Vol 16 No 3, March 2007
1009-1963/2007/16(03)/0611-10

Chinese Physics

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The extended trace identity and its application*

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(Received 3 June 2006; revised manuscript received 12 September 2006)

The trace identity is extended to the general loop algebra. The Hamiltonian structures of the integrable systems concerning vector spectral problems and the multi-component integrable hierarchy can be worked out by using the extended trace identity. As its application, we have obtained the Hamiltonian structures of the Yang hierarchy, the Korteweg-de–Vries (KdV) hierarchy, the multi-component Ablowitz–Kaup–Newell–Segur (M-AKNS) hierarchy, the multi-component Ablowitz–Kaup–Newell–Segur Kaup–Newell (M-AKNS–KN) hierarchy and a new multi-component integrable hierarchy separately.

Keywords: loop algebra, Killing form, trace identity, Hamiltonian structure
PACC: 0340K, 0220, 0365G

1. Introduction

The theory of integrable Hamiltonian systems with infinite dimensions has rapidly developed since the late 1960s. In 1989, Tu[1] proposed an efficient approach to searching for integrable Hamiltonian hierarchy, which was usually called Tu scheme. Nowadays, by using the Tu scheme, a lot of integrable systems with physical significance have been obtained.[1−7] The multi-component integrable systems have been developed in Refs.[8,9]. Recently, a new 3M-dimensional Lie algebra has been constructed and a simple method of producing the multi-component integrable hierarchy of soliton equation has been proposed in Ref.[10]. At present, many multi-component integrable systems such as the multi-component Korteweg-de–Vries (M-KdV) hierarchy, Tu hierarchy, etc. have been proposed[11−17] while the Hamiltonian structures of the obtained multi-component integrable systems have not been developed because the trace identity is not suitable for them. In this paper, we extend the trace identity to the general loop algebra. As its application, the Hamiltonian structures of the Yang hierarchy, the KdV hierarchy, the multi-component Ablowitz–Kaup–Newell–Segur (M-AKNS) hierarchy, the multi-component Ablowitz–Kaup–Newell–Segur Kaup–Newell (M-AKNS–KN) hierarchy and a new multi-component integrable hierarchy which can be reduced to the well-known heat-conduction equation are developed. The method proposed in this paper can be used for general purposes.

2. The extended trace identity

Let $G = \{U|U = (u_{ij})_{M\times N}\}$, where u_{ij} are real or complex; M and N are arbitrary positive integers. Obviously, G is a linear space. For arbitrary elements U, V of G, we can determine a kind of operation $[U, V]$ with the following properties:

(1°) antisymmetry: $[U, V] = -[V, U]$,
(2°) bilinear: $[aU + bV, W] = a[U, V] + b[V, W], U, V, W \in G, a, b = \text{const}$,
(3°) Jacobi identity: $[[U, V], W]+[[V, W], U]+[[W, U], V] = 0, \forall U, V, W \in G$.

Thus, we refer to $[U, V]$ as the commutation operation of U and V, denoted by

$$[,] : G \times G \to G.$$

Hence, G becomes a Lie algebra. Its corresponding loop algebra is denoted by $\tilde{G}$

$$\tilde{G} = G \otimes C[\lambda, \lambda^{-1}],$$

where $C[\lambda, \lambda^{-1}]$ is the set of Laurent polynomials in λ.

*Project supported by the National Natural Science Foundation of China (Grant Nos 10371070 and 10547123).
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Consider the following Lax pair:

$$\begin{cases} \varphi_x = [U, \varphi], \ \lambda_t = 0, \\ \varphi_t = [V, \varphi], \ \varphi, U, V \in \tilde{G}, \end{cases} \tag{1}$$

whose compatibility $\varphi_{xt} = \varphi_{tx}$ gives rise to

$$[U_t, \varphi] - [V_x, \varphi] + [U, [V, \varphi]] + [V, [\varphi, U]] = 0.$$

By employing property (3°), the above formula can be written as

$$[U_t, \varphi] - [V_x, \varphi] + [[U, V], \varphi] = 0.$$

Since φ is arbitrary, the following equation holds:

$$U_t - V_x + [U, V] = 0, \tag{2}$$

i.e. the compatibility of the expression (1) leads to the zero curvature equation.

Definition 1 For arbitrary elements U, V of G, we can define a kind of so-called Killing form and make it have the following properties:

(a) symmetry: $< U, V >=< V, U >$,
(b) bilinear: $< aU + bV, W >= a < U, W > +b < V, W >, a, b \in C$,
(c) invariance: $< [U, V], W >=< U, [V, W] >$.

Setting U and V to be two elements of G and $U = U(u, \lambda)$, we consider the integral function

$$\Delta = \int_{-\infty}^{+\infty} < V, \frac{\partial U}{\partial \lambda} > \mathrm{d}x \tag{3}$$

with the following constraint condition:

$$V_x = [U, V]. \tag{4}$$

The one-order variations of the expressions (3) and (4) are expressed as

$$\delta\Delta = \int_{-\infty}^{+\infty} \left(< \delta V, \frac{\partial U}{\partial \lambda} > + < V, \delta \left(\frac{\partial U}{\partial \lambda} \right) > \right) \mathrm{d}x, \tag{5}$$

$$\delta V_x = [\delta U, V] + [U, \delta V]. \tag{6}$$

Making use of Lagrange method of multipliers in the expression (6), we have

$$\int_{-\infty}^{+\infty} < \Lambda, \delta V_x - [\delta U, V] - [U, \delta V] > \mathrm{d}x = 0. \tag{7}$$

