau=7$ s, and $L=5000$ m.)

Thus, we can obtain some information of aggregation in freeway traffic based on the equations that we have derived.

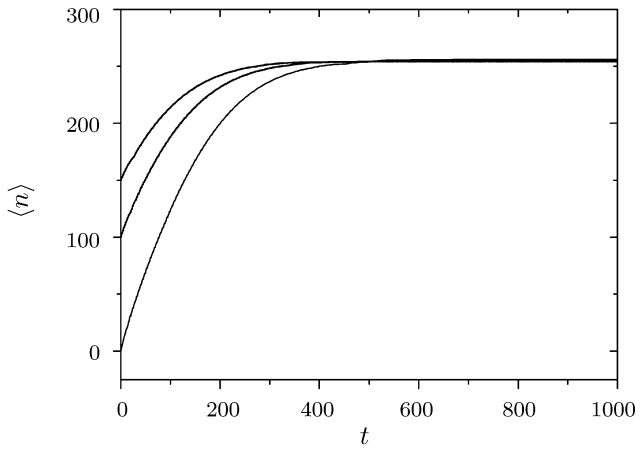


Fig.3. The curves of the mean cluster size $\langle n \rangle$ versus time t at different values of $n_0 (N = 270)$

5. Conclusion

We extend the stochastic master-equation approach to the case of congestion in traffic flow desired in Ref.[11] by means of the birth–death process. The rate of the attachment of 'free' cars to the cluster is reconstructed by using Greenshields' model. Different from the sigmoidal optimal velocity curve adopted by Mahnke, an equilibrium size is found to exist over critical densities. We derive the values of mean $\langle n \rangle$ and the second degree moment $\langle n^2 \rangle$ at any time t. Furthermore, we acquire the variance error and the relative error of the size of the cluster at any time t. We give both the curves of the mean cluster size $\langle n \rangle$ versus time t in an initially homogeneous traffic steam at different values of N and the curves of the mean cluster size $\langle n \rangle$ versus time t at different values of n_0, these figures are very agreeable with general situations of traffic jam problems.

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