From the expressions (5) and (7), we obtain

$$\delta\Delta = \int_{-\infty}^{+\infty} \left(< \Lambda, \delta V_x > + < \Lambda, [V, \delta U] > + < \Lambda, [\delta V, U] > + < \delta V, \frac{\partial U}{\partial \lambda} > + < V, \delta \left(\frac{\partial U}{\partial \lambda} \right) > \right) \mathrm{d}x. \tag{8}$$

In terms of Definition 1, the expression (8) can be written as

$$\begin{aligned} \delta\Delta = & \int_{-\infty}^{+\infty} (- < \delta V, \Lambda_x > + < [\Lambda, V], \delta U > + < \delta V, [U, \Lambda] > + < \delta V, U_\lambda > + < V, \partial_\lambda(\delta U) >) \mathrm{d}x \\ = & \int_{-\infty}^{+\infty} (< [\Lambda, V], \delta U > + < V, \partial_\lambda(\delta U) > + < \delta V, -\Lambda_x + [U, \Lambda] + U_\lambda >) \mathrm{d}x. \end{aligned} \tag{9}$$

Choose an appropriate value of Λ such that

$$-\Lambda_x + [U, \Lambda] + U_\lambda = 0, \tag{10}$$

and use $U = U(u, \lambda)$ that depends only on u and λ, then we will obtain

$$\begin{aligned} \delta\Delta = & \int_{-\infty}^{+\infty} \left(< [\Lambda, V], \frac{\partial U}{\partial u} > \delta u \right. \\ & \left. + < V, \partial_\lambda \left(\frac{\partial U}{\partial u} \right) > \delta u \right) \mathrm{d}x, \end{aligned} \tag{11}$$

where $u = (u_1, u_2, \ldots, u_N)$, and $u_i = (u_{i1}, u_{i2}, \ldots, u_{iM})^{\mathrm{T}}$. On the other hand

$$\delta\Delta = \int_{-\infty}^{+\infty} \frac{\delta < V, \frac{\partial U}{\partial \lambda} >}{\delta u} \delta u \mathrm{d}x. \tag{12}$$

Comparing the expression (11) with expression (12) gives

$$\begin{aligned} \frac{\delta < V, \frac{\partial U}{\partial \lambda} >}{\delta u} = & < [\Lambda, V], \frac{\partial U}{\partial u} > \\ & + < V, \partial_\lambda \left(\frac{\partial U}{\partial u} \right) > . \end{aligned} \tag{13}$$

From Eqs.(4) and (10), we have

$$\begin{aligned} [\Lambda, V]_x &= [\Lambda_x, V] + [\Lambda, V_x] \\ &= [U_\lambda + [U, \Lambda], V] + [\Lambda, [U, V]] \\ &= [U_\lambda, V] + [[U, \Lambda], V] + [\Lambda, [U, V]] \\ &= [U_\lambda, V] + [[V, \Lambda], U]], \end{aligned} \tag{14}$$

$$V_{x\lambda} = [U_\lambda, V] + [U, V_\lambda]. \tag{15}$$

Subtracting the expression (15) from the expression (14) leads to

$$([\Lambda, V] - V_\lambda)_x = [U, [\Lambda, V] - V_\lambda].$$

Thus, $[\Lambda, V] - V_\lambda$ satisfies the constraint condition. Therefore, there exists a constant γ satisfying

$$[\Lambda, V] - V_\lambda = \frac{\gamma}{\lambda} V. \tag{16}$$

Inserting Eq.(16) into Eq.(13) gives

$$\begin{aligned}\frac{\delta < V, \frac{\partial U}{\partial \lambda} >}{\delta u} &=< V_\lambda + \frac{\gamma}{\lambda} V, \frac{\partial U}{\partial u} > + < V, \partial_\lambda \left(\frac{\partial U}{\partial u}\right) >=< \partial_\lambda V, \frac{\partial U}{\partial u} > + < \frac{\gamma}{\lambda} V, \frac{\partial U}{\partial u} > + < V, \partial_\lambda \left(\frac{\partial U}{\partial u}\right) > \\ &= \partial_\lambda < V, \frac{\partial U}{\partial u} > + \frac{\gamma}{\lambda} < V, \frac{\partial U}{\partial u} >= \lambda^{-\gamma} \partial_\lambda \lambda^\gamma < V, \frac{\partial U}{\partial u} > . \end{aligned} \tag{17}$$

Summing up the above result yields the following conclusions:

Theorem 1 Let $U = U(u, \lambda)$ ($u = (u_1,\ u_2,\ \ldots,\ u_N), u_i = (u_{i1},\ u_{i2},\ \ldots,\ u_{iM})^{\mathrm{T}}$) be an element of $\tilde{G}$ and suppose we have introduced the rank of u_i, ∂, λ and $x \in \tilde{G}$, and the solution V of Eq.(4), which is given homogeneous rank, will be unique up to a constant multiplier. Under Definition 1, the equation

$$\frac{\delta}{\delta u} < V, \frac{\partial U}{\partial \lambda} >= \lambda^{-\gamma} \partial_\lambda \lambda^\gamma < V, \frac{\partial U}{\partial u} > \tag{18}$$

holds, which is called the extended trace identity.

If $M = N$, thus G is a matrix semi-simple Lie algebra. It is well known that the Killing form $< U, V >$ is equal to the trace, $\mathrm{tr}(UV)$. Thus, Eq.(18) is just the trace identity in Ref.[1].

Proposition 1 Let J and L be two linear operators mapping S^{M}($S^{\mathrm{M}} = S \bigotimes S \ldots \bigotimes S$, S is the Schwartz space) into itself. Suppose that

(i) both J and JL are skew symmetric, i.e.

$$J^* = -J, \quad JL = L^* J, \tag{19}$$

(ii) there exists a series of scalar functions $\{H_n\}$ satisfying

$$L^n f(u) = \frac{\delta H_n}{\delta u} \tag{20}$$

for some $f(u) \in S^{\mathrm{M}}$, then $\{H_n\}$ will be a common series of conserved densities for the whole hierarchy of equation

$$u_t = JL^n f(u),$$

and we have

$$\{H_n, H_m\} = 0.$$

3. The Hamiltonian structure of the integrable system concerning vector spectral problem

In this section, we take the Yang hierarchy and KdV hierarchy as examples to illustrate that the integrable systems also can be obtained by solving the vector spectral problems and their Hamiltonian structures can be presented by the extended trace identity.

If $M = 1, N$ is arbitrary positive integer, thus G denotes a vector set as follows:

$$\boldsymbol{G}_N = \{\boldsymbol{a} = (\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_N)\},$$

where $\boldsymbol{a}$ denotes a row vector, $\boldsymbol{a}_i \in C(i = 1, 2, \ldots, N)$. For an arbitrary element of $\boldsymbol{G}_N$, we can also define a commutation operator that has the properties (1°)–(3°) in Section 2. Hence $\boldsymbol{G}_N$ becomes a Lie algebra, and its corresponding loop algebra is denoted by $\tilde{G}_N$. Next, we consider only the case of $N = 3$.

3.1. The Hamiltonian structure of Yang hierarchy

Definition 2 Let $a = (a_1, a_2, a_3)$, and $b = (b_1, b_2, b_3)$ be two elements of G_3 and define a commutation operator $[a, b]$ as

$$[a, b] = (a_2 b_3 - a_3 b_2, a_1 b_3 - a_3 b_1, a_1 b_2 - a_2 b_1). \tag{21}$$

It is easy to verify that G_3 along with the expression (21) becomes a Lie algebra.

A corresponding loop algebra is given as

$$\tilde{G}_3 = \{a\lambda^m, a \in G_3, m = 0, \pm 1, \pm 2 \quad \ldots \ \}, \tag{22}$$

with a commutation operation being

$$[a\lambda^m, b\lambda^n] = [a, b]\lambda^{m+n}, \quad \forall a, b \in G_3. \tag{23}$$

Consider the isospectral problem

$$\varphi_x = [U, \varphi], \quad \lambda_t = 0, \quad U = (s, q + \lambda, r) \tag{24}$$

and set $V = \sum_{m=0}^{\infty} (a_m, b_m, c_m)\lambda^{-m}$, then solving the auxiliary equation

$$V_x = [U, V] \tag{25}$$

will give rise to

$$\begin{cases} a_{mx} = c_{m+1} + qc_m - rb_m, \quad b_{mx} = sc_m - ra_m, \quad c_{mx} = -a_{m+1} + sb_m - qa_m, \\ a_0 = c_0 = 0, \quad b_0 = \alpha, \quad a_1 = \alpha s, \quad b_1 = 0, \quad c_1 = \alpha r. \end{cases} \tag{26}$$

By employing the Tu scheme, the Yang hierarchy is presented as

$$u_t = \begin{pmatrix} q \\ r \\ s \end{pmatrix}_t = J \begin{pmatrix} -b_{n+1} \\ c_{n+1} \\ a_{n+1} \end{pmatrix} = \cdots = JL^n \begin{pmatrix} 0 \\ \alpha r \\ \alpha s \end{pmatrix}, \tag{27}$$

where $J = \begin{pmatrix} -\partial & -s & r \\ s & 0 & -1 \\ -r & 1 & 0 \end{pmatrix}$ is Hamilton operator,

$L = \begin{pmatrix} 0 & 2\partial^{-1}qs - \partial^{-1}r\partial & -\partial^{-1}s\partial - 2\partial^{-1}qr \\ -r & -q & \frac{\partial}{2} \\ -s & -\frac{\partial}{2} & -q \end{pmatrix}$ is a recurrence operator.

Definition 3 Set $U = (U_1, U_2, U_3)$ and $V = (V_1, V_2, V_3)$ as two elements of $\tilde{G}_3$, and their so-called Killing form is defined as

$$< U, V >= U_1V_1 - U_2V_2 + U_3V_3. \tag{28}$$

It is easy to check that $< U, V >$ has the properties (a)–(c) in Section 2.

Note $V = (a,\ b,\ c)$, $a = \sum_{m\geq 0} a_m\lambda^{-m}$, $b = \sum_{m\geq 0} b_m\lambda^{-m}$, and $c = \sum_{m\geq 0} c_m\lambda^{-m}$.

By using Definition 3, a direct calculation gives

$$< V, \frac{\partial U}{\partial \lambda} >= -b, \quad < V, \frac{\partial U}{\partial q} >= -b, \quad < V, \frac{\partial U}{\partial r} >= c, \quad \text{and} \quad < V, \frac{\partial U}{\partial s} >= a. \tag{29}$$

Inserting the expressions (29) into the extended trace identity (18) and comparing the coefficient of λ^{-n-1} lead to

$$\frac{\delta}{\delta u}(-b_{n+1}) = (\gamma - n) \begin{pmatrix} -b_n \\ c_n \\ a_n \end{pmatrix}. \tag{30}$$

Setting $n = 1$ in Eq.(30) gives $\gamma = 0$.

Thus, the Hamiltonian function is presented as

$$\frac{\delta H_n}{\delta u} = \begin{pmatrix} -b_n \\ c_n \\ a_n \end{pmatrix}, \ H_n = \frac{b_{n+1}}{n}. \tag{31}$$

Hence, the Yang hierarchy has the following Hamiltonian structure:

$$u_t = J\frac{\delta H_{n+1}}{\delta u}. \tag{32}$$

3.2. The Hamiltonian structure of KdV hierarchy

Definition 4 Let $a = (a_1,\ a_2,\ a_3)$, and $b = (b_1,\ b_2,\ b_3)$ be two elements of G_3, a commutation operator $[a, b]$ is defined as

$$[a, b] = (a_2b_3 - a_3b_2, 2(a_1b_2 - a_2b_1), 2(a_3b_1 - a_1b_3)). \tag{33}$$

It is easy to verify that G_3 along with (33) also becomes a Lie algebra. Its corresponding loop algebra

and commutation operation are respectively similar to the expressions (22) and (23).

Consider the following isospectral problem:

$$\varphi_x = [U, \varphi], \quad \lambda_t = 0, \quad U = (0, 1, \lambda - q), \tag{34}$$

and set $V = \sum_{m=0}^{\infty} (a_m, b_m, c_m)\lambda^{-m}$, then solving the equation similar to Eq.(25) will yield

$$\begin{cases} a_{mx} = -b_{m+1} + c_m + qb_m, \quad b_{mx} = -2a_m, \quad c_{mx} = -b_{m+1,x} - 2qa_m, \\ a_0 = b_0 = 0, \quad c_0 = \beta, \quad a_1 = 0, \quad b_1 = \beta, \quad c_1 = -\dfrac{\beta}{2}q. \end{cases} \tag{35}$$

By utilizing the Tu scheme, the KdV hierarchy is presented as

$$q_t = -2b_{n+1,x} = J(-b_{n+1}) = \cdots = JL^n\beta, \tag{36}$$

where $J = 2\partial$ is the Hamilton operator, and $L = \frac{1}{4}\partial^2 - \frac{1}{2}\partial^{-1}q_x + q$ is a recurrence operator.

Definition 5 Set $U = (U_1,\ U_2,\ U_3)$, and $V = (V_1,\ V_2,\ V_3)$ as two elements of $\tilde{G}_3$, and their so-called Killing form is defined as

$$< U, V >= 2U_1V_1 + U_2V_3 + U_3V_2. \tag{37}$$

It is easy to test that $< U, V >$ also has the properties (a)–(c) in Section 2.

By making use of Definition 5, a direct calculation gives

$$< V, \frac{\partial U}{\partial \lambda} >= b, \quad \text{and} \quad < V, \frac{\partial U}{\partial q} >= -b. \tag{38}$$

Inserting Eq.(38) into the extended trace identity (18) and comparing the coefficients of λ^{-n-1} yield

$$\frac{\delta}{\delta q}(b_{n+1}) = (\gamma - n)(-b_n). \tag{39}$$

Setting $n = 1$ in Eq.(39) gives $\gamma = \frac{1}{2}$. Thus, the Hamiltonian function is presented as

$$\frac{\delta H_n}{\delta q} = -b_n, \quad \text{and} \quad H_n = -\frac{2b_{n+1}}{n-1}. \tag{40}$$

Consequentially, the Hamiltonian structure of KdV hierarchy is obtained as follows:

$$q_t = J\frac{\delta H_{n+1}}{\delta u}. \tag{41}$$

4. The Hamiltonian structure of the multi-component integrable system

Let G_M denote a matrix set as follows:

$$G_M = \{a = (a_{ij})_{M\times N} = (a_1, a_2, \ldots, a_N)\}, \tag{42}$$

where a_{ij} are real or complex functions; $a_i (i = 1, 2, \ldots, N)$ denote the ith column of the matrix a; M and N are arbitrary positive integers.

Definition 6[8] Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_M)^{\mathrm{T}}$, and $\beta = (\beta_1, \beta_2, \ldots, \beta_M)^{\mathrm{T}}$ be two column vectors. Their vector product $\alpha * \beta$ is defined as

$$\alpha * \beta = \beta * \alpha = (\alpha_1\beta_1, \alpha_2\beta_2, \ldots, \alpha_M\beta_M)^{\mathrm{T}}. \tag{43}$$

Definition 7 Setting $X = (X_1, X_2, \ldots X_N)$, and $Y = (Y_1, Y_2, \ldots, Y_N)$ as two elements of G_M, where $X_i = (X_{i1}, X_{i2}, \ldots, X_{iM})^{\mathrm{T}}$, and $Y_i = (Y_{i1}, Y_{i2}, \ldots, Y_{iM})^{\mathrm{T}}$, we can always define an operation $[X, Y]$ and make it have the properties (1°)–(3°) in Section 2. Thus, G_M becomes a Lie algebra and dim $G_M = MN$. A corresponding loop algebra and commutation operation are respectively similar to the expressions (22) and (23), and the loop algebra is denoted by $\tilde{G}_M$.

4.1. The bi-Hamiltonian structure of the M-AKNS hierarchy

In this section, we consider only the case of $N = 3$, thus the expression (42) can be written as

$$G_M = \{a = (a_{ij})_{M\times 3} = (a_1, a_2, a_3)\}. \tag{44}$$

Definition 8[8] Set $a = (a_1,\ a_2,\ a_3)$, and $b = (b_1,\ b_2,\ b_3)$ to be two elements of G_M; a commutation

operation is defined as

$$[a,b]=(a_2*b_3-a_3*b_2,2(a_1*b_2-a_2*b_1),\\ 2(a_3*b_1-a_1*b_3)). \tag{45}$$

It is easy to verify that the operation (45) has the properties $(1^\circ)-(3^\circ)$ in Section 2, hence, G_M is a Lie algebra along with the commutation operation (45) and $\dim G_M=3M$. Its corresponding loop algebra is also denoted by $\tilde{G}_M$ whose commutation operation is similar to the operation (23).

Consider an isospectral problem

$$\varphi_x=[U,\varphi],\quad \lambda_t=0,\quad U=(-I_{\rm M}\lambda,q,r), \tag{46}$$

where

$$I_M=(1,1,\ldots,1)^{\rm T}_{1\times M},$$

$$q=(q_1,q_2,\ldots,q_M)^{\rm T},$$

$$r=(r_1,r_2,\ldots,r_M)^{\rm T}.$$

Set

$$V=\sum_{m=0}^{\infty}(a_m,b_m,c_m)\lambda^{-m},$$

where $a_m=(a_{m1},\ a_{m2},\ \ldots,\ a_{mM})^{\rm T}$, $b_m=(b_{m1},\ b_{m2},\ \ldots,\ b_{mM})^{\rm T}$, and $c_m=(c_{m1},\ c_{m2},\ \ldots,\ c_{mM})^{\rm T}$. Solving the equation similar to Eq.(25) yields

$$\begin{cases} a_{mx}=q*c_m-r*b_m,\quad b_{mx}=-2b_{m+1}-2a_m*q,\quad c_{mx}=2c_{m+1}+2r*a_m,\\ a_0=\alpha,\quad b_0=c_0=0. \end{cases} \tag{47}$$

Making use of the Tu scheme, the multi-component AKNS hierarchy can be worked out as follows:

$$u_t=\begin{pmatrix} q\\ r\end{pmatrix}_t=\begin{pmatrix} 0 & -2I_M*\\ 2I_M* & 0\end{pmatrix}\begin{pmatrix} c_{n+1}\\ b_{n+1}\end{pmatrix}=J\begin{pmatrix} c_{n+1}\\ b_{n+1}\end{pmatrix}=JL^n\begin{pmatrix} -\alpha*r\\ -\alpha*q\end{pmatrix}, \tag{48}$$

where J is a Hamiltonian operator,

$$L=\begin{pmatrix} \frac{1}{2}\partial-r*\partial^{-1}q* & r*\partial^{-1}r*\\ -q*\partial^{-1}q* & q*\partial^{-1}r*-\frac{1}{2}\partial\end{pmatrix}$$

is a recurrence operator.

Definition 9 Let $U=(U_1,U_2,U_3)$, and $V=(V_1,V_2,V_3)$ be two elements of $\tilde{G}_M$, where $U_i=(U_{i1},U_{i2},\ldots U_{iM})^{\rm T}$, and $V_i=(V_{i1},V_{i2},\ldots V_{iM})^{\rm T}, i=1,2,3$, their so-called Killing form is defined as

$$<U,V>=2U_1*V_1+U_2*V_3+U_3*V_2. \tag{49}$$

It is easy to verify that $<U,V>$ has the properties (a)–(c) in Section 2.

Note

$$V=(a,b,c),\quad a=\sum_{m\geq 0}a_m\lambda^{-m},\\ b=\sum_{m\geq 0}b_m\lambda^{-m},\ \text{ and }\ c=\sum_{m\geq 0}c_m\lambda^{-m}.$$

By utilizing Definition 9, a direct computation gives

$$<V,\frac{\partial U}{\partial\lambda}>=-2a,\quad <V,\frac{\partial U}{\partial q}>=c,\\ \text{and}\quad <V,\frac{\partial U}{\partial r}>=b. \tag{50}$$

Inserting the expressions (50) into the extended trace identity (18) and comparing the coefficient of λ^{-n-1} lead to

$$\frac{\delta}{\delta u}(-2a_{n+1})=(\gamma-n)\begin{pmatrix} c_n\\ b_n\end{pmatrix}. \tag{51}$$

Setting $n=1$ in the expression (51) gives $\gamma=0$.

Thus, we may present the Hamiltonian function as follows:

$$\frac{\delta H_n}{\delta u}=\begin{pmatrix} c_n\\ b_n\end{pmatrix},\quad H_n=\frac{2a_{n+1}}{n}. \tag{52}$$

Therefore, M-AKNS hierarchy has the following bi-Hamiltonian structure:

$$u_t=J\frac{\delta H_{n+1}}{\delta u}=K\frac{\delta H_n}{\delta u}, \tag{53}$$

where

$$K=JL=\begin{pmatrix} q*\partial^{-1}q* & -2q*\partial^{-1}r*+\partial\\ \partial-2r*\partial^{-1}q* & 2r*\partial^{-1}r*\end{pmatrix}.$$

It is easy to verify that $JL=L^*J$. Hence, the system (48) is integrable in Liouville sense.

4.2. The Hamiltonian structure of the M-AKNS–KN hierarchy

In Ref.[10], an isospectral problem is given as

$$\varphi_x = [U, \varphi], \quad \lambda_t = 0, \quad \text{and} \quad U = (\lambda^2 I_M, u_1 + \lambda u_3, u_2 + \lambda u_4), \tag{54}$$

where

$$I_M = (1, 1, \ldots, 1)^{\mathrm{T}}_{1\times M}, \quad \text{and} \quad u_i = (u_{i1}, u_{i2}, \ldots, u_{iM})^{\mathrm{T}},\ i = 1, 2, 3, 4.$$

Set

$$V = \sum_{m=0}^{\infty} (a(0,m) + \lambda a(1,m), b(0,m) + \lambda b(1,m), c(0,m) + \lambda c(1,m))\lambda^{-2m},$$

where
$a(i,m) = \left(a^{(i)}_{m1}, \quad a^{(i)}_{m2}, \ldots, a^{(i)}_{mM}\right)^{\mathrm{T}}, \quad b(i,m) = \left(b^{(i)}_{m1}, b^{(i)}_{m2}, \ldots, b^{(i)}_{mM}\right)^{\mathrm{T}}, \quad \text{and } c(i,m) = \left(c^{(i)}_{m1}, c^{(i)}_{m2}, \ldots, c^{(i)}_{mM}\right)^{\mathrm{T}},$
$(i = 0, 1)$.
Solving the equation similar to Eq.(25) gives rise to

$$\begin{cases} b(1,m+1) = \dfrac{1}{2} b_x(1,m) + u_1 * a(1,m) + u_3 * a(0,m), \\ c(1,m+1) = -\dfrac{1}{2} c_x(1,m) + u_2 * a(1,m) + u_4 * a(0,m), \\ a_x(0,m) = u_1 * c(0,m) - u_2 * b(0,m) + u_3 * c(1,m+1) - u_4 * b(1,m+1), \\ a_x(1,m) = u_1 * c(1,m) - u_2 * b(1,m) + u_3 * c(0,m) - u_4 * b(0,m), \\ b(0,m+1) = u_3 * a(1,m+1) + \dfrac{1}{2} b_x(0,m) + u_1 * a(0,m), \\ c(0,m+1) = u_4 * a(1,m+1) - \dfrac{1}{2} c_x(0,m) + u_2 * a(0,m), \\ a(0,0) = \beta = (\beta_1, \beta_2, \ldots, \beta_M)^{\mathrm{T}}, \quad b(1,1) = \beta * u_3, \quad c(1,1) = \beta * u_4, \\ b(0,1) = \beta * u_1, \quad c(0,1) = \beta * u_2. \end{cases}$$

Similarly, we have the M-AKNS–KN hierarchy

$$\begin{aligned} u_t &= \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}_{\mathrm{t}} = J_1 \begin{pmatrix} c(1,n+1) \\ b(1,n+1) \\ c(0,n+1) \\ b(0,n+1) \end{pmatrix} = J_1 L \begin{pmatrix} c(0,n) \\ b(0,n) \\ c(1,n+1) \\ b(1,n+1) \end{pmatrix} = J_2 \begin{pmatrix} c(0,n) \\ b(0,n) \\ c(1,n+1) \\ b(1,n+1) \end{pmatrix} \\ &= J_1 L^2 \begin{pmatrix} c(1,n) \\ b(1,n) \\ c(0,n) \\ b(0,n) \end{pmatrix} = \cdots = J_1 L^{2n} \begin{pmatrix} \beta * u_4 \\ \beta * u_3 \\ \beta * u_2 \\ \beta * u_1 \end{pmatrix}, \end{aligned} \tag{55}$$

where

$$J_1 = \begin{pmatrix} -2u_1 * \partial^{-1} u_3 * - 2u_3 * \partial^{-1} u_1 * & 2u_3 * \partial^{-1} u_2 * + 2u_1 * \partial^{-1} u_4 * & -2u_3 * \partial^{-1} u_3 * & 2I_M * + 2u_3 * \partial^{-1} u_4 * \\ 2u_2 * \partial^{-1} u_3 * + 2u_4 * \partial^{-1} u_1 * & -2u_2 * \partial^{-1} u_4 * - 2u_4 * \partial^{-1} u_2 * & -2I_M * + 2u_4 * \partial^{-1} u_3 * & -2u_4 * \partial^{-1} u_4 * \\ -2u_3 * \partial^{-1} u_3 * & 2I_M * + 2u_3 * \partial^{-1} u_4 * & 0 & 0 \\ -2I_M * + 2u_4 * \partial^{-1} u_3 * & -2u_4 * \partial^{-1} u_4 * & 0 & 0 \end{pmatrix},$$

$$J_2 = \begin{pmatrix} 2u_1 * \partial^{-1} u_1 * & \partial - 2u_1 * \partial^{-1} u_2 * & 0 & 0 \\ \partial - 2u_2 * \partial^{-1} u_1 * & 2u_2 * \partial^{-1} u_2 * & 0 & 0 \\ 0 & 0 & -2u_3 * \partial^{-1} u_3 * & 2I_M * + 2u_3 * \partial^{-1} u_4 * \\ 0 & 0 & -2I_M * + 2u_4 * \partial^{-1} u_3 * & -2u_4 * \partial^{-1} u_4 * \end{pmatrix},$$

are Hamiltonian operators, L is the recurrence operator $L=(l_{ij})_{4\times4}$

$$\begin{cases} l_{11}=l_{12}=l_{21}=l_{22}=0, \quad l_{13}=l_{24}=I_M, \\ l_{31}=-\frac{1}{2}\partial+u_2*\partial^{-1}u_1*-\frac{1}{2}u_4*\partial^{-1}u_3*\partial+u_4*\partial^{-1}(u_2*u_3-u_1*u_4)*\partial^{-1}u_1*, \\ l_{32}=-u_2*\partial^{-1}u_2*-\frac{1}{2}u_4*\partial^{-1}u_4*\partial-u_4*\partial^{-1}(u_2*u_3-u_1*u_4)*\partial^{-1}u_2* \\ l_{33}=u_2*\partial^{-1}u_3*+u_4*\partial^{-1}u_1*+u_4*\partial^{-1}(u_2*u_3-u_1*u_4)*\partial^{-1}u_3*, \\ l_{34}=-u_2*\partial^{-1}u_4*-u_4*\partial^{-1}u_2*-u_4*\partial^{-1}(u_2*u_3-u_1*u_4)*\partial^{-1}u_4*, \\ l_{41}=u_1*\partial^{-1}u_1*-\frac{1}{2}u_3*\partial^{-1}u_3*\partial+u_3*\partial^{-1}(u_2*u_3-u_1*u_4)*\partial^{-1}u_1*, \\ l_{42}=\frac{1}{2}\partial-u_1*\partial^{-1}u_2*-\frac{1}{2}u_3*\partial^{-1}u_4*\partial-u_3*\partial^{-1}(u_2*u_3-u_1*u_4)*\partial^{-1}u_2*, \\ l_{43}=u_3*\partial^{-1}u_1*+u_1*\partial^{-1}u_3*+u_3*\partial^{-1}(u_2*u_3-u_1*u_4)*\partial^{-1}u_3*, \\ l_{44}=-u_1*\partial^{-1}u_4*-u_3*\partial^{-1}u_2*-u_3*\partial^{-1}(u_2*u_3-u_1*u_4)*\partial^{-1}u_4*. \end{cases}$$

Note

$$V=(a(0,m)+\lambda a(1,m),b(0,m)+\lambda b(1,m),c(0,m)+\lambda c(1,m)),$$

$$a(i,m)=\sum_{m\geq0}a(i,m)\lambda^{-2m}, \quad b(i,m)=\sum_{m\geq0}b(i,m)\lambda^{-2m}, \quad c(i,m)=\sum_{m\geq0}c(i,m)\lambda^{-2m}, \quad i=1,2.$$

By utilizing Definition 9, a direct calculation yields

$$\begin{aligned} &<V,\frac{\partial U}{\partial\lambda}>=4\lambda a(0,m)+4\lambda^2a(1,m)+u_4*b(0,m)+\lambda u_4*b(1,m)+u_3*c(0,m)+\lambda u_3*c(1,m), \\ &<V,\frac{\partial U}{\partial u_1}>=c(0,m)+\lambda c(1,m), \quad <V,\frac{\partial U}{\partial u_2}>=b(0,m)+\lambda b(1,m), \\ &<V,\frac{\partial U}{\partial u_3}>=\lambda c(0,m)+\lambda^2c(1,m), \quad <V,\frac{\partial U}{\partial u_4}>=\lambda b(0,m)+\lambda^2b(1,m). \end{aligned} \tag{56}$$

Substituting the expressions (56) into the extended trace identity (18) and comparing the coefficients of λ^{-2n-2} and λ^{-2n-1} respectively, yield

$$\frac{\delta}{\delta u}(4a(1,n+2)+u_4*b(0,n+1)+u_3*c(0,n+1))=(\gamma-2n-1)\begin{pmatrix} c(1,n+1) \\ b(1,n+1) \\ c(0,n+1) \\ b(0,n+1) \end{pmatrix}, \tag{57}$$

$$\frac{\delta}{\delta u}(4a(0,n+1)+u_4*b(1,n+1)+u_3*c(1,n+1))=(\gamma-2n)\begin{pmatrix} c(0,n) \\ b(0,n) \\ c(1,n+1) \\ b(1,n+1) \end{pmatrix}. \tag{58}$$

Substituting the initial values into Eqs.(57) and (58) gives $\gamma=0$. Therefore, we may present two Hamiltonian functions as follows:

$$\frac{\delta H_1(n)}{\delta u}=\begin{pmatrix} c(1,n+1) \\ b(1,n+1) \\ c(0,n+1) \\ b(0,n+1) \end{pmatrix}, \ H_1(n)=-\frac{4a(1,n+2)+u_4*b(0,n+1)+u_3*c(0,n+1))}{2n+1}, \tag{59}$$

$$\frac{\delta H_2(n)}{\delta u}=\begin{pmatrix} c(0,n) \\ b(0,n) \\ c(1,n+1) \\ b(1,n+1) \end{pmatrix}, \ H_2(n)=-\frac{4a(0,n+1)+u_4*b(1,n+1)+u_3*c(1,n+1)}{2n}. \tag{60}$$

Thus, the M-AKNS-KN hierarchy has the following Hamiltonian structure:

$$u_t = J_1\frac{\delta H_1(n)}{\delta u} = J_2\frac{\delta H_2(n)}{\delta u}. \tag{61}$$

4.3. The bi-Hamiltonian structure of a new multi-component integrable system

In this section, we consider the case of $N = 4$. Thus the expression (42) can be written as

$$G_M = \{a = (a_{ij})_{M\times 4} = (a_1, a_2, a_3, a_4)\}. \tag{62}$$

Definition 10 Set $a = (a_1,\ a_2,\ a_3,\ a_4)$, $b = (b_1,\ b_2,\ b_3,\ b_4)$ to be two elements of $G_{\rm M}$, and define a commutation operation $[a, b]$ as

$$\begin{aligned}[a,b] = \ & (2(a_2 * b_4 - a_4 * b_2), 2(a_4 * b_1 - a_1 * b_4)\\ & + a_4 * b_3 - a_3 * b_4, 0, 2(a_2 * b_1 - a_1 * b_2)\\ & + a_2 b_3 - a_3 * b_2).\end{aligned} \tag{63}$$

It is easy to test that (63) has the properties (1°)–(3°) in Section 2, so G_M along with the operation (63) also becomes a Lie algebra and $\dim G_{\rm M} = 4M$. Its corresponding loop algebra is also denoted by $\tilde{G}_{\rm M}$ and it has the commutation operation similar to the operation (23).

Consider an isospectral problem

$$\varphi_x = [U, \varphi], \quad \lambda_t = 0, \quad U = (q, r, \lambda I_M, s), \tag{64}$$

where

$$\begin{aligned} I_M = \ & (1, 1, \ldots, 1)^{\rm T}_{1\times M},\\ q = \ & (q_1, q_2, \ldots, q_M)^{\rm T},\\ r = \ & (r_1, r_2, \ldots, r_M)^{\rm T},\\ s = \ & (s_1, s_2, \ldots, s_M)^{\rm T}. \end{aligned}$$

Set

$$V = \sum_{m=0}^{\infty}(a_m, b_m, 0, c_m)\lambda^{-m}$$

where $a_m = (a_{m1},\ a_{m2},\ \ldots,\ a_{mM})^{\rm T}$, $b_m = (b_{m1},\ b_{m2},\ \ldots,\ b_{mM})^{\rm T}$, and $c_m = (c_{m1},\ c_{m2},\ \ldots,\ c_{mM})^{\rm T}$. Solving the equation similar to Eq.(25) yields

$$\begin{cases} a_{mx} = 2r * c_m - 2s * b_m,\\ b_{mx} = 2s * a_m - 2q * c_m - c_{m+1},\\ c_{mx} = 2r * a_m - 2q * b_m - b_{m+1},\\ b_0 = c_0 = 0, \quad a_0 = \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_M)^{\rm T}. \end{cases}$$

Making use of the Tu scheme, a multi-component integrable hierarchy is presented as follows:

$$u_t = \begin{pmatrix} q\\ r\\ s\end{pmatrix}_t = \begin{pmatrix} \partial I_M * & 0 & 0\\ 0 & -\partial I_M * & q*\\ 0 & -q* & \partial I_M *\end{pmatrix}\begin{pmatrix} a_n\\ b_n\\ -c_n\end{pmatrix} = J\begin{pmatrix} a_n\\ b_n\\ -c_n\end{pmatrix} = JL^n\begin{pmatrix} 0\\ 2\alpha * r\\ -2\alpha * s\end{pmatrix}, \tag{65}$$

where J is Hamiltonian operator, and

$$L = \begin{pmatrix} 0 & 4\partial^{-1}s * q * -2\partial^{-1}r * \partial & \partial^{-1}r * q * -2\partial^{-1}s * \partial\\ 2r* & -2q* & \partial\\ -2s* & \partial & -2q*\end{pmatrix}$$ is a recurrence operator.

Taking $n = 2$, the expression (65) reduces to the following multi-component nonlinear evolution equation:

$$\begin{cases} q_t = 4\alpha(s * s_x - r * r_x),\\ r_t = 2\alpha(4q * r_x + 2r * q_x + s_{xx} + 4s * q * q),\\ s_t = 2\alpha(4r * q * q + 4q * s_x + 2s * q_x + r_{xx}. \end{cases}$$

Further reducing and setting $s = r$, the well-known multi-component heat-conduction equation can be obtained as

$$r_t = 2\alpha r_{xx}.$$

Definition 11 Let $U = (U_1, U_2, U_3, U_4)$, and $V = (V_1, V_2, V_3, V_4)$, where $U_i = (U_{i1}, U_{i2}, \ldots,\ U_{iM})^{\rm T}$, and $V_i = (V_{i1}, V_{i2}, \ldots, V_{iM})^{\rm T}, i = 1, 2, 3, 4$, and the so-called Killing form is defined as

$$\begin{aligned} < U, V > = \ & U_1 * V_1 + U_2 * V_2 + \frac{1}{2}(U_3 * V_1)\\ & + \frac{1}{2}(U_1 * V_3) - U_4 * V_4. \end{aligned} \tag{66}$$

It is easy to verify that $< U, V >$ has the properties (a)–(c) in Section 2.

Note

$$V=(a,b,0,c),\quad a=\sum_{m\geq 0}a_m\lambda^{-m},\qquad b=\sum_{m\geq 0}b_m\lambda^{-m},\quad \text{and}\quad c=\sum_{m\geq 0}c_m\lambda^{-m}.$$

By using Definition 11, a direct calculation leads to

$$<V,\frac{\partial U}{\partial\lambda}>=\frac{1}{2}a,\quad <V,\frac{\partial U}{\partial q}>=a,\quad <V,\frac{\partial U}{\partial r}>=b,\quad \text{and}\quad <V,\frac{\partial U}{\partial s}>=-c. \tag{67}$$

Inserting the expressions (67) into the extended trace identity (18) and comparing the coefficient of λ^{-n-1} yields

$$\frac{\delta}{\delta u}\left(\frac{1}{2}a_{n+1}\right)=(\gamma-n)\begin{pmatrix}a_n\\ b_n\\ -c_n\end{pmatrix}. \tag{68}$$

Setting $n=0$ in Eq.(68) gives $\gamma=0$.
Thus, we may present the Hamiltonian function as follows:

$$\frac{\delta H_n}{\delta u}=\begin{pmatrix}a_n\\ b_n\\ -c_n\end{pmatrix},\quad H_n=-\frac{a_{n+1}}{2n}. \tag{69}$$

Therefore, the bi-Hamiltonian structure of the multi-component system (65) is presented as

$$u_t=\begin{pmatrix}q\\ r\\ s\end{pmatrix}_t=J\frac{\delta H_n}{\delta u}=K\frac{\delta H_{n-1}}{\delta u}, \tag{70}$$

where $K=JL=\begin{pmatrix}0 & 2s*q*-r*\partial & 2r*q*-s*\partial\\ -\partial r*-2s*q* & \partial q*+q*\partial & -\frac{1}{2}\partial^2-2q*q*\\ -2q*r*-\partial s* & 2q*q*+\frac{1}{2}\partial^2 & -q*\partial-\partial q*\end{pmatrix}$.

It is easy to verify that $JL=L^*J$, hence, the system (65) is integrable in Liouville sense.

Acknowledgments

The first author expresses her appreciation to the Soliton Research Team of Shanghai University, China for useful discussion.

